

YuMi Deadly Maths

Geometry

Prep to Year 9

Prepared by the YuMi Deadly Centre
Queensland University of Technology
Kelvin Grove, Queensland, 4059

Prep to Year 9: Resource 5 – Geometry





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The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

DEVELOPMENT OF THIS BOOK

This version of the YuMi Deadly Maths Geometry book is a modification and extension of a book developed as part of the Teaching Indigenous Mathematics Education (TIME) project funded by the Queensland Department of Education and Training from 2010–12. The YuMi Deadly Centre acknowledges the Department’s role in the development of YuMi Deadly Maths and in funding the first version of this book.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at QUT which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

“YuMi” is a Torres Strait Islander Creole word meaning “you and me” but is used here with permission from the Torres Strait Islanders’ Regional Education Council to mean working together as a community for the betterment of education for all. “Deadly” is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre’s motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre’s vision: *Growing community through education*.

The YuMi Deadly Centre can be contacted at ydc@qut.edu.au. Its website is <http://ydc.qut.edu.au>.

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ABOUT YUMI DEADLY MATHS

From 2000–09, researchers who are now part of the YuMi Deadly Centre (YDC) collaborated with principals and teachers predominantly from Aboriginal and Torres Strait Islander schools and occasionally from low socio-economic status (SES) schools in a series of small projects to enhance student learning of mathematics. These projects tended to focus on a particular mathematics strand (e.g. whole-number numeration, operations, algebra, measurement) or on a particular part of schooling (e.g. middle school teachers, teacher aides, parents). They resulted in the development of specialist materials but not a complete mathematics program (these specialist materials can be accessed via the YDC website, <http://ydc.qut.edu.au>).

In October 2009, YDC received funding from the Queensland Department of Education and Training through the Indigenous Schooling Support Unit, Central-Southern Queensland, to develop a train-the-trainer project, called the **Teaching Indigenous Mathematics Education** or **TIME** project. The aim of the project was to enhance the capacity of schools in Central and Southern Queensland Indigenous and low SES communities to teach mathematics effectively to their students. The project focused on Years P to 3 in 2010, Years 4 to 7 in 2011 and Years 7 to 9 in 2012, covering all mathematics strands in the Australian Curriculum: Number and Algebra, Measurement and Geometry, and Probability and Statistics. The work of the TIME project across these three years enabled YDC to develop a cohesive mathematics pedagogical framework, **YuMi Deadly Maths**, that covers all strands of the *Australian Curriculum: Mathematics* and now underpins all YDC projects.

YuMi Deadly Maths (YDM) is designed to enhance mathematics learning outcomes, improve participation in higher mathematics subjects and tertiary courses, and improve employment and life chances. YDM is unique in its focus on creativity, structure and culture with regard to mathematics and on whole-of-school change with regard to implementation. It aims for the highest level of mathematics understanding and deep learning, through activity that engages students and involves teachers, parents and community. With a focus on big ideas, an emphasis on connecting mathematics topics, and a pedagogy that starts and finishes with students' reality, it is effective for all students. It works successfully in different schools/communities as it is not a scripted program and encourages teachers to take account of the particular needs of their students. Being a train-the-trainer model, it can also offer long-term sustainability for schools.

YDC believes that changing mathematics pedagogy will not improve mathematics learning unless accompanied by a whole-of-school program to challenge attendance and behaviour, encourage pride and self-belief, instil high expectations, and build local leadership and community involvement. YDC has been strongly influenced by the philosophy of the Stronger Smarter Institute (C. Sarra, 2003) which states that any school has the potential to rise to the challenge of successfully teaching their students. YDM is applicable to all schools and has extensive application to classrooms with high numbers of at-risk students. This is because the mathematics teaching and learning, school change and leadership, and contextualisation and cultural empowerment ideas that are advocated by YDC represent the best practice for **all** students.

YDM is now available direct to schools face-to-face and online. Individual schools can fund YDM in their own classrooms (contact ydc@qut.edu.au or 07 3138 0035). This Geometry resource is part of the provision of YDM direct to schools and is the fifth in a series of resources that fully describe the YDM approach and pedagogical framework for Prep to Year 9. It focuses on teaching geometry and covers 3D and 2D shape, line and angle, coordinates and graphing, Euclidean transformations, and projective and topology. It overviews the mathematics and describes classroom activities for Prep to Year 9. Because YDM is largely implemented within an action-research model, the resources undergo amendment and refinement as a result of school-based training and trialling. The ideas in this resource will be refined into the future.

YDM underlies three projects available to schools: YDM Teacher Development Training (TDT) in the YDM pedagogy; YDM AIM training in remedial pedagogy to accelerate learning; and YDM MITI training in enrichment and extension pedagogy to build deep learning of powerful maths and increase participation in Years 11 and 12 advanced maths subjects and tertiary entrance.

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1 Purpose and Overview

Human thinking has two aspects: verbal logical and visual spatial. Verbal logical thinking, associated in some literature with the left hemisphere of the brain, is the conscious processing of which we are always aware. It tends to operate sequentially and logically and to be language and symbol (e.g. number) oriented. On the other hand, visual spatial thinking, associated in some literature with the right hemisphere of the brain, can occur unconsciously without us being aware of it. It tends to operate holistically and intuitively, to be oriented towards the use of pictures and to be capable of processing more than one thing at a time – as such it can be associated with what some literature calls simultaneous processing.

Our senses and the world around us have enabled both these forms of thinking to evolve and develop. To understand and to modify our environment has required the use of logic, the development of language and number, an understanding of the space that the environment exists in and an understanding of shape, size and position that enables these to be visualised (what we call geometry). Thus, as it is a product of human thinking that has emerged from solving problems in the world around us, mathematics has, historically and presently, two aspects at the basis of its structure: number and geometry.

It is very difficult to find two people who agree on a precise definition of geometry. However, the following viewpoints provide a basis for discussion and indicate clearly that the scope of geometry is far wider than the narrow horizons of Euclidean geometry that many of us remember from our high school days.

Geometry is sometimes thought of as investigation or discovery of pattern and relationship in shape, size and place (position). These are observed in and derived from the immediate environment and the much wider world, both natural and manmade.

Geometry is the exploration of space. A learner, from the moment he is born, explores space. First he looks at it, then he reaches out into it, and then he moves in it. It takes a long time for him to develop the idea of perspective, of distance and depth, and notions such as inside and outside, back and front, before and after, and so on. In school, the development that has already taken place should naturally be encouraged and extended through the learner's own experience.

Geometry is used in life when you read a map, give directions, arrange furniture, reinforce a sagging door or stabilise a windblown tree. Geometry also serves many people in their professional lives; for example, scientists, engineers, architects, carpenters and elementary school teachers all use geometry in their daily work. In particular, there is a lot of geometry in construction, fitting and turning, commercial artist and design work, engineering, automotive repair, and so on.

This resource looks at the teaching of the Geometry strand of mathematics. It provides the framework and detail of Years P to 9 Geometry as perceived by YuMi Deadly Maths (YDM). (*Note: School geometry appears to be the strand which is not so confronting to non-Western cultures, and to be a field in which nearly all cultures have excelled, particularly with respect to the geometric aspects of art.*)

1.1 Connections and big ideas

This section overviews the role geometry plays in the structure of mathematics, describing the important **connections** in geometry, how it is based on a series of **big ideas** that recur across Years P to 9, and how **relationship vs change** is central to the structure of geometry.

Big ideas are mathematical ideas that recur across the years of schooling and can be used in many situations. They are built slowly over time and not in one lesson. Big ideas can be (a) **global** – that is, so big they cover most of mathematics (e.g. part-whole); (b) **concepts** – for example, the concepts of square and rectangle that play a

large part in models for multiplicative situations such as arrays, fractions and area; (c) **principles** – for example, relationships where meaning is encapsulated in the arrangement of the parts not what the parts are, such as interior angle sums of n -sided polygons are $(n-2) \times 180^\circ$ degrees; (d) **strategies** – for example, “rules of thumb” that point towards solutions such as using proportion in similar shapes; and (e) **teaching big ideas** – for example, the Reality–Abstraction–Mathematics–Reflection (RAMR) model and reversing.

1.1.1 Geometry connections

YDM advocates that mathematics should be taught so that it is accessible as well as available, that is, learnt as a rich schema containing knowledge of when as well as how (see YDM Overview book). Rich schema has knowledge as connected nodes which facilitates recall (it is easier to remember a structure than a collection of individual pieces of information) and problem solving (content that solves problems is usually peripheral, along a connection from the content on which the problem is based). The Reality and Mathematics components of the RAMR cycle are built, in part, around connections.

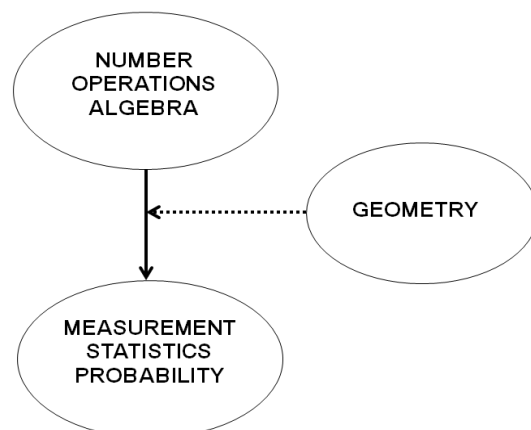
As a consequence, YDM argues that knowledge of the structure of mathematics, particularly of connections and big ideas, can assist teachers to be effective and efficient in teaching mathematics. This is because it enables teachers to:

- (a) **determine what mathematics is important to teach** – mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present;
- (b) **link new mathematics ideas to existing known mathematics** – mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem solving;
- (c) **choose effective instructional materials, models and strategies** – mathematics that is connected to other mathematics or based around a big idea commonly can be taught with similar materials, models and strategies; and
- (d) **teach mathematics in a manner that makes it easier for students’ future teachers to teach more advanced mathematics** – by preparing the linkages to other ideas and the foundations for the big ideas the future teacher will use.

Thus it is essential that teachers know the mathematics that precedes and follows what they are teaching: it enables them to build on the past and prepare for the future.

European mathematics grew out of two views of reality: the first was number, the amount of discrete objects present; and the second was the world around us, the shapes and the spaces we live in, that is, geometry. The basis of number was the unit, the one. Large numbers were formed by grouping these ones, and small numbers (e.g. fractions) by partitioning these ones into equal parts. The operations of addition and multiplication, and the inverse operations of subtraction and division, were actions on these ones which joined and separated sets of numbers. Algebra was constructed by generalising number and arithmetic, and representing general results with letters.

When geometry is added to the mix, number, operations and algebra give rise to applications within measurement, and statistics and probability. This relationship is diagrammatically represented on the right. This gives a framework for Years P to 9 mathematics that enables mathematics as a whole to be considered. It provides an overview and sequence for the connections upon which teaching should be built (e.g. number and geometry before measurement; the relationship between fractions and probability).



1.1.2 Big ideas underlying geometry

Big ideas are very important to teaching mathematics – they are generalities and, if learnt well, assist recall, last the learner many years, provide mathematics to cover many situations, and can be applied to solve many mathematics problems. Thus, they are a powerful way to learn mathematics.

The important big ideas for geometry are listed in the YDM Supplementary Resource Big Ideas book. Some important ones are listed below.

Global big ideas

1. **Change vs relationship.** Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. Everything can be seen as a change (e.g. 2 goes to 5 by +3, similar shapes are formed by “blowing one up” using a projector) or as a relationship (e.g. 2 and 3 relate to 5 by addition, similar shapes have angles the same and sides in proportion or equivalent ratio). Geometry can be the study of shape or transformations between shapes (e.g. flips, slides and turns).
2. **Interpretation vs construction.** Things can either be interpreted (e.g. what operation for this problem, what principles for this shape) or constructed (write a problem for $2+3=5$; construct a shape of four sides with two sides parallel). This is particularly true for geometry – shapes can be interpreted or constructed.
3. **Parts vs wholes.** Parts (these can also be seen as groups) can be combined to make wholes (this can also be seen as a total), and wholes can be partitioned to form parts (e.g. fraction is part to whole, ratio is part to part; addition is knowing parts and wanting whole, division is partitioning a whole into many equal parts – the whole area can be partitioned into equal square units). In geometry, this big idea is particularly applicable to dissections and tessellations.

Concept big ideas

1. **Turn, angle, line, parallel, perpendicular.** The meanings of these five geometric ideas.
2. **2D or plane shapes, diagonals, rigidity.** The meanings of these terms including the meanings of common shapes (e.g. circles, squares, triangles, and so on).
3. **3D or solid shapes, vertices, edges, surfaces.** The meanings of these terms including the meanings of common shapes (e.g. prism, pyramid, cylinder, cone, cube, sphere, and so on).
4. **Polar and Cartesian coordinates, maps, graphs.** The meanings of these terms.
5. **Symmetries, flips-slides-turns, tessellations, dissections.** The meanings of these terms.
6. **Projections, topological changes, similarity, perspective, networks.** The meanings of these terms.

Principle big ideas

1. **Angle formulae.** Polygon interior angle sum = $180^\circ \times (\text{number of sides} - 2)$; polygon exterior supplementary angle sum = 360° ; line crossing a parallel has corresponding and alternate angles equal; and so on.
2. **Length, diagonal and rigidity relationships.** Pythagoras’ theorem (square of hypotenuse equals square of adjacent side plus square of opposite side for right-angle triangles); the number of diagonals in a polygon is equal to half the product of number of sides and number of sides minus three; number of diagonals to make a polygon rigid is three less than the number of sides; and so on.
3. **Pythagoras’ theorem.** For a right-angle triangle, the square of the length of the hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides.
4. **Transformational invariance.** Topological transformations change straightness and length, projective transformations change length but not straightness, and Euclidean transformations change neither. (*Note:* Affine projections leave parallelness unchanged, while similarity projections leave parallelness unchanged)

and sides in ratio.) This gives implications for shape, angle and what does not change (invariance) for each of the types of transformations.

5. **Line graph relationships.** Relationships for line graphs based on $y = mx + c$ where slope = m and y -intercept = c . These include: distance between (a, b) and $(c, d) = \sqrt{(c - a)^2 + (d - b)^2}$, and slope = $\frac{d-c}{b-a}$.
6. **Transforming shapes relationships.** These include (a) *identity* – actions that leave things unchanged, for example, in geometry, a 360° turn does not change things for flips, slides, turns; and (b) *inverse* – actions that undo other actions, for example, in geometry, 90° turns are the inverse of 270° turns, and a flip is the inverse of itself.
7. **Line symmetry–reflection relationships.** Number of lines of symmetry equals number of rotations of symmetry when there are two or more lines of symmetry, with the angle between lines half the angle between rotations; two reflections through the centre equal one rotation where the angle between reflections is half the angle of rotation; two reflections equal one translation where the reflection lines are perpendicular to the translations and the distance between lines is half the distance of the translation.
8. **Trigonometric relationships.** Similar right-angled triangles (having the same angles) have the same three ratios for their angles other than the right angle:
 - sine = opposite length/hypotenuse length
 - cosine = adjacent length/hypotenuse length
 - tangent = opposite length/adjacent length
9. **Euler’s formula.** Nodes/corners plus regions/surfaces equals lines/edges plus two (holds for 3D shapes and networks).

Strategy big ideas

1. **Visualisation.** Visualising shapes in the mind and mentally rotating the shape or yourself around the shape.
2. **Manipulation of shapes.** Body \rightarrow hand as a strategy – dropping back to a more real environment.

Teaching big ideas

1. **RAMR model.** The sequence reality–abstraction–mathematics–reflection continues to hold. In particular, reversing remains important (e.g. shape to symmetries **and** symmetries to shape).
2. **Recording with words and drawings.** Assists the mind to structure ideas in the mind.
3. **Van Hiele levels.** Move through experience (play, materials, language) to informal/analysis (determining properties) to formal/synthesis (determining relationships) to deductive (proof).
4. **Part \rightarrow whole and whole \rightarrow part.** This can be seen in the two ways of teaching relationship geometry.

For YDM, curricula should be taught so that connections and principles are emphasised. One of YDC’s ambitions is to design teaching frameworks that contain the knowledge of how to do this by specifying and sequencing topics and big ideas within each year level and drawing attention to connections. This will take some time to finalise as connections and big ideas are developed across the years by looking at similarities between different forms of the same mathematics in the same and different topics and year levels. It is planned that the next draft of all the books will have teaching frameworks for Years P to 9 (in line with the Australian Curriculum) but emphasising big ideas and connections.

1.1.3 Relationship geometry vs transformational geometry

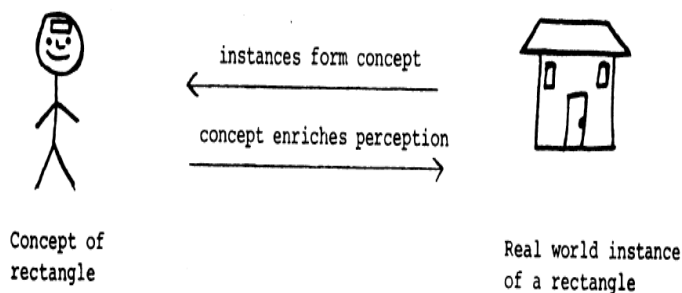
These two global approaches to mathematics have an important implication in geometry. YDM has used them to structure its sections and sequencing (see section 1.2).

Relationship geometry

Relationship geometry is concerned with how shapes relate to each other and this is best seen through studying 3D and 2D shape, line and angle. There are two approaches to teaching relationship geometry that are worth discussing here: the environmental approach which starts from reality and moves from 3D → 2D → properties; and the sub-concept approach which starts from position and direction and moves from properties → 2D → 3D (the inverse direction).

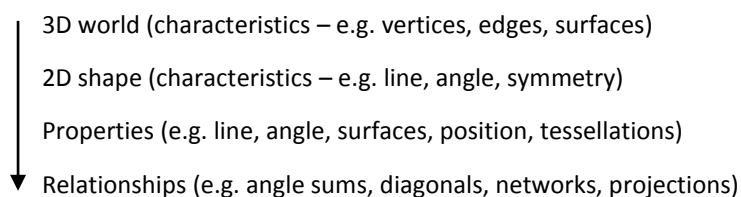
The environmental approach (or 3D approach)

With this approach, the starting point and organising imperative for teaching is the environment – the everyday 3D world around the learners. Ideas are first developed from instances in the world and these ideas then improve the observation powers, as on the right.



In developing geometric concepts and processes, learners are given practical

experiences classifying, constructing and manipulating solids. Discussion of these experiences leads to identifying key distinguishing properties: corners, flat and curved surfaces, edges, ability to roll, and so on. Two-dimensional shapes are presented as they are encountered in the three-dimensional solids: rectangles from table tops, circles from clocks, triangles from roofs of houses, and so on. These 2D shapes are then investigated for their properties. Discussion of the positioning of 3D solids leads to language development relating to position. Thus the sequence is:



After the practical experiences with the 3D solids and 2D shapes, there is more specialised investigation of these shapes. As appropriate, certain broad unifying concepts are developed which allow more efficient and action-oriented investigation of properties (e.g. parallel lines, angle sums).

Overall, the development involves the following: (a) intuitive experience with concepts and materials; (b) construction of appropriate language, more detailed investigation of particular instances, identification of attributes and properties, construction of shapes through a variety of techniques and the combination of shapes to give larger ones; (c) construction of a systematic scheme of properties, which shows shared principles and connections between concepts and shapes; and (d) applications of the above understandings to problem situations and the development of visual imagery.

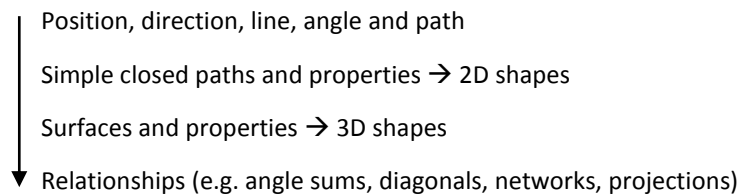
The sub-concept approach

Here the starting point for teaching is to analyse a task into its prior abilities and to order these abilities the way they should be developed in learners. The ideas are then taught, building from sub-concepts and sub-processes to final concepts or processes. For a rectangle, the sub-concepts are line (straight and parallel), turn and angle, right angle, closed, simple (lines do not cross), and path – the rectangle is a simple closed path or boundary of four sides with opposite sides equal and parallel and all angles 90 degrees.

We start with the concepts of position and direction which lead to line and angle. Lines are joined at angles to give paths. The ends of these paths are joined to give closed boundaries. These closed boundaries enclose regions and, if simple, one region whose boundary is a 2D shape. Simple closed paths are seen as products of their paths and, unlike the 3D approach, it is not necessary to investigate these 2D shapes for their properties and principles

to the same extent. For example, a figure formed from four equal straight lines (composed of parallel pairs) with four right angles is a square. These do not have to be found as the properties of a square, they are given before the shapes are constructed.

The resulting 2D shapes are studied for relationships (including Pythagoras' theorem) and then used as surfaces to assemble 3D solids (with properties being given before the solid shape is constructed), and these solids are then studied for their relationships. The sequence is:



Since the environmental approach is more experiential, it is commonly taught earlier. Then, the sub-concept approach is used to reverse the direction and build deeper understandings of 2D and 3D shapes.

Transformational geometry

Transformation geometry is concerned with three changes:

- (a) topological (how living things change – length and straightness change);
- (b) projective (how our eyes see the world – length changes, straightness does not change); and
- (c) Euclidean (how the human-made world changes – length and straightness both do not change).

The Australian Mathematics Curriculum emphasises Euclidean and these changes are flips, slides and turns. They are similar to symmetries and lead to tessellations and dissections. They also underpin congruence. Thus, these topics are covered in the Euclidean Transformations chapter of this book (see section 1.2 below).

Projective and topological changes are not a strong part of the Australian Mathematics Curriculum but are **common in NAPLAN items**, where they are seen as part of Location. They begin with visual experiences, and lead to maps, coordinates, simple study of projections and topological change, perspective and networks. One form of projection leads to similarity and scale, and this is the basis of trigonometry. Thus, these topics are covered in the Projective and Topology chapter of this book (see section 1.2 below).

1.2 Sequencing

This section looks at the structure of geometry topics that organise this book – the sections that the book is divided into and what is under each chapter. This mathematics content decision is based on the two types of geometry that can be part of a curriculum, **relationship** and **transformation**, and the extent to which the ideas should be in a Prep to Year 9 Geometry curriculum (see **Appendix A**).

Note: At this point, it must be understood that YDM's proposal for the Prep to Year 9 Geometry curriculum is an **enrichment** of what is in the Australian Mathematics Curriculum.

1.2.1 Overall sequencing

By its very nature, geometry does not have the dominating sequential nature of arithmetic and much more teacher choice is available in determining appropriate teaching sequences. There are also many experiences in geometry not directly connected to the development of rules and general procedures but rather to the development of imagery and intuition and as such they may not be recognised by teachers to be important.

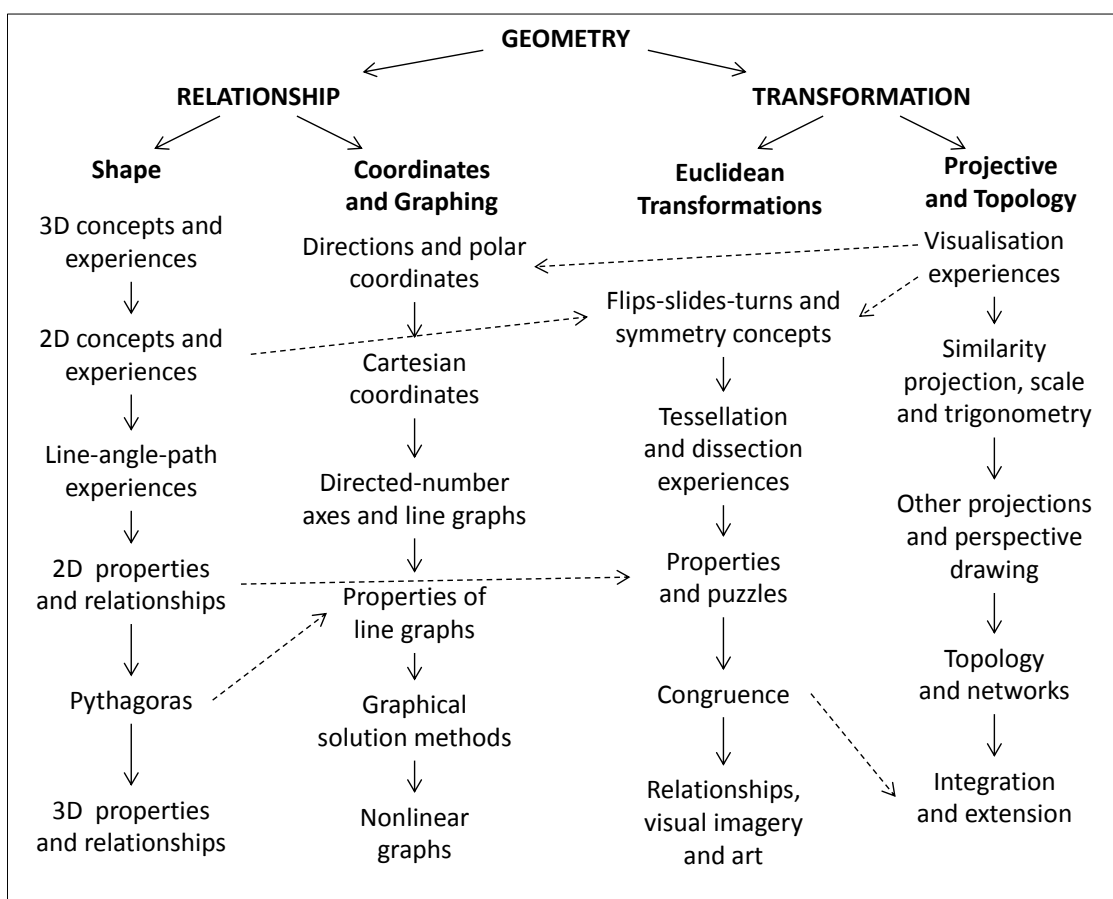
Thus, YDM has developed a sequence for geometry that is based on, but enriches, the geometry in the Australian Curriculum and in NAPLAN testing. YDM recommends that the study of geometry includes the following:

- (a) two overall approaches: relationship geometry and transformational geometry;
- (b) two sections in relationship geometry: shape (line, angle, 2D and 3D shape, and Pythagoras' theorem); and coordinates and graphs (polar and Cartesian coordinates, line graphs, slope and y-intercept, and graphical solutions to unknowns); and
- (c) two sections in transformational geometry: Euclidean (flips, slide, turns and congruence); and projective and topology (projections, similarity, trigonometry, perspective and networks).

This has resulted in there being four components to the structure of the book:

1. **Shape** – 3D and 2D shape, line and angle and their properties (using both environmental and sub-concept approaches, and including Pythagoras' theorem);
2. **Coordinates and graphing** – directions and polar coordinates, Cartesian coordinates, directed numbers, axes and line graphs, properties of line graphs, graphical solution methods, nonlinear graphs;
3. **Euclidean transformations** – the geometry of flips, slides and turns, symmetry, tessellations and dissections (puzzles) leading to congruence; and
4. **Projective and topology** – covering visualisation, similarity projection, other projections, topology and networks, and integration and extension across topics.

The four components form a chapter each, with the overall sequence as in the figure below.



1.2.2 Topics and within-section sequencing

Within the four main teaching chapters, YDM Geometry has the following topics and sub-topics. Full descriptions of each of these are given in **Appendix A**.

1. Shape

- (a) 3D shape (solids), particularly polyhedra, prisms, pyramids, cones, cylinders and spheres;
- (b) 2D shape (plane shapes), particularly polygons (e.g. triangles, rectangles, pentagons) and circles, angle and diagonal principles, rigidity and Pythagoras; and
- (c) line and angle, particularly straight and parallel lines, right, acute, obtuse and reflex angles and angle principles.

2. Coordinates and graphing

- (a) early position, language, reading and drawing maps, and moving on to Cartesian and polar coordinates, orienteering, and latitude and longitude;
- (b) coordinate systems (polar and Cartesian), length and direction, x and y axes (including directed-number axes);
- (c) properties of line graphs such as slope, y -intercept, distance and midpoint; and
- (d) graphical methods to find solutions for unknowns plus nonlinear graphs.

3. Euclidean transformations

- (a) flips (reflections), slides (translations) and turns (rotations), their properties and the relationships between them;
- (b) symmetry (line and angle) covering identification, change, construction and relationship between line and angle symmetry;
- (c) tessellations (tiling patterns) covering shapes that tessellate, properties, tessellation grids, 3D tessellations, art, fabric design and shape puzzles;
- (d) dissections (like jigsaw puzzles) of two types (parts \rightarrow shape and shape \rightarrow parts \rightarrow new shape) developing visual imagery; and
- (e) congruence meaning same size and shape (two shapes are congruent if you can get from one to the other by flips, slides and turns alone).

4. Projective and topology

- (a) visualisation – mental rotation of shape and self, computer simulations, virtual special effects and cartooning;
- (b) similarity, scale and trigonometry – similarity as enlargement or scale (same angles and sides in ratio), similar right-angled triangles, height measurement and trigonometry;
- (c) projections and perspective – studying properties and principles of three projections (similarity or enlargement, affine or parallel light/sunlight, and diverging light as eyes see world), leading to perspective drawings; and
- (d) topology and networks – studying the change where bending, stretching and twisting but no cutting or joining is allowed, leading to puzzles (e.g. Möbius strip) and to networks and Euler's formula.

For ease of presentation, the activities in the chapters have sometimes been **divided by these topics** instead of overall sequential development (however development has been shown within topics in these cases). An example of this is Chapter 2 Shape, where the division is around information, environmental approach and sub-concept approach. However, in the lower secondary year levels, topics are often integrated to produce the more advanced knowledge and the more overall objectives of relationships and visual thinking (e.g. coordinates and flips/slides/turns, relationships between flips and slides/turns and line and rotational symmetry).

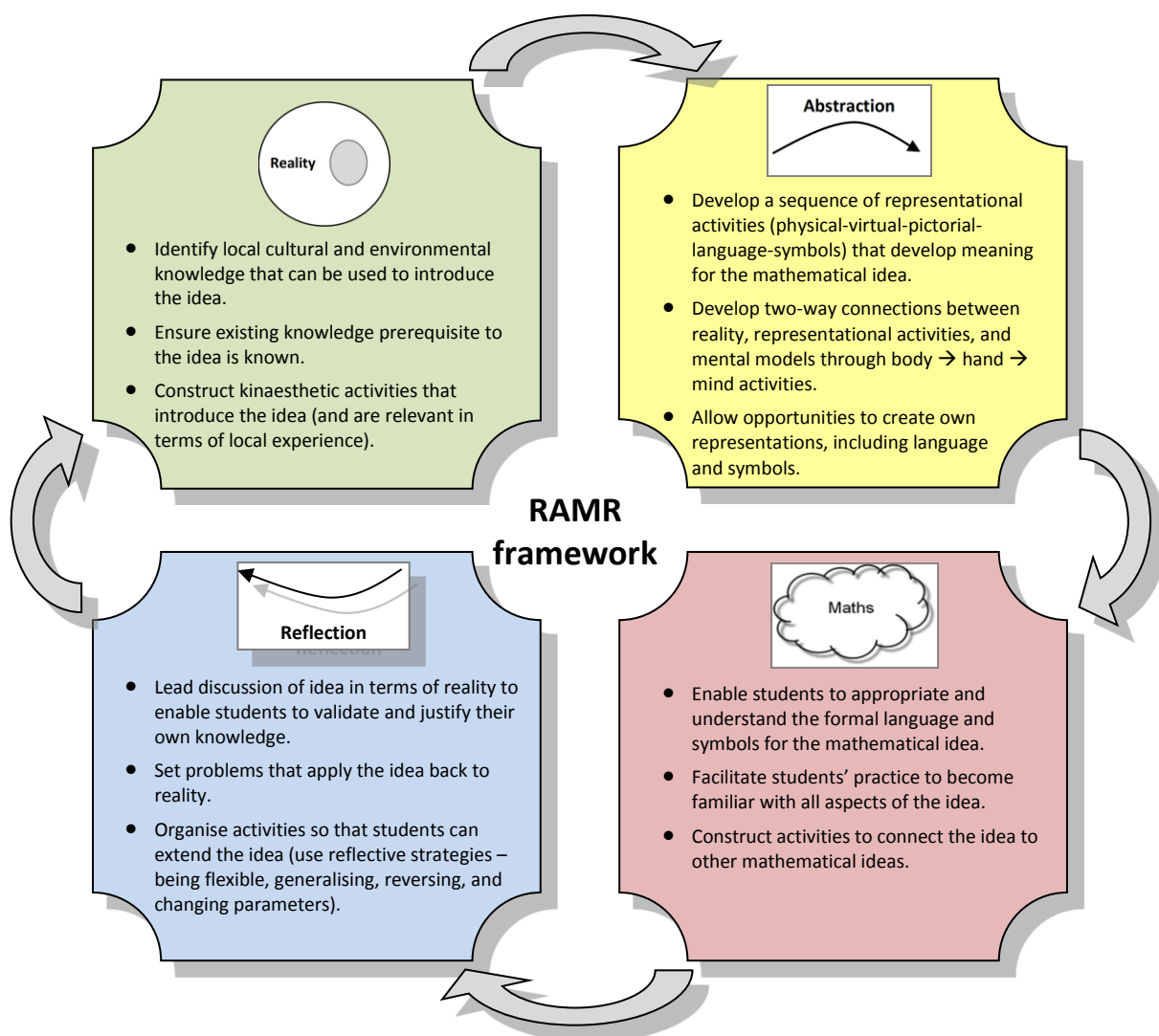
Thus, each chapter will **conclude with a section on integration and extension** which will cover work at the end of the schooling years on which these booklets focus. This will bring the **sequence in line with the Van Hiele levels** as stated in sections 1.1.2 and 1.4.2.

1.3 Teaching and cultural implications

This section looks at the implications of geometry for teaching and culture. In this regard, geometry is similar to algebra in that it is easily taught whole to part. Further, sequence, although evident, is not as important as for number. In fact, an effective way to teach geometry is by **themes** – for example, just picking a topic like triangles, and doing everything that one can do with triangles regardless of complexity.

1.3.1 Generic teaching strategies

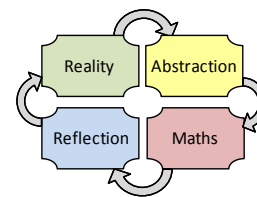
YDM sees mathematics teaching as having three components – **technical** (handling materials), **domain** (the particular pedagogies needed for individual topics) and **generic** (pedagogies that work for all mathematics). Interestingly, and fortunately, the domain section is not as complicated as it could be because mathematical ideas that are structurally similar can be taught by similar methods. For example, fractions and division are similar and both are taught by partitioning sets into equal parts – except that the set is seen as one whole for fractions and a collection of objects for division. There are also some generic teaching methods that hold for any topic.



The **RAMR framework** (see the figure above) is very useful for geometry because of the generic teaching ideas contained in the framework. For a start, it grounds all mathematics in reality and provides many opportunities for connections, flexibility, reversing, generalisations and changing parameters, as well as body → hand → mind.

The idea is to use the framework and all its components throughout the years of schooling and this will help prevent learning from collapsing back into symbol manipulation and the quest for answers by following procedures.

Within the chapters of this book, activities are sometimes provided using the RAMR framework. This is to enable these activities to act as exemplars of the RAMR framework. Some activities are just described without the stages. Activities that are given in RAMR framework form can be identified as they will begin with the symbol on right. These activities will also be written in italics so they are easy to distinguish from the rest of the text.



1.3.2 Cultural implications for geometry

As described in the YDM Number and YDM Operations books, Indigenous cultures from around the world followed a different path from number-oriented cultures in the development of mathematics; for Indigenous cultures, people were seen as more important than number so their mathematics specialised in other areas. This different focus could be seen as emanating from cultural beliefs with regard to group rather than individual ownership. Thus, the teaching of number, operations and measurement can bring Australian mainstream Eurocentric school teaching into conflict with Indigenous students, both Australian and immigrant; it can be a topic that can designate these cultures as primitive. It must be taught with care as part of a European culture that Indigenous people need to understand. It should not be celebrated as something that raises people above others.

For low socio-economic status (SES) and non-Indigenous students in Australia, the situation is different, but the outcome is the same. These people are part of Australia's Eurocentric number-oriented culture, but at the lower unsuccessful end. The number systems created as part of Eurocentric mathematics have always benefited high SES people at the expense of low SES people, promulgating the idea that bigger numbers (e.g. money, house cost, cars) are better and indicate that the person with the bigger numbers is more successful. The way numbers function within Eurocentric societies achieves two outcomes simultaneously: (a) benefits one class of people at the expense of the other; and (b) puts the blame for their lack of benefit on the actions of the class that is not benefited. Thus, the mathematics of number, operations and measurement must be taught with care to low SES students to avoid designating such students as failures in a similar way that its teaching can designate Indigenous peoples as primitive.

However, the teaching of geometry appears not to have the same problems for Indigenous and low SES students as the teaching of number, operations and measurement. The first reason for this is that the learning style and thinking strengths of both Aboriginal and Torres Strait Islander and low SES students are **holistic-intuitive** and **visual-spatial** rather than **verbal-logical**, meaning that they have an affinity for geometry because this is mathematics taught as a structural whole and in a visual-spatial form. Second, traditional Australian teaching of mathematics is procedural and rote and focuses on number and arithmetic as algorithmic procedures, while geometry lends itself to hands-on, investigative, and thematic learning. The third reason is that geometry teaching, if done expertly, involves construction and pattern, and a strong connection to trades and art, both endeavours that attract Aboriginal, Torres Strait Islander and low SES students. Finally, geometry is a strong cultural tradition for Aboriginal and Torres Strait Islander people, one in which they cannot be considered as primitive and simplistic.

The YDM framework is designed to enrich the Australian Mathematics Curriculum. This bolstering does not result in new work, just a better way of developing teaching that **reflects big ideas** and is more in harmony with Aboriginal, Torres Strait Islander and low SES students.

1.4 Overview of book

This section looks at the book in overview and discusses the teaching approach to geometry it advocates.

1.4.1 Overview of sections

The book has six sections, including four central chapters looking at the major components. The six sections are:

Chapter 1: Purpose and overview – this chapter, covering connections, big ideas, sequencing, teaching and culture, and overview of book;

Chapter 2: Shape – covering 3D and 2D shape, line and angle using both relationship approaches (environmental and sub-concept);

Chapter 3: Coordinates and graphing – covering polar and Cartesian coordinates and graphing linear equations;

Chapter 4: Euclidean transformations – covering flips/slides/turns, symmetry, tessellations, dissections and congruence (using transformational approach and focusing on visual imagery);

Chapter 5: Projective and topology – covering visualisation, similarity projection including trigonometry, other projections and perspective, topology and networks; and

Appendix A: Major geometric ideas for each of the above four main topics.

With respect to teaching, there are four components as follows (in line with all YDM booklets):

1. **Culture, community and whole school.** YDM projects take account of culture, involve community and are based on whole-school programs. Geometry is important here because it is a strength of Aboriginal, Torres Strait Islander and low SES students; their understanding is often stronger for the holistic–intuitive or visual–spatial activities.
2. **Connections and big ideas.** YDM sees mathematics as a structure, a language and a tool for problem solving and reasoning. Visual problem solving and reasoning are important parts of this and the basis of many jobs from brain surgeon to carpenter.
3. **RAMR framework.** YDM teaching is based on a cycle of teaching built around four stages: reality, abstraction, mathematics, and reflection. This is well explained in the YDM Overview book.
4. **Teaching framework.** YDM is endeavouring to provide a framework of content (i.e. topics and sequences) and proficiencies (i.e. understanding and fluency). This is based on the Australian Mathematics Curriculum but adds richer information to show how big ideas and connections link into the year levels and the curriculum content statements.

1.4.2 Specific teaching of geometry

Similar to the other strands of mathematics, geometry can be seen as a structure, as a language and as a tool for problem solving. Too often in the past teachers have focused on the language aspect – developing the names for various shapes (e.g. prisms, polygons, cylinders), rules for relationships such as similarity (e.g. equal angles) and procedures for constructions (e.g. bisecting an angle). Yet some of the more interesting activities are associated with structure (e.g. Euler’s formula, the relationship between slides/turns and flips) and development of problem-solving skill (e.g. dissections, tessellations), particularly with respect to visual imagery.


Thus, geometry can be one of the most exciting and interesting parts of mathematics. It provides an opportunity for motivating learners that should not be missed. It can be colourful and attractive. Pattern and shape can be created and admired. Success can be enjoyed by the majority.

To allow the best development of geometric understandings, the following are important for effective sequencing of geometry teaching and learning.

1. The focus of geometry should be **from and to the everyday world** of the learner (as in the RAMR framework that is advocated in YDM).
2. There should be a balance between **geometry experiences** which enable learners to **interpret** their geometric world, and **geometry processes** where **problems are solved** with visual imagery; that is, geometry should be within a problem setting.
3. Learners' activities should be **multisensory** (using the students' bodies and actual physical materials and moving and transforming them – as in the body → hand → mind of the RAMR framework) and **structuring** (recording results on paper in words and pictures) – the “typical” geometry classroom would have groups, physical materials and pens and books ready to record, and there should also be opportunities for learners to display what they have made.
4. Teaching activities can also be **thematic**. For example, triangles could be studied across a time interval and include the following:
 - 2D and 3D shapes based on triangles (e.g. equilateral triangles, triangular prisms, stellated polyhedra);
 - role of triangles in rigidity and interior angle sums, plus Pythagoras' theorem;
 - flips-slides-turns with triangles, symmetries of triangles, triangle-based tessellations, triangle-based dissections (e.g. hexiamonds, McMahan 4-coloured triangles), and Escher patterns with triangles; and
 - congruence and similarity properties and trigonometry.

Van Hiele levels

Teaching activities should move through **three levels of development** (based on Van Hiele levels):

- 
- the **experiential level**, at which learners learn through interaction with their environment (shapes are identified and named – e.g. this is a triangle);
 - the **informal/analysis level**, in which certain shapes and concepts are singled out for investigation at an active, non-theoretical level (e.g. triangles have three sides and three angles); and
 - the **formal/synthesis level**, where a systematic study is undertaken and relationships identified (e.g. interior angle sum of triangles is 180 degrees).

At the experiential level, learners should be allowed to learn through experience with materials, not the teacher's words. Shapes can be labelled and described but not broken into their component parts. Learners should not be expected to be accurate in their statements.

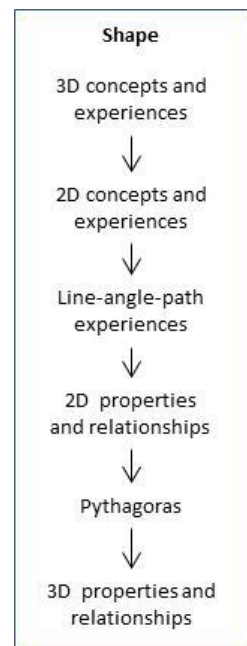
At the informal level, experiences can include analysing shapes for their properties/principles and constructing shapes from their properties. The sub-concept approach discussed earlier (section 1.1.3) would be appropriate here.

At the formal level, the focus should be on synthesis and relationships; principles such as congruence and similarity can be investigated, and formulae discovered. There should be no attempt at deductive proof and posing abstract systems.

This chapter covers three-dimensional shape (3D or solid shapes), two-dimensional shape (2D or plane shapes), and line and angle. This is the traditional geometry of the primary school. As such, it contains many types of shapes, lines and angles. Therefore, we have divided the chapter into four parts:

1. an overview of 3D solids, 2D plane shapes, lines and angles – the various types, their names, their properties, and a short summary of teaching ideas;
2. a selection of activities to teach shape, line and angle by the environmental approach;
3. a selection of activities to teach shape, line and angle by the sub-concept approach; and
4. a selection of activities to integrate and extend geometry.

The sequence for teaching shape is on the right and involves an integration of two approaches – the **environmental approach** that moves 3D → 2D → line/angle → properties → relationships; and the **sub-concept approach** that starts later and moves line/angle → properties → 2D → 3D → relationships.



Note: Many more ideas can be found in the old geometry book, *Space and Shape in the Primary School*, in the Resources section of the YDC website.

2.1 Overview of 3D shape, 2D shape, line and angle

This section provides information on the components of geometry (3D and 2D shapes, lines and angles).

2.1.1 Types of 3D shape

Formally, a solid or 3D shape is something that encloses a portion of space. It may or may not be filled (or “solid”). For example, a matchbox, whether it is filled or not, is a solid shape. True solid shapes must be completely closed. That is, they must have a “top”, a “bottom” and “sides”. These words have been placed in inverted commas because they are not the proper words to use. We will now develop the correct terms.

Three-dimensional shapes are important in vocational education and training (VET) because many trades are about building 3D constructions and knowing what is possible is always an advantage.

Solid shapes can be divided into two types: polyhedra (which have flat surfaces and straight-line edges, i.e. they have polygons for surfaces) and non-polyhedra (which have curved surfaces). Important examples of polyhedra are:



Prisms



Pyramids



Platonic Solids

Important examples of non-polyhedra are:



Cylinders



Cones



Spheres

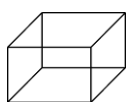
The major terms with regard to these solid shapes are:

- (a) *surface* – all solid shapes are enclosed by surfaces which are either curved or flat;
- (b) *face* – a flat surface of a solid shape;
- (c) *base* – a face on which the solid shape rests – a sphere has no base but all prisms and cylinders have two bases;
- (d) *edge* – an edge is formed on a solid shape when one surface meets another – an edge may be curved or straight; and
- (e) *vertex* – a special point on a solid shape when three or more straight edges meet – the cone also has a vertex but this is a special case; the plural of vertex is vertices.

Prisms

Each prism is made up of a certain number of flat surfaces called faces. The two parallel and congruent faces of each of the prisms illustrated below are called bases. There are many types of prisms because the base of a prism is a polygon and there are an infinite number of polygons. Each prism is named according to the shape of its base. All the other faces of a prism must be rectangular or square in shape.

Rectangular prism



Pentagonal prism



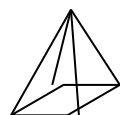
Hexagonal prism



Note: All prisms have at least five faces, two bases (at each end of the shape) which are congruent polygons and parallel, and all other faces rectangular.

Pyramids

Pyramids differ from prisms in that they have only one base and all their other faces are triangular. One end is a point or vertex. Each pyramid is named according to the shape of its base.



Rectangular Pyramid



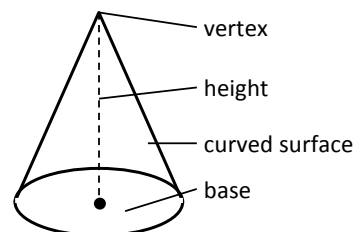
Pentagonal Pyramid



Hexagonal Pyramid

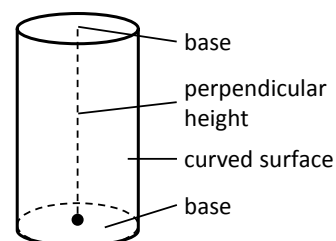
Cones

When the number of sides on the base of a pyramid increases without limit until the base finally becomes a “smooth” curve (a circle) without vertices, we have a cone. A cone has a base which is circular, a pointed top which is called a vertex, a curved surface from circle to vertex and a height which is found by a perpendicular line from vertex to base, as in the diagram on right.



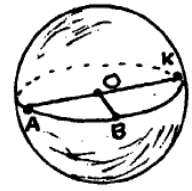
Cylinders

A cylinder is similar to a cone in having bases that are smooth curves. However, the cylinder is related to the prism in the same way the cone is related to the pyramid. When the sides of a prism increase without limit the bases become circular and we have a cylinder. A cylinder has two congruent and parallel bases, which are circular, a curved surface joining both bases and a height which is found by drawing a perpendicular from top base to bottom base, as in the diagram on right.



Spheres

The sphere is the most difficult solid to construct but probably the easiest to identify. Anything shaped like a round ball is a sphere, e.g. a tennis ball, golf ball, cricket ball, and so on. The diagram of the sphere on right has a completely curved surface. It has no polygonal base, no edges, and no vertices. The centre (O) of the sphere is the same distance from any point on its surface. Any line segment from the centre to the surface is the radius and any line segment through the centre from one side to the other is called a diameter of the sphere.



Platonic solids

These are polyhedra that are composed of congruent surfaces. There are only five of them and they are named by the number of surfaces.



Tetrahedron
(a pyramid)
4 triangles



Hexahedron
(a cube)
6 squares



Octahedron
8 triangles



Dodecahedron
12 pentagons

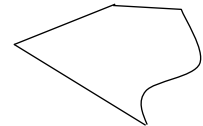


Icosahedron
20 triangles

2.1.2 2D shapes

Our world is three-dimensional. But we observe, perceive and represent the world two-dimensionally. Through evolution and experience, we have developed the ability to operate in our three-dimensional world using two-dimensional representations.

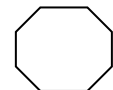
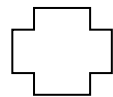
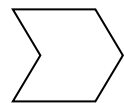
A 2D shape lies in a plane and is a closed simple boundary consisting of straight and curved lined segments. For example, the figure on the right is a shape because: (a) it is a boundary in a plane; (b) it is closed, not open; (c) it is simple, not crossing like a figure eight; and (d) it is composed of curved and straight lines. (*Note:* The point at which two straight lines or a curved and straight line meet or intersect is called a vertex, similarly to a 3D shape).



Polygons

Any closed simple plane figure consisting of three or more vertices and straight line segments (sides) joining them is called a polygon. The name comes from *poly*, a Greek word for many, and *gonos*, a Greek word meaning angles. There are many examples of polygons in everyday life – for example, the stop sign. Polygons can be **convex** or **non-convex**. A convex polygon does not “jut in”, while a non-convex polygon does “jut in” – the 12-sided figure on right is a non-convex polygon, while the 8-sided figure is a convex polygon. Polygons can be different sizes, and can have sides of unequal length and angles of unequal size. Polygons are named from their number of sides (using Latin). The names are:

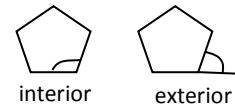
- 3 sides – triangle
- 4 sides – quadrilateral
- 5 sides – pentagon
- 6 sides – hexagon
- 7 sides – heptagon
- 8 sides – octagon, and so on.



Polygons with sides of equal length and angles of equal size are called **regular polygons**. The hexagon on the right is an example of a regular polygon.



Polygons have interior and exterior angles as shown on right. The exterior angle is the turn if one was walking around the shape.



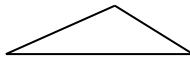
interior

exterior

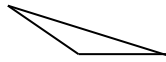
Triangles have three sides and three angles. Triangles are named by their angles and lengths of sides:



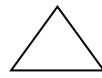
Right triangle



Acute triangle



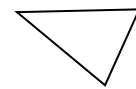
Obtuse triangle



Equilateral triangle



Isosceles triangle



Scalene triangle

Quadrilaterals are four-sided figures. They can be categorised according to whether or not they have sides in parallel. Their names are determined by their angles and lengths of sides as well as parallelness. The most important are:



Square (equal sides)



Rectangle



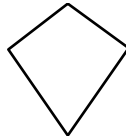
Rhombus (equal sides)



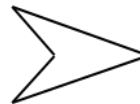
Parallelogram



Trapezium



Kite



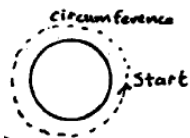
Arrowhead



Irregular

Circles

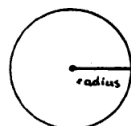
Circles are a boundary which is equidistant from a centre point. Circles have important characteristics and parts – for example, arc is a part of a circle. Diagrams showing them are:



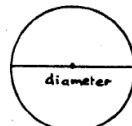
Circumference



Centre



Radius



Diameter



Chord



Semi-circle

Regions and discs

2D shapes are the boundary – what is inside the boundary is the region (or disc for circle). Some examples are below:



Triangular region



Square region



Quadrant



Sector

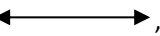
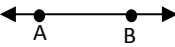


Segment

2.1.3 Points, lines and angles

Points and lines

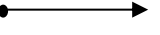
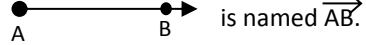
It is useful to begin by looking at line in relation to point and plane. We represent a **point** by a circular region but a point as having no size at all. That is, we say a point has no dimensions. However, a small dot is a good “model” of the idea we have in mind. The point of a needle or pin, or the sharp top of a well-made pyramid or cone, would also be models of points.

We represent a **line** by a long, thin, shaded rectangular region, which again never actually becomes a line, because mathematically a line has length but no thickness. (*Note:* In spatial knowledge, we understand line to mean a straight line unless stated otherwise.) Therefore, we can say a line has one dimension. A line can be extended infinitely in both directions. We could never find such a thing in practice but we use a thin rectangle with arrows at each end, e.g. , as a model of a line. The arrows at both ends show that the line can be extended without ending in both directions. Other models of lines are the sharp edge of a ruler, or the edge of a box (if we imagine the edge, in these cases, extending without ending in both directions). Since we want to show direction, we always name a line with two points. For example,  is named AB.

A **plane** (2D) is a flat surface which is very thin and extending without ending in all directions. A table top, a sheet of glass, and a skating rink are models of planes (if we imagine them extending without ending in all directions). Each page in this book is a model of a plane. Each surface of a box, or a pyramid, is a model of a plane. We do not usually state exactly what the words *point*, *line*, and *plane* mean. Instead, we have a picture in the mind of what we mean by point or line or plane — and we find models of these things in the world around us which help us use these ideas. It should be noted that only three points, if they are not in a line, are necessary to define a plane.

Some examples of points, lines and planes are (a) a town in Australia is a point on a map; (b) telephone wires stretched between poles are lines; and (c) the top of a table is a plane.

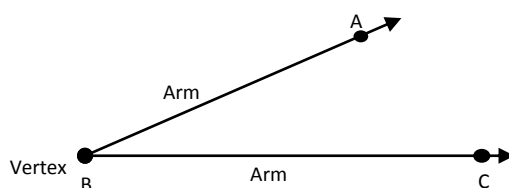
A **line segment** is only part of a line. For example, when you rule a line in your exercise book, you are really only drawing a part of a line. You start at one point and finish at another point. This part of a line is called a line segment (the word *segment* means *a part of*). In notation, we represent a line segment by capital letters at the two ends of the line segment.

A **ray** is a line that has a starting point but no ending point. In other words, it can extend in only one direction, without ending. A ray can be thought of as a “half-line”, e.g. . A good model of a ray is the light from a torch. Each ray of light has a starting point (i.e. the bulb), and extends in one direction in a straight line. We draw rays with two points. For example,  is named \overrightarrow{AB} .

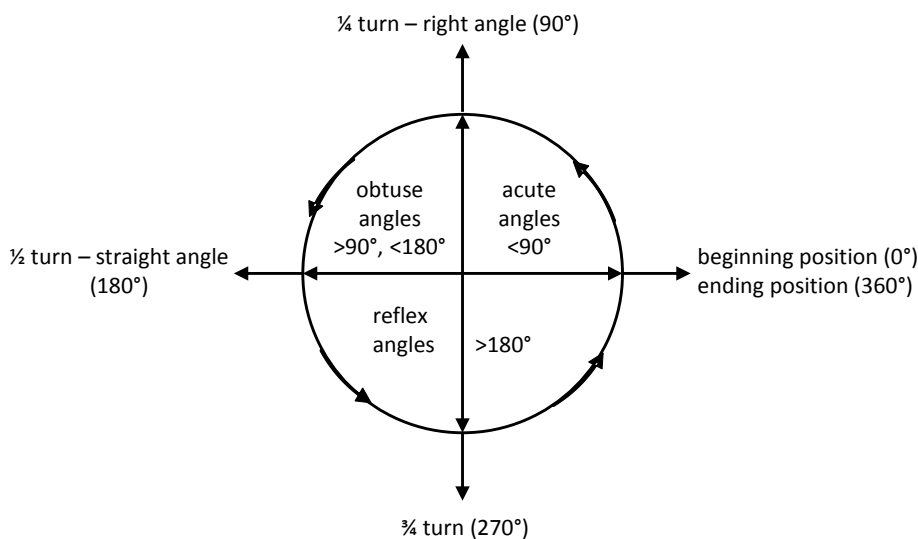
There are many types of lines: (a) when two lines meet at a point, these two lines are called **intersecting**; (b) when two lines meet at right angles, they are called **perpendicular**; and (c) when two lines never meet, they are called **parallel**.

Angles

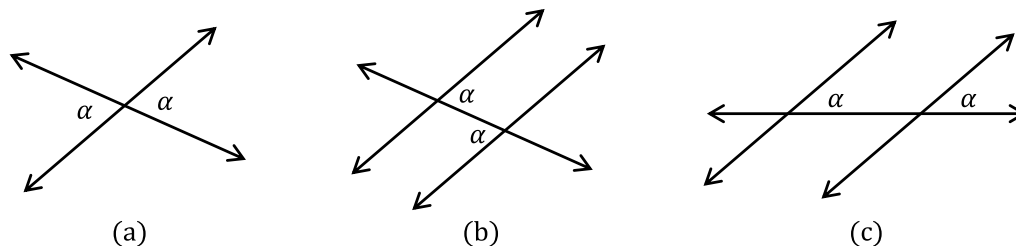
An **angle** is the opening between two rays that have a common end point; it is the amount of turn, the measure of the change of direction between the rays. The point is called the vertex and the two rays that form the angle are called the arms of the angle. We can name an angle by naming a point on each arm of the angle and the point at the vertex. The point at the vertex is always stated in the middle of the angle name. For example, in the diagram below, this angle could be labelled ABC or CBA.



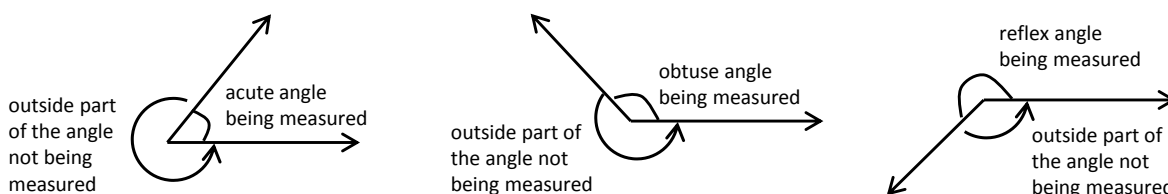
The angles which have particular emphasis are: (a) the **right angle** (rays meet at 90° – $\frac{1}{4}$ turn), (b) the **straight angle** (180°), (c) the **acute angle** (less than 90°), (d) the **obtuse angle** (between 90° and 180°), and (e) the **reflex angle** (over 180°).



With regard to parallel lines, there are: (a) the **vertically opposite angle**, (b) the **alternate angle**, and (c) the **corresponding angle** (see diagram below). Vertically opposite, alternate and corresponding angles are congruent (equal).



There are also: (a) **interior angles** (between the arms), and (b) **exterior angles** (the other part of the angle):



A **curved line** is a two-dimensional boundary that does not go on continually in the same direction, but turns constantly. A curve cannot be defined by a fixed number of points (unless it is regular, e.g. a circle or ellipse); it has to be drawn.

2.1.4 Properties and principles of 3D and 2D shapes

Once shapes are constructed they can be examined for properties and principles.

3D shapes

The properties that are worth investigating in solids are (a) characteristics of the surface of the solid (its vertices, edges and surfaces), (b) the shape of cross-sections, (c) the way shapes pack together, and (d) the relative strength of different shapes.

Classifying solids. Attributes upon which classification may be based include roll or not roll, rock or not rock, rough, smooth, pointed, vertices, faces, edges, regularity, volume, surface area, base area, shape of base, shape of faces, flat or curved surfaces, solid construction, open construction, hollow construction, mass, etc. Relationships such as the ratio of vertices to faces, edges to faces and edges to vertices can be explored. Venn and Carroll diagrams can be used to record the classifications.

Euler's formula. There is a relationship between the number of surfaces, vertices and edges of polyhedra called Euler's formula. It is: number of surfaces + number of vertices = number of edges + 2. Once this relationship is known, learners can tell whether a polyhedra of a certain number of faces, edges and corners can exist.

Cross-sections. Cross-sections are two-dimensional shapes that are formed by cutting across a solid shape. Learners can (a) cut solids made from potatoes, plasticine, etc. and study their shape; (b) predict the shape of cross-sections; and (c) predict which solids can have given cross-sections.

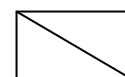
Packing and strength. Which solid shapes tessellate – pack together without gaps or overlaps? This is a crucial characteristic of packaging – the boxes, tins, etc., into which goods are placed for sale. What solid shapes are strong? What shapes can hold material under pressure? Or can stand rough treatment? Again this is important for packaging. Triangular, square and rectangular prisms and pyramids tessellate. Cones, cylinders and spheres do not. Yet spheres and cylinders are much stronger than prisms and pyramids because edges and vertices are weaknesses.

2D shapes

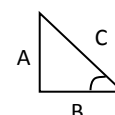
Angle properties and principles. The interior angle sum of a triangle is 180° and of a quadrilateral is 360° . The interior angle sum of any polygon is (the number of sides subtract 2) $\times 180^\circ$. The exterior angle sum is always 360° for any polygon, since walking once around a polygon is a complete turn.

Diagonal properties. Quadrilaterals have particular properties with regard to diagonals: a square has equal and perpendicular-bisected diagonals, a rectangle has equal bisected diagonals, a rhombus has perpendicular-bisected diagonals but not equal, and a parallelogram has bisected diagonals but not equal.

Rigidity. Triangles make polygons rigid so they can be used in construction – the number of triangles is equal to the number of sides subtract 2. This is particularly important for the rectangle where a diagonal is needed to make it rigid (as on right). The number of diagonals to make a shape rigid is 2 less than the number of sides.



Pythagoras' theorem. Right-angle triangles obey Pythagoras' theorem – the sum of the squares of the two sides adjacent to the right angle equals the square of the side opposite the right angle (the hypotenuse). For the example on right, $A^2 + B^2 = C^2$.



Tangent. Most construction uses a horizontal floor and a vertical wall, i.e. a right triangle. Triangles of this type are different in relation to the opposite over adjacent side length. This division is called the tangent, e.g. for the triangle above, the tangent is A/B . The tangent can be used to find height if distance is known and vice versa. This can be useful for building ramps and stairs.

2.1.5 Overview of teaching 3D and 2D shape

At the beginning, the best way to teach geometry is through the environmental approach. Learners should experience many 3D shapes. It is important that care is taken with the examples students experience to understand 3D shapes. The teacher should ensure that learners see: (a) many different examples of a solid type; and (b) examples that are not that type of solid as well as examples of the solid. Furthermore, activities should be structured so that both the following are done: (a) the teacher says or writes the names of the solid – the learner finds an example or model of it; and (b) the teacher shows a model – the learner says or writes its name.

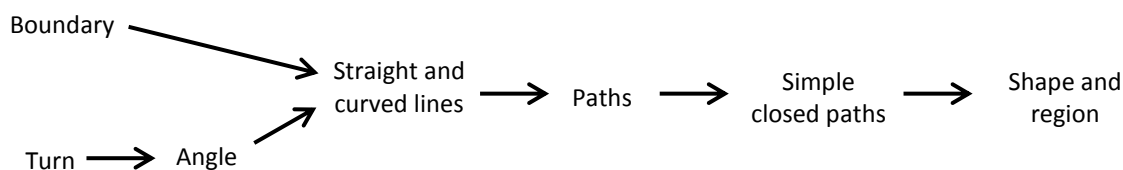
In later learning, the sub-concept approach should be used. 2D shapes should be joined to make 3D shapes, and 3D shapes should be de-constructed to form faces. There should be a focus on the relationship between faces and 3D shape. Learners should look, observe, describe, experiment, analyse, dissect, construct and infer.

It is essential that students have a balance of activities that describe (interpret) solids and that construct solids. A construction is a useful starting point to discuss what will happen in real life, i.e. to infer. A construction gives insight into what surfaces a shape will dissect to. Construction requires a focus on the faces, edges, vertices of the shape and their particular properties — a starting point for the formal analysis of a solid shape. Constructions, particularly when insufficient detail is given in instructions and the students have to work out what to do themselves, are great opportunities for observations, experimentations, and actions.

Construction techniques to achieve this include: (a) **solid techniques** like making shapes out of clay, plasticine or LEGO™; (b) **closed techniques** like nets (diagrams of faces that are cut out and folded to make the shape) and cardboard and plastic faces that staple or “click” to form solid shapes; and (c) **open techniques** like straws and string that build edges and vertices and show the skeleton of the shape (triangles are often required for rigidity).

Learners should be given many opportunities to construct 2D shapes, to analyse such shapes, and to manipulate such shapes. The introduction of 2D shapes should contain many opportunities for learners to experience many examples (and non-examples) of the different shape types. This should be initially with the environmental approach by looking at the surfaces of 3D shapes in the environment and investigating properties and principles. The environmental approach uses the world around us to find and classify examples of shapes.

However, a richer approach is to use the idea that a 2D shape can be considered as a simple closed boundary composed of straight or curved lines with the inside being called a region. We can use the sub-concept approach to develop the notion of 2D and 3D shape through these stages. (*Note: a path is composed of lines and angles.*)

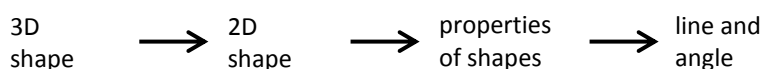


It seems appropriate to use this environmental approach as the starting point for instruction but to fall back to the sub-concept approach for more formal analysis. Thus, the idea is to let the learners first experience shape in their everyday world and then build it up more formally from sub-concepts. The sub-concept approach has the advantage that properties are developed before names. If we make 4-sided shapes with one opposite set of sides parallel, then everything we make is a trapezium – thus the name can be attached to the properties.

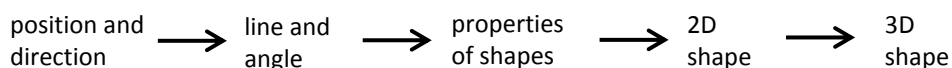
It is also important to **reverse** activities. We tend to make shapes and then explore the diagonals. Why not make diagonals and then see what shapes emerge as we move them, change their length, and their point of intersection? Similarly, give a shape and ask for lines of symmetry – then ask to draw a shape with a given number, say three, lines of symmetry. Try to allow students to explore change as in the two scenarios. It is also important to **be flexible** and to show shapes in non-prototypic ways (e.g. rectangle at angle to horizontal) and to look for **generalities** (e.g. all squares are rectangles, all rectangles are parallelograms, all parallelograms are quadrilaterals).

Thus, there are **two ways of teaching shape** that flow in opposite directions:

Environmental approach



Sub-concept approach



2.2 Activities for the environmental approach

Below is a collection of activities that follow the environmental approach. The main aim of this resource is to provide these ideas. They are not presented in the RAMR format – suitable reality, kinaesthetic and reflection activities should be devised around these ideas.

2.2.1 3D construction activities

Constructions from nets

Purposes:

- To develop the notion that 3D shapes are closed and have three dimensions (length, width, height).
- To develop the notion that solid shapes have faces which are plane shapes and that many of the faces are congruent.
- To develop the language associated with solid shapes (e.g. face, base, closed, open, edge, vertex).
- To develop the notion that 2D and 3D shapes are related.

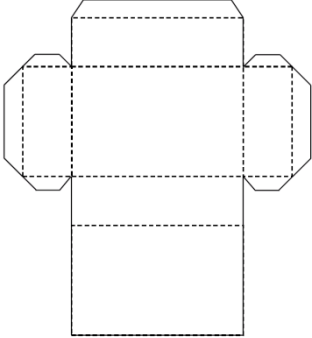
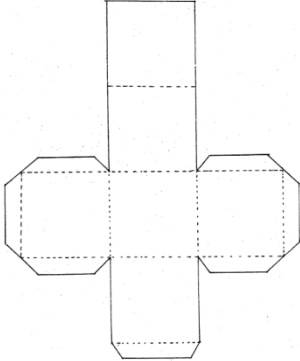
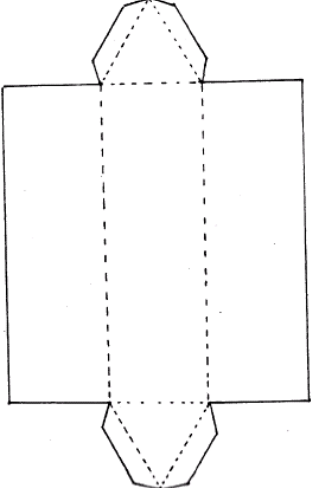
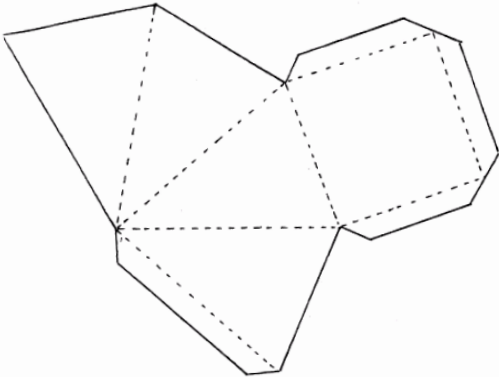
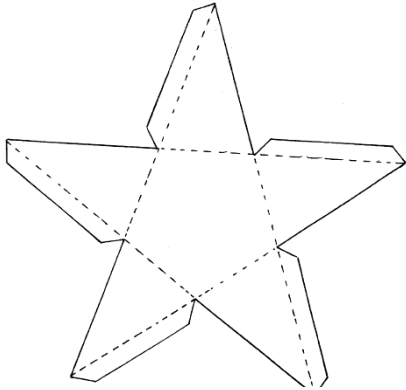
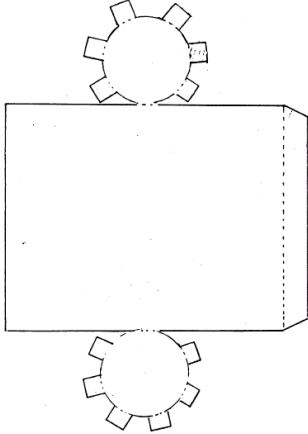
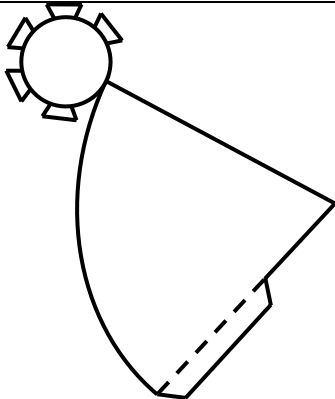
Materials: Nets of the following 3D shapes – rectangular prism, cube, triangular-based prism; square-based pyramid, pentagonal-based pyramid; cylinder (six nets altogether); a sphere; scissors, tape, ruler, compasses; waste paper basket.

Processes: Folding, visualising.

Problem-solving strategy: Following directions; guess and check (cone); thinking visually.

Directions:

- Construct the 3D shapes from the nets provided on the next page and obtain a sphere.
- Divide your set of shapes (the ones you constructed and the sphere) into two groups according to some criteria. Draw the two sets of shapes and write an explanation of the basis for your classification. For example: *I put all the shapes with ---- in one group and all the shapes with ---- in the other group.*
- Put all your solid shapes into one group and then form another two groups with the shapes but use a different basis for this classification. Draw the new two groups of shapes and write your reason for classifying them as you did.
- Compare your classifications with other group members. Discuss the utilitarian properties of the shapes.
- Challenge – make a net into a solid, place symbols/pictures on each side of solid, ask the students to place the same symbols/pictures on their nets before folding them so that when folded, the resulting solid has symbols/pictures the same as the original (very hard).

<p style="text-align: center;">RECTANGULAR PRISM</p>	<p style="text-align: center;">CUBE</p>
	
<p style="text-align: center;">TRIANGULAR-BASED PRISM</p>	<p style="text-align: center;">SQUARE-BASED PRISM</p>
	
<p style="text-align: center;">PENTAGONAL-BASED PYRAMID</p>	<p style="text-align: center;">CYLINDER</p>
	
<p style="text-align: center;">CONE</p>	
	

2.2.2 Construction from other materials

Purposes:

- To develop creativity and the language associated with solid shapes (e.g. face, base, closed, open, edge, point/vertex).
- To develop the notion that 3D shapes are closed, have three dimensions (length, width, height), have faces which are plane shapes and that many of the faces are congruent.

Materials: Zaks, polydrons, construct-o-straws, other commercial materials. (Students need a lot of free play with materials right throughout their school life – they never tire of such creative activities.) A special material for constructing 3D shapes is the Maths Mat – this is a 6×10 grid on hessian with different-coloured long elastic bands – it is extremely effective as students can quickly build large 3D shapes, get inside them, and see them from all directions.

Processes: Flipping, turning, sliding, visualising, assessing, monitoring.

Problem-solving strategy: Guess and check; thinking visually and creatively.

General directions:

- Create as many different solid shapes as you can. Think about what you would call your shape and how you would describe it to someone else.
- Make basic shapes (prisms, pyramid) and look for properties (e.g. the type of faces).
- Make shapes out of play dough or potatoes and cut cross-sections – look at relation between shape and cross-section.
- Discover Euler's formula ($\text{edges} + 2 = \text{surfaces} + \text{vertices}$).

Directions for Maths Mat:

- Use the first elastic band and students holding corners to construct a 2D base on the mat (e.g. triangles, square, rectangle).
- Use other students and other bands to build a prism or pyramid on the base (students have to hold more than one band at a vertex) – a good idea is to make a rectangle, add two more bands and another student and turn into a pyramid, then add another student and no more bands and turn the pyramid into a triangular prism on its side, then add another band and change to a rectangular prism – then make these shapes higher, lower, etc.
- Give more complex shapes to make (e.g. a trapezium prism on its side – like a folder box).
- Count edges, surfaces and vertices (and discover Euler's formula – $\text{edges} + 2 = \text{surfaces} + \text{vertices}$) – note how the vertices equal (in number) the people holding the bands.
- Put one student in charge – get a 3D shape to copy – student has to organise/direct other students to make the shape.

2.2.3 Construction and properties/principles of 2D shapes

Purposes:

- To construct plane shapes.
- To consolidate the properties of plane shapes through investigations.
- To develop principles that hold true for shapes.

Materials: Maths Mat, geoboards, geostrips, A4 paper, circles, scissors, rotagrams, protractors, ruler, coloured texta, coloured squares.

Processes: Constructing, folding, visualising.

Problem-solving strategy: Modelling.

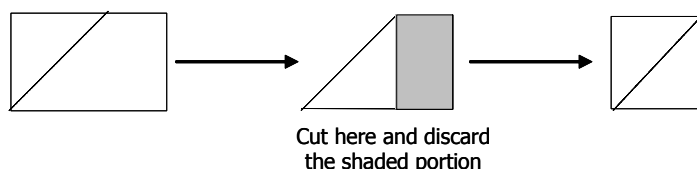
Directions:

Constructing 2D shapes

- Use elastic bands on Maths Mat, rubber bands on geoboards to make shapes by students/nails holding the corners of the shapes – make basic shapes then more complicated ones – which can you make and which can you not make?
- Draw a shape and then get students to copy it, then change its length and orientation (always use at least three orientations to prevent shapes always being made parallel to vertical and horizontal lines).
- Repeat above for geostrips but join plastic strips with fasteners.
- Count sides and angles and name the shapes.

Investigating the diagonals, angles and sides of a square

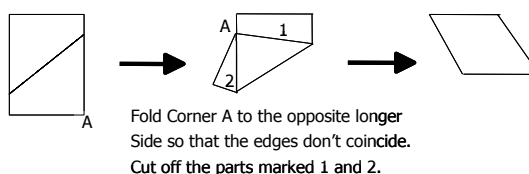
- Fold a rectangular sheet of paper as shown to produce a square.



- Fold the square to show that: its opposite angles are equal; its adjacent angles are equal; its opposite sides are equal; and its adjacent sides are equal.
- Mark the diagonals in texta colour and then fold the square along the diagonals to show that the diagonals bisect each other. Use a rotagram set at 90° , a protractor or a right-angle reckoner to show that all the angles in the square (including those where the diagonals intersect) are right angles.

Investigating the diagonals, angles and sides of a rhombus

- Fold a rectangular sheet of paper to produce a rhombus, as shown on right. By folding, show that the opposite angles are equal, the adjacent angles are not equal, and the adjacent sides are equal.



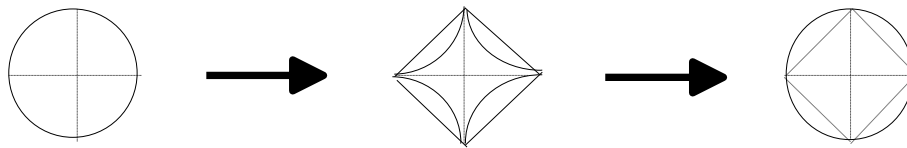
- Mark the diagonals with a texta. Fold to show that the diagonals bisect each other and are perpendicular (use right-angle reckoner to validate perpendicularity). Can you show, by folding, whether the opposite sides are equal/unequal? What would you need to do?

Determining the diameter, centre and radius of a circle

- Fold a circle in half; open it. What part of the circle is shown by the fold line?
- Fold the circle in half again, using a different fold line (does not have to be perpendicular to the first fold line); open it. Can you see the centre of the circle?
- How many radii do you see?

Inscribing a square in a circle

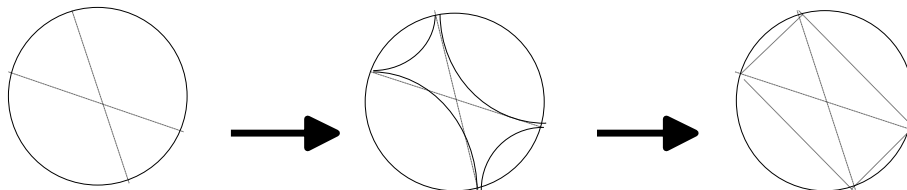
Fold a circle in half; fold in half again; open the circle. You should see two diameters bisecting each other at right angles. Fold in the edges of the circle to form four equal arcs (as shown in the diagram below). These fold lines will be chords of the circle. Open the circle; you should see a square formed by the chords.



Inscribing a rectangle in a circle

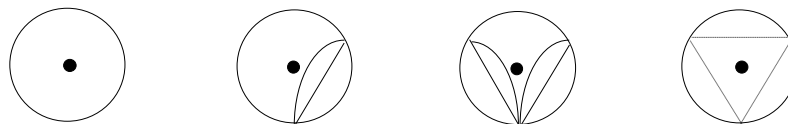
Fold a circle in half; open it; fold again but not at right angles to the first fold; open it. You should see two diagonals that bisect each other but are not perpendicular to each other.

Fold in the edges of the circle to form four arcs. Open the circle. You should see a rectangle formed by the fold lines (chords of the circle). This is a similar process to the example above.



Inscribing an equilateral triangle in a circle

Find the centre of the circle (fold in half, then in half again). Fold one side to the centre to form an arc; repeat this for the second and third sides (as shown below). Open the circle and you should see an equilateral triangle inscribed in the circle.



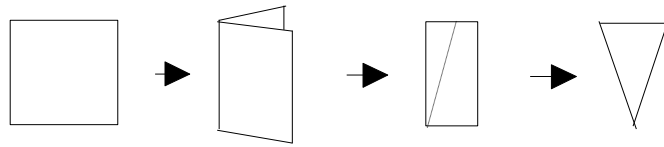
Check that all sides are equal and that all angles are equal (use a rotagram to check angles).

Inscribing a regular hexagon in a circle

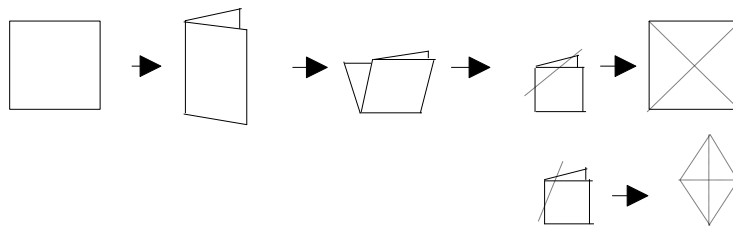
Fold to find the centre; open. Fold opposite edges to the centre line; open. Fold in edges to make six arcs; open. You should see a hexagon formed by the chords (fold lines).

Constructing polygons through cutting and folding (use the coloured squares)

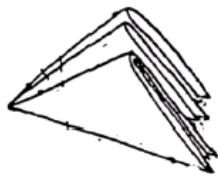
Construct an isosceles triangle. Fold the coloured square in half either horizontally or vertically; cut from one corner of the fold line to the opposite edge (see the diagram below); open the shape and you should see an isosceles triangle (two equal sides and angles).



Construct a square and a rhombus. Fold the square in half vertically; fold in half again horizontally. If you cut along any diagonal, you will produce a rhombus. However, to produce the square, you need to cut across the fold at an angle of 45°.



Construct a regular octagon and an irregular octagon (a star). (1) Fold a square in half vertically and then fold again horizontally (as for the rhombus). (2) Fold the paper again but along the diagonal. (You have now folded the paper in eighths.) Hold the paper so that the diagonal is in the position shown below. Cut through the folds at an angle of 45°. Open the paper and you should see a regular octagon.



Repeat steps (1) and (2) with another square. Hold the paper in the same position as before but this time cut so that you have one long side and one short side (see the diagram above right). Open the paper and you should see a star with eight sides.

To make a square with eight equal parts, fold into eighths as before but cut from the diagonal fold to the top at a right angle (see the diagram at right.)



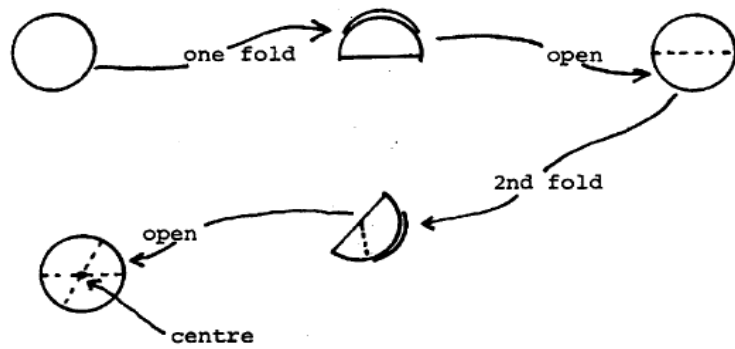
2.2.4 Circle patterns

Inscribing a square

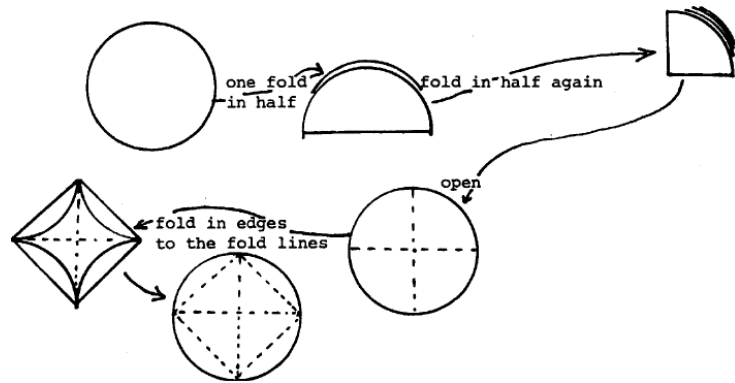
From a piece of A4 paper, cut out a circle that is about 25 cm across. Fold the circle, as shown below, to find the diameter of the circle.



Fold the circle, as shown below, to find the centre and radius of the circle.



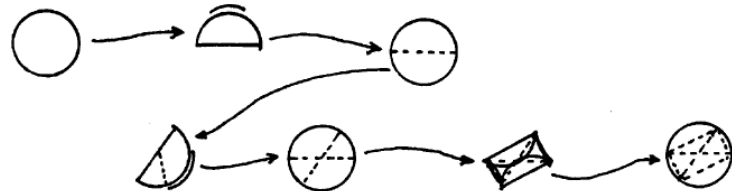
Fold the circle, as shown below, to inscribe a square in the circle.



What properties of a square does this show? (Hint: Look at the diagonals!)

Inscribing a rectangle

Fold the circle, as shown below, to inscribe a rectangle in the circle. What properties of a rectangle does this show?



Inscribing a triangle

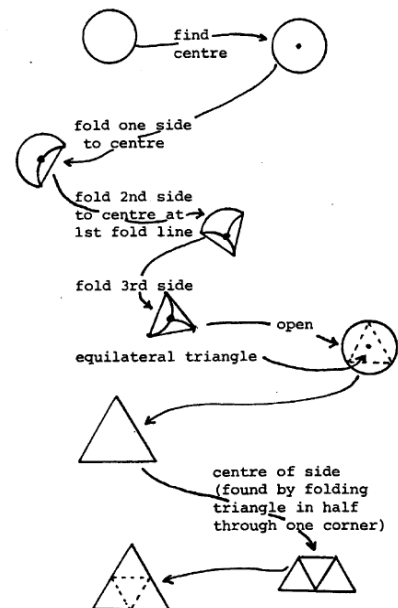
Fold the circle, as shown on right, to inscribe a triangle in the circle.

What are the properties of an equilateral triangle?

Show all three lines of symmetry in the triangle.

Fold the equilateral triangle into four smaller equilateral triangles.

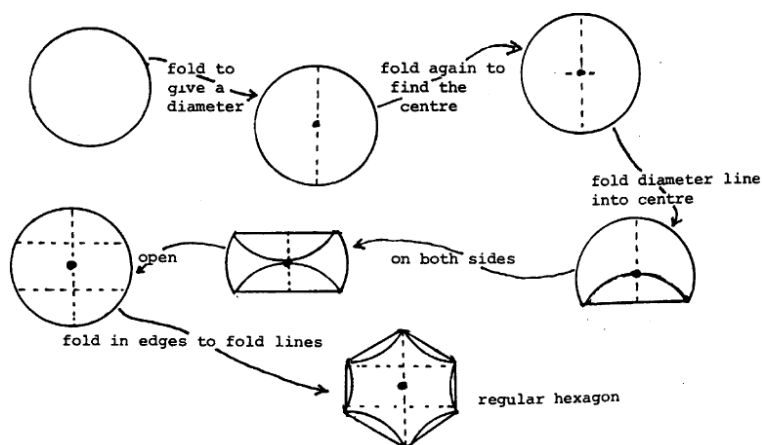
Construct a tetrahedron from the folded triangle.



Inscribing a hexagon

Fold the circle, as shown on right, to inscribe a regular hexagon in the circle. (Construct the hexagon by folding the corners of the triangle into the centre.)

What can we say about the diagonals of a regular hexagon?



Investigations

There are wonderful designs that can be made from a circle using a compass and pen. Search the Internet and books to find some of these designs. Make one and colour it. Make a large poster of your design.

Circles are the basis of designs in Indigenous art. Approach local Indigenous artists and learn about the role of circles in art. Construct your own Indigenous circle design.

Why are circles good for wheels? Does this relate to line and rotational symmetry? What are the line symmetries and rotational symmetries for a circle?

2.2.5 Exploring 2D shape properties/principles with geostrips

Purposes:

- To develop the notion of rigidity and the relationship between rigidity and diagonals.
- To develop relationship between polygons and interior angles and polygons and total number of diagonals.
- To develop the notion that space and number are related.
- To develop, informally, the language associated with polygons (e.g. regular, irregular, diagonals, angles).

Materials: Geostrips; worksheet with Table 1 and Table 2.

Processes: Constructing.

Problem-solving strategy: Investigating, looking for patterns.

Directions:

Rigidity and interior angles

Using the geostrips, construct a door (a rectangle). Can you move the sides or are they fixed/rigid? When you move the sides of the rectangle, what part changes – the length of each side? The parallelness of the opposite sides? The size of the angles? How can you make the shape rigid?

Construct a triangle, a square, a trapezium, a pentagon, a hexagon. Which shapes have rigidity? Add the minimum number of diagonals needed to make each of the above shapes rigid. (Don't forget the diagonals cross.) How many triangles did the diagonal/s divide the original shape into? (Hint: This process will divide the polygon into triangles.)

Complete Table 1 using the information you have gained from the previous activities. Find a pattern and then predict how many diagonals are required to make a heptagon (7-sided polygon) and an octagon rigid? What would be the sum of the interior angles of each shape? Do the same patterns hold for both regular and irregular polygons? Find out where the property of rigidity is used in real life.

Table 1: Relationships between polygons and the sum of the interior angles

Name of polygon	No. of sides	No. of diagonals required for rigidity	No. of triangles made	Sum of the interior angles
Triangle	3	0	1	180°
Rectangle	4	1	2	360°
Square				
Trapezium				
Pentagon				
Hexagon				

Number of diagonals

Construct the shapes listed in Table 2 and add in all the possible diagonals (the diagonals will cross each other). Complete Table 2.

Find a pattern that will enable you to determine all of the different possible diagonals for a 100-sided polygon. Write an algebraic equation for the pattern.

Table 2: Relationship between polygons and the number of different diagonals

Name of polygon	No. of sides	No. of vertices (A)	Diagonals from each vertex (B)	A × B	Total no. of different diagonals
Triangle	3	3	0	0	0
Rectangle	4	4	1	4	2
Square					
Pentagon					
Hexagon					
Heptagon					
Octagon					

Investigating shapes built from diagonals (part → whole)

Take two equal geostrips and join them at their centre. Considering these as the diagonals, build the shape that encloses them, using the fewest number of strips. What shape have you made? Repeat the process above but, this time, use two unequal geostrips. What shape have these diagonals produced?

Join two equal geostrips with the centre of one joined to anything but the centre or the end of the other. What shape have these diagonals produced? Repeat the process but use two unequal geostrips as the diagonals. What shape have these diagonals produced?

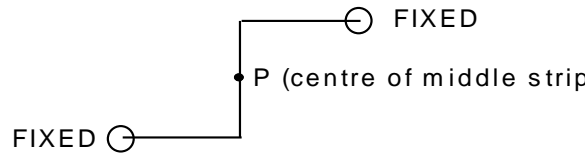
Experiment with different diagonals to answer the following:

- Can the diagonals of a rectangle intersect at right angles without forming a square?
- Can the diagonals of a parallelogram be equal without forming a rectangle?
- When do the diagonals of a parallelogram intersect at right angles?
- Can a kite have equal diagonals intersecting at right angles?

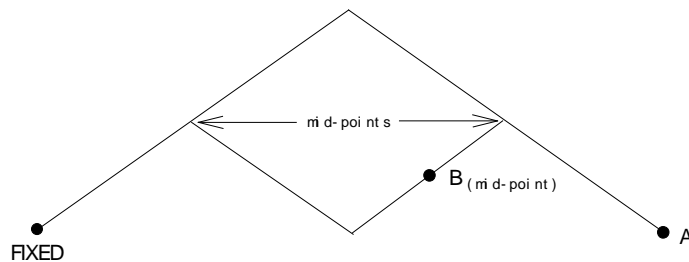
- (e) Can a quadrilateral have equal diagonals intersecting at right angles and yet not be a kite, a parallelogram or a trapezium?

Linkages (work in pairs for these activities)

Using three equal geostrips, construct the *James Watt linkage* shown below. Place a sharp pencil through the centre point of the middle strip (P). Keeping the end points of the other geostrips fixed, move the geostrips as much as possible. What path does the pencil at P trace out?



Using two long and equal geostrips and two short and equal geostrips, construct a pantograph as shown below.



Copy a picture. Insert a sharp pencil through A. Place the picture under B and move point B around the outline of the picture. Is the copy larger or smaller than the original? Insert a sharp pencil at B. Place the picture under A and move point A around the outline of the picture. Is the copy larger or smaller than the original?

2.2.6 Exploring line and angle

Straight and parallel

Students can use reality to study line and angle. For example, they can identify as many occurrences of straight lines in their life as possible. For each, they can describe the occurrence and analyse why they think the line is straight. The same can be done for parallel lines.

Get students to try to describe a straight line to a classmate without relating to physical objects. Get them to list those attributes of a straight line which distinguish it from a curved line. Discuss in what situations each is most useful. For example, which is easier to measure and why?

Lead discussion to wider meanings of straight, for example, what attributes of a straight line is Johnny Cash evoking when he sings, “Because you’re mine, I’ll walk the line”?

Get students to explore what straightness means in reality. For example, discuss what is a straight line when walking on the equator. Experiment with a globe to determine which curves on the globe share which properties of straight lines. For example, on a sphere what are the analogues to parallel lines?

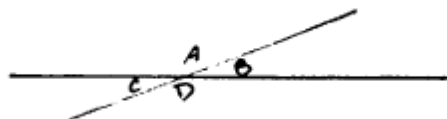
Angles at intersections

When two straight lines intersect in a point, several angles are formed. Are all of the angles always the same? Explore these angles. Are all of the angles sometimes the same? Are some of the angles always the same? Are some of the angles sometimes the same? Experiment with some straight lines in order to arrive at answers to these questions. Illustrate your answers with examples. Discover the parallel line-angle principles.

Problems to solve:

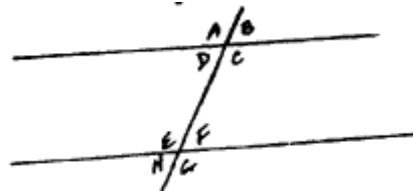
- Two straight lines intersect in such a way that the measure of angle A is 145° . Find the measures of angles B, C, and D. Use reasoning rather than a protractor to arrive at your answer.

Measure of angle A = 145° Measure of angle C = 35°
 Measure of angle B = 35° Measure of angle D = 145°



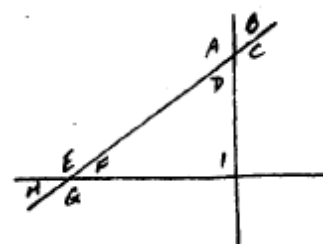
- Two parallel lines are intersected by a transversal in such a way that the measure of angle F is 72° . Find the measures of the other lettered angles.

Measure of angle A = Measure of angle E = Measure of angle C = Measure of angle G = 108° .
 Measure of angle B = Measure of angle F = Measure of angle D = Measure of angle H = 72° .



- Two perpendicular lines are intersected by a transversal in such a way that the measure of angle D is 54° . Find the measures of the other lettered angles.

Measure of angle B = 54° . Measure of angles A and C = 126° . Measure of angle I = 90° . Measure of angles F and H = 36° . Measures of angles E and G = 144° .



2.3 Activities for the sub-concept approach

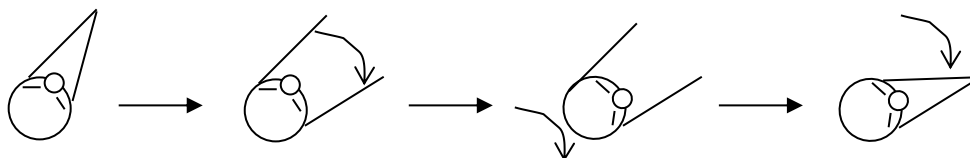
Below is a collection of activities that follow the sub-concept approach: position/direction/turn \rightarrow line/angle \rightarrow path/properties \rightarrow 2D shape \rightarrow 3D shape. The main aim of this resource is to provide the ideas that need to be abstracted. Suitable reality, abstraction (body, hand, mind), mathematics and reflection activities should be devised.

2.3.1 Body (position/direction \rightarrow 2D shape)

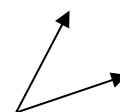
Angle as turn

Stand and pick two directions. Place arms together in front of body and aim at the first direction. Turn body (holding arms in same position in front of body) to second position.

Repeat this turn in a different way. Point towards the first direction, turn one hand to the second direction (without moving body), turn body, and then turn second hand (in four steps as below).



Relate the change from one direction to another to the drawing of angle – stress that angle is the turn between the two directions. Use body to make large angles (turning a long way) and small angles (turning only a short amount). Use body to make a right angle – put arms straight out forward, swing one arm around to the side (so it makes a straight line from fingertips to across the shoulders), then complete this turn – this is close to a right angle.



Introducing lines and paths

For a straight line, walk forward without turning. For a curved line, walk forward turning constantly. For parallel lines, link up with a partner and walk forward side by side in the same direction without turning. For intersecting lines, walk forward with your partner so your paths cross (don't bump into each other). For paths, walk forward without turning, stop and turn, walk forward, stop and turn, and so on. Repeat this until you have the straight lines that your path requires.



Introducing 2D shape

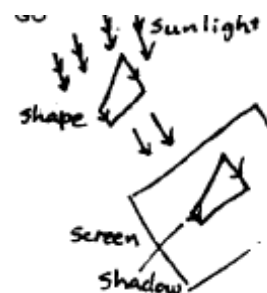
Walk simple open paths (starting and stopping at different positions, with no crossing) and complex open paths (starting and stopping at different positions, with crossing). Walk complex closed paths (starting and stopping at same place, with crossing) and finally walk simple closed paths (starting and stopping at same place without crossing). Discuss how simple closed paths are 2D shapes.

Make shapes from characteristics. For example, walk a simple closed path of three straight-line sides (result is always a triangle); walk a simple closed path of four sides with all angles equal to 90 degrees (a rectangle or square – depending on whether you walk the same or different length sides). Lay out shapes by placing objects at turning points.

2.3.2 Sunlight (parallel lines)

Parallel lines can be advantageously studied by shadows in sunlight. Since the rays of the sun are parallel, then parallel sides remain parallel regardless of the way we tilt the shape and screen. (This does not happen in shadows by torchlight.)

Cut out a figure which has one pair of parallel sides. Go out into the sunlight. Cast shadows on the screen. Tilt screen and shape. Do the parallel sides remain parallel in the shadow? Go inside and cast shadows on the screen using a torch. Can you now get non-parallel sides? You can also study straight lines in this way. If your screen is flat a shadow of a straight stick is always straight.



2.3.3 Circle wheel/Rotagrams (turn → angle)

Angle as turn

Construct an angle wheel by cutting out two identical circles of different colours (shade one in). Put one circle on top of the other (the coloured one underneath) and cut a slit to the centre of both circles. With the slits in line and at 3 o'clock slip the lower right part of the top circle underneath the upper right part of the bottom circle. Turn the top circle anticlockwise and you will see an angle whose arms are the slits in the two circles, as shown below.



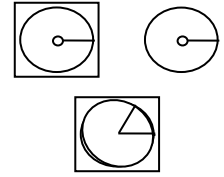
Use your angle wheel to make an acute angle, a right angle, an obtuse angle, a straight angle, and a reflex angle. Use your **right-angle reckoner** (see 2.3.4 Paper folding). (It may be useful to mark, with lines, 90°, 180°, 270° points on the top circle of your angle wheel.)

Look at angles on the circle wheel – which of the angles below are acute, obtuse or reflex?

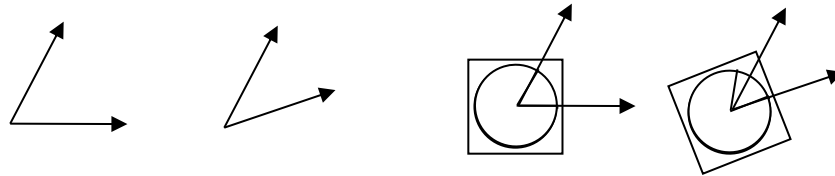


Comparing angles

Do one of the following. Build an **angle wheel** out of a square of plastic with a circle and radius line drawn on it and a circle of plastic with a radius line as on right. Pin the two pieces together through the centre of both circles so that the circle will turn on top of the square (and as it turns, the two radius lines will turn apart showing an angle). Or get hold of rotagrams – a commercial form of angle wheels that are plastic and see-through.

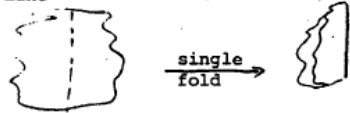
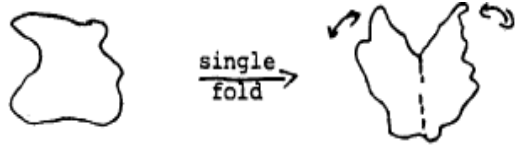
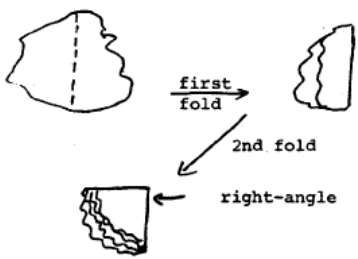
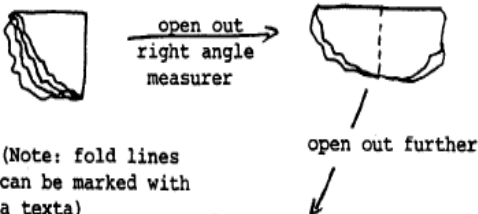


Use the wheel to compare size of angles (the bigger angle has the most turn) as below – turn the wheel so that it equals the first angle, then move wheel to second angle and check whether the wheel has to be turned more (first angle is smaller) or less (first angle is larger).

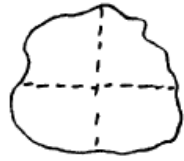


2.3.4 Paper folding (line/angle → shape)

Use scrap paper to complete the following paper folding activities.

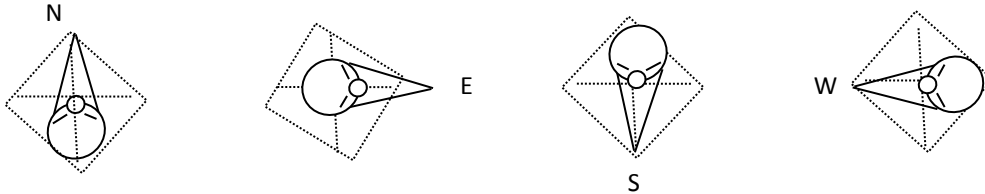
Representation with paper	Development
<p>1. Line</p> 	<p>Test for occurrence of straight lines, flatness (of desks), straightness (of doors), etc.</p>
<p>2. Angle as turn</p>  <p>Open and close folded sheet (or book or hinged strips).</p>	<p>Progress to angle as amount of turn. Classify turns as more or less than a right angle (obtuse and acute) – classify angles as more than a straight line (reflex).</p>
<p>3. Right angle (making a right-angle reckoner)</p> 	<p>Classify angles as being equal to a right angle, smaller than a right angle (acute angle), and larger than a right angle (obtuse angle). Classify triangles: one right angle = right triangle; all angles less than right angle = acute triangle; one angle larger than a right angle = obtuse triangle.</p>
<p>4. Two right angles meeting in a straight line</p>  <p>(Note: fold lines can be marked with a texta)</p>	<p>Test for where two right angles meet in a straight line (e.g. door frame, window). Use to also classify angles as larger than a straight line (reflex angle).</p>

5. Right angles meeting in a complete turn

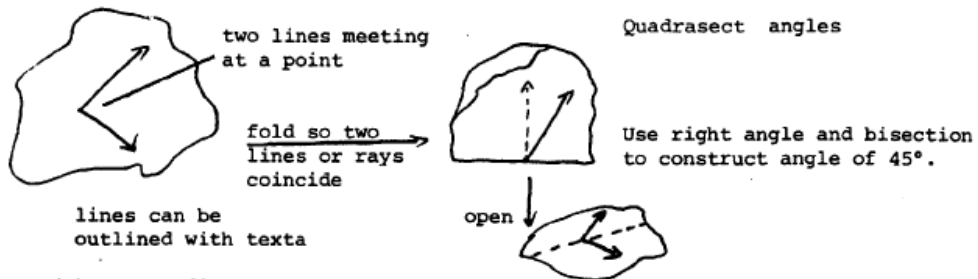


Test for occurrence (e.g. window). Put paper on floor. Stand where fold lines meet. Turn from one fold line to the next, i.e. rotate body through right angle or quarter turn (showing that four such turns give a complete 360° turn – see below). Introduce compass bearings of north, south, east and west.

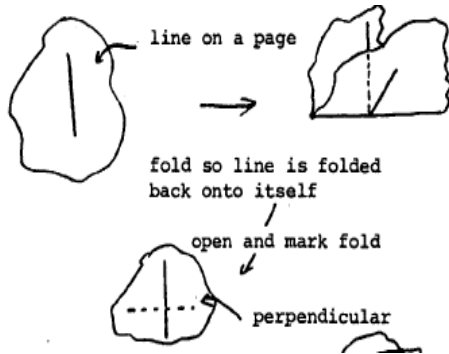
The four turns on the right-angle reckoner (fully opened) that can lead to compass bearings (north, east, south and west):



6. Bisector of an angle



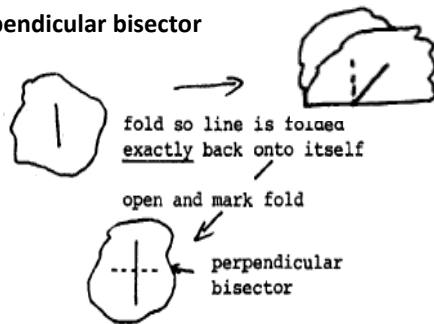
7. Perpendicular



Construct perpendiculars from a point to a line (ensure fold goes through point). Construct perpendicular bisectors (see 8 below).

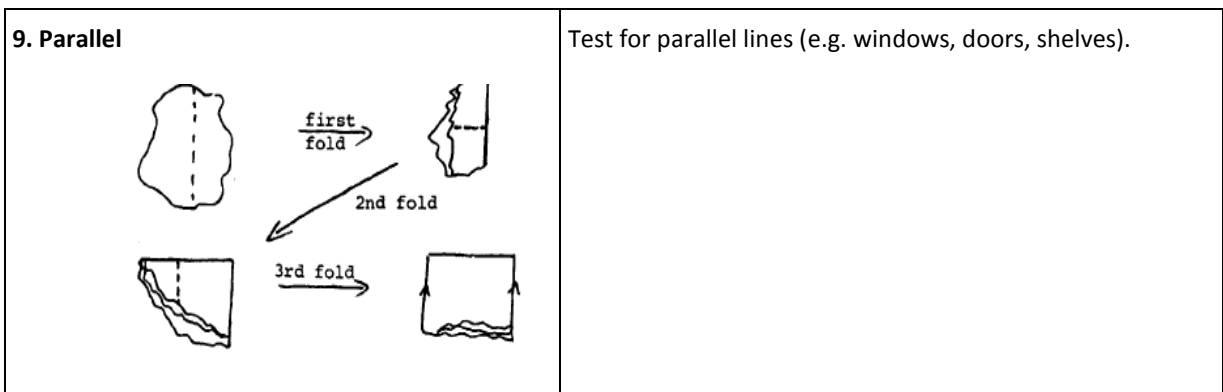
Use these to construct squares, rectangles, etc.

8. Perpendicular bisector



Construct isosceles triangles





2.3.5 Geoboards/Dot paper/Maths Mat (line/angle → 2D shape)

Purposes:

- To develop the basic geometric ideas in sub-concept approach (e.g. points, angle, line, path, region and shape) and, informally, the language associated with plane shapes (e.g. side, closed–open, parallel, equal length, angle, simple–complex, concave–convex).
- To show a sequence of activities involving problem solving that move from angle to shape.
- To develop the notion that *plane shapes* have a boundary with no gaps (and therefore enclose a region with two dimensions – length and width).

Materials: At least 5 × 5 geoboards; rubber bands or dot paper.

Processes: Creating, constructing.

Directions:

Lines

Make the shortest *line segment* (part of a line) possible on your geoboard. Make the longest line segment possible. Make three line segments of different lengths (different rubber bands). Make as many different line segments as you can in 30 seconds. How many times did your line segments cross? What shapes did your lines make? (*Note:* Make sure students realise that straight lines do not have to be only vertical, horizontal, or 45°.)

Angles

Make a narrow angle then a wide angle. (Use different rubber bands for each ray/arm.) Make an angle from a starting point: at the edge of the geoboard and near the centre.

Make an angle like the corner of a square (a right angle). Make an angle with two nails between the rays and with zero nails between the rays. Make an angle with six nails outside the rays, and with four nails inside and five nails outside the rays.

Use right-angle reckoner to help make acute, obtuse, reflex and right angles – make the right angle so its rays are not vertical, horizontal or at 45°. (*Note:* The way to do this is to have opposite nail movements, e.g. if one ray is two nails up-down and three nails left-right or across, then the other ray must be the opposite – three nails up-down and two nails across).

Paths and region

Choose one nail. Choose another nail as far away as possible. Make a path of line segments from one nail to the other (use double rubber band method). How many times did your lines cross? How many angles did you make? Repeat above but return to the first nail. Try again without any crossings (to make a region). How many lines, how many angles?

Introduce types of paths. If lines cross, path is complex; if lines do not cross, path is simple. If path returns to starting nail, path is closed; if path does not return to starting nail, path is open. Make examples – simple open path, complex open path with one crossing, complex closed path with two crossings.

Focus on simple closed paths that make shapes (the boundary) and regions (boundary and inside). Students can now try switching to one rubber band.

Make a region with five nails on the outside, only one nail inside it, zero nails inside it, one nail outside it, one nail outside and three nails inside, and three nails on the boundary.

Solve puzzles. For example, if the geoboard has 25 nails and there are two nails on the inside of a region and three nails on the outside, where are the other 20? Make such a region on your board.

Shape

Using one rubber band, try to make a shape that has: (a) one side; (b) two sides; (c) three sides. What is the three-sided shape called? How many triangles can you make with: (a) two rubber bands? (b) three rubber bands? Who has the most triangles? Try to make a star from several triangles.

Make shapes from properties, for example, four equal sides with opposite sides parallel – a rhombus; four sides with opposite sides equal and parallel – a parallelogram; one pair of sides parallel – a trapezium. What do all of these shapes have in common? Make a three-sided shape with two sides equal – isosceles triangle. What about the angles?

Make the following: (a) a square inside a square; (b) a triangle inside a rectangle; (c) a square inside a triangle inside a square; and (d) a parallelogram overlapping a trapezium.

Challenges

Give angles and sides and ask for shape to be made, for example: five sides, one reflex angle, one right angle, one obtuse angle, and two acute angles.

Introduce concave shapes – no reflex angle, and convex shapes – at least one reflex angle. Concave shapes have “jut ins” – like chevrons.

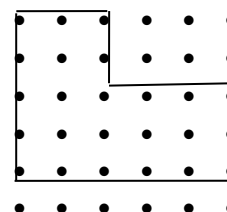
Puzzles

Select a 3×3 set of nails (as shown).

Construct five isosceles triangles that have different sizes in this small area



Make the shape shown on the geoboard. Divide the region into four congruent parts (i.e. same size and shape).



2.3.6 Geostrips (line/angle → 2D shapes)

Geostrips are lengths of plastic with holes regularly along them that can be joined by fasteners. They are very useful for the sub-concept sequence as the following shows:

- (a) each strip is a straight line;
- (b) two strips joined at end form an angle which can be easily turned to form acute, right, obtuse and reflex angles;

- (c) strips joined end to end form paths (which can be simple and complex) and if the first strip is joined to the last they form simple closed paths or shapes/regions;
- (d) properties can be used to make shapes – all geoboard activities can be emulated; and
- (e) angle, diagonals and rigidity properties/principles can be explored (see earlier work) – in particular, diagonal properties can be represented and shown to lead to shapes (e.g. two unequal length diagonals bisecting each other gives a parallelogram).

2.3.7 Surface construction material (2D shapes → 3D shapes)

Construction material for 3D shapes can be solid (e.g. play dough), made from nets, made from faces (e.g. polydrons) and made from edges (e.g. construct-o-straws). The material that builds from surfaces can be used to show the last step of the sub-concept journey – 2D shapes → 3D shapes.

The type of material used to construct the 3D shape will affect the focus students have. For example, polydrons (solid variety) and Zaks provide a focus on how faces as 2D shapes are connected to form 3D shapes. Construct-o-straws, open-face polydrons, and geoframes have open faces and provide a natural focus on edges and vertices as the faces are transparent. Straws/toothpicks and Blu Tack have a similar effect.

A sequence of activities is as follows:

- (a) explore using 2D shapes to make 3D shapes (2D shapes become surfaces);
- (b) use properties of 2D shapes to make special 3D shapes (teaching properties → shapes instead of shapes → properties) – for example, a base shape joined with squares or rectangles to a second base shape is a prism; a base shape with triangles attached is a pyramid;
- (c) make special shapes (e.g. the platonic solids); and
- (d) use Euler’s formula to make challenges (e.g. Euler’s formula is that edges + 2 = surfaces + vertices, so ask students to make shapes for which this holds – *Make a solid with 5 surfaces, 4 vertices and 7 edges.*)

2.4 Integration and extension of shape ideas

So far in this chapter, we have looked at definitions, environmental approach and sub-concept approach. Now, we look at integrating and extending. For instance, the environmental approach takes 3D to 2D and the sub-concept approach takes 2D to 3D but stronger learning comes from continuously reversing between the two directions (i.e. $2D \leftrightarrow 3D$). Further, we start to look across examples for generalisations that form principles and relationships (e.g. “all prisms have rectangles between base and top”; “a base shape joined with squares or rectangles to a second base shape is a prism, and a base shape with triangles attached is a pyramid”).

Examples of extension/integration activities are: (a) dividing polygons into triangles with diagonals and using this to determine interior angle sums for shapes with 4, 5, 6, sides and so on, and looking for a pattern and general rule for interior angle sum of an n -sided polygon; (b) making platonic solids and stellating them (i.e. replacing the flat surfaces of polyhedra with pyramids); and (c) using Euler’s formula to make challenges as described in 2.3.7 (d) above.

2.4.1 Developing principles for 2D shapes

Line and angle principles can be developed for 2D shapes. These cover:

- (a) relationships between different-sided polygons (e.g. squares are rectangles are parallelograms are trapeziums are quadrilaterals are polygons);

- (b) angle, diagonal and rigidity principles – e.g. interior angle sum of an n -sided polygon is $(n - 2) \times 180^\circ$; minimum number of diagonals for rigidity of an n -sided polygon is $n - 3$; the total number of diagonals possible in an n -sided figure is $1 + 2 + 3 \dots$ up to $(n - 3) + (n - 3)$.

The activities for these are in subsection 2.2.5.

2.4.2 Constructions and principles for 3D shapes

Constructing 3D shapes can assist in developing principles (relationships and formulae) for these 3D shapes.

Constructing and relating

Use construction methods to make 3D shapes out of faces (e.g. nets) or out of edges (e.g. mat and elastic bands).

- Construct all the same 3D shapes (e.g. prisms) and look for similarities (all have rectangles).
- Construct different 3D shapes (e.g. a prism and a pyramid) and look for differences (one has rectangles, the other has triangles).
- Construct 3D shapes from four or more 2D shapes. Look up names for these 3D shapes in terms of the number of surfaces using the generic name for a 3D shape made out of flat surfaces [polydrons]. Example – make a shape out of six squares – what is its name based on the number of surfaces? [hexadron] – what is its other better known name? [cube].
- Repeat (c) but for curved shapes. What is the difference between cone and cylinder? What is similar about cone and pyramid, and cylinder and prism?

Constructions from nets

- Construct the 3D shapes from the nets provided in subsection 2.2.1.
- Find an example of a sphere (not available as a net).
- Challenge – make a net into a solid, place symbols/pictures on each side of solid, ask the students to place the same symbols/pictures on their nets before folding them so that when folded, the resulting solid has symbols/pictures the same as the original (very hard).

Defining shapes

Use the findings for above to define/determine what a shape is from its surfaces, etc. For example, two pentagons on each end, and rectangles between the pentagons – what is it?

Euler's formula/principle

Construct different 3D shapes and record surfaces, vertices (corners) and edges. Record in a table like below.

Shape	Surfaces	Vertices	Edges
Cube	6	8	12
Triangular pyramid	4	4	6
and so on			

If students have problems discovering the formula (surfaces + vertices = edges + 2), ask them to add a column with surfaces + vertices in it and to compare with edges.

Euler's formula for 3D solids is related to Euler's formula for networks (because a 3D shape made out of edges becomes a network if a light is used to project the edges and vertices onto a flat piece of paper). Euler's formula is also useful in setting up problems – see subsection 2.3.7.

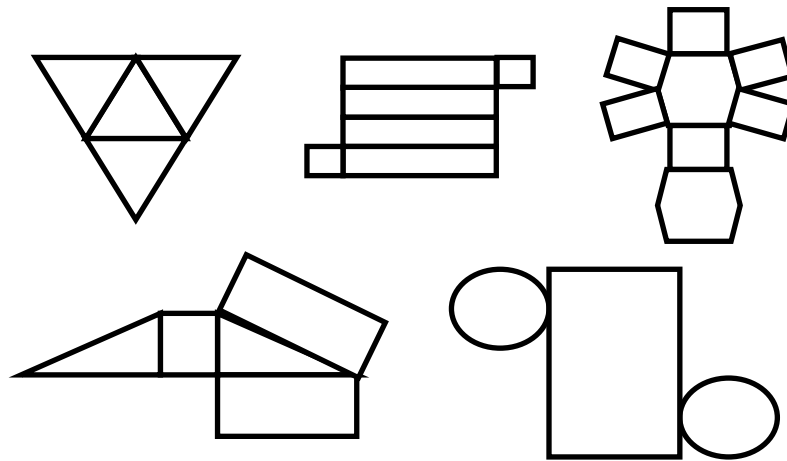
2.4.3 Relating 2D and 3D shapes

There are relationships between 3D shapes and their 2D or curved surfaces. For examples, prisms join the two ends/bases by rectangles while pyramids join the end/base to a point by triangles. We need to relate 3D shapes to their surfaces.

2D shape to 3D solid

The objective of this activity is to reinforce that 3D shapes are put together using 2D faces and that different combinations of 2D faces will result in different 3D shapes but only certain combinations will work together.

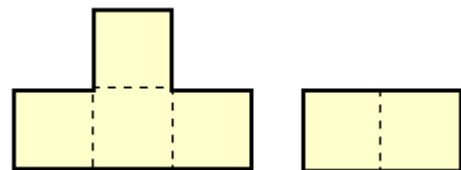
1. Provide the nets below (enlarged). Ensure that the edges of a net are congruent (the same) to make the shape work.
2. Ask students to determine what 3D shapes can be made from the nets. Ask students to say why as well as what.
3. Let the students make the shapes. Discuss relationships between nets and shapes (e.g. triangles give pyramids, rectangles give prisms).



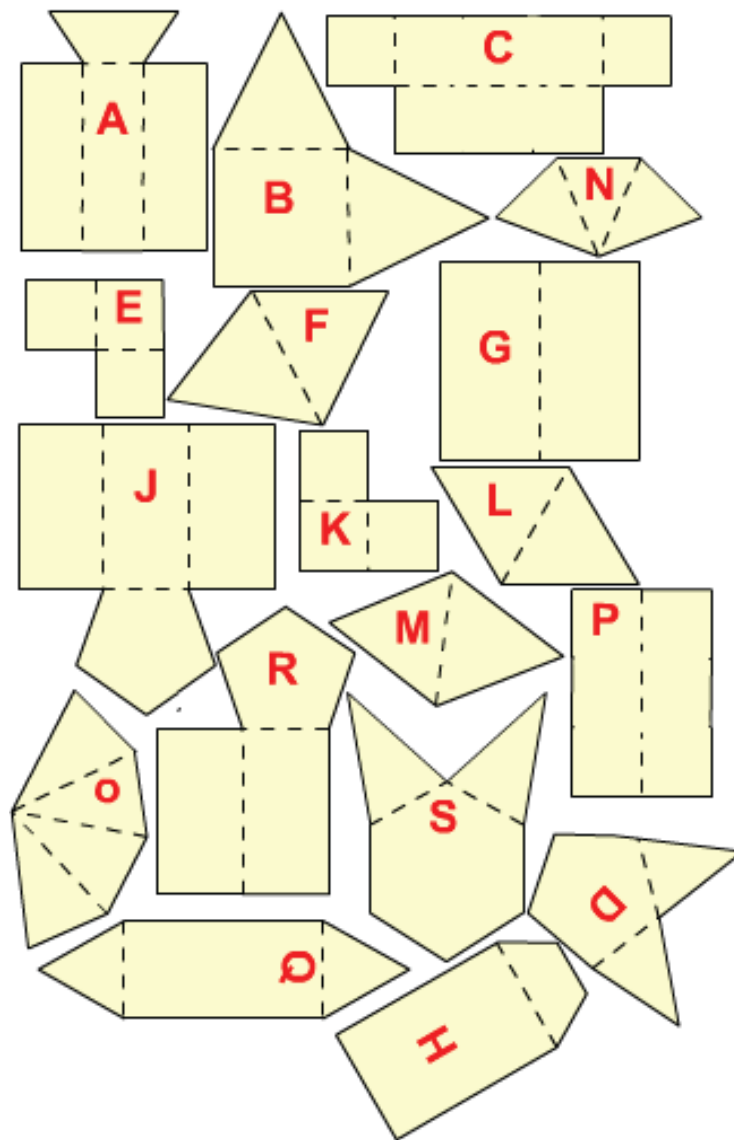
Cut nets problems

The objective of this activity is to provide nets which have been cut into two. Students have to choose two that can be put together to make a 3D solid, and then find more than one way to put them together. The cut nets below could be enlarged, cut out and manipulated and made with sticky tape. Students should also be asked to visualise and state the formed solid shape before it is made.

1. The two shapes on right are a net of a cube that has been cut into two. Put it together to form the cube. Can you do this in more than one way?



2. Below are the nets of nine solid shapes. Each one of these has been cut into two pieces, like the net of the cube.
 - Most pieces will not go together to form a shape; which ones will?
 - What is it about those combinations that make them work?
 - Enlarge the shapes and make the solids that are possible. Can you do this more than one way?
 - Make a table of the shapes that went together. Does another student have a different table?



Visualising cube nets

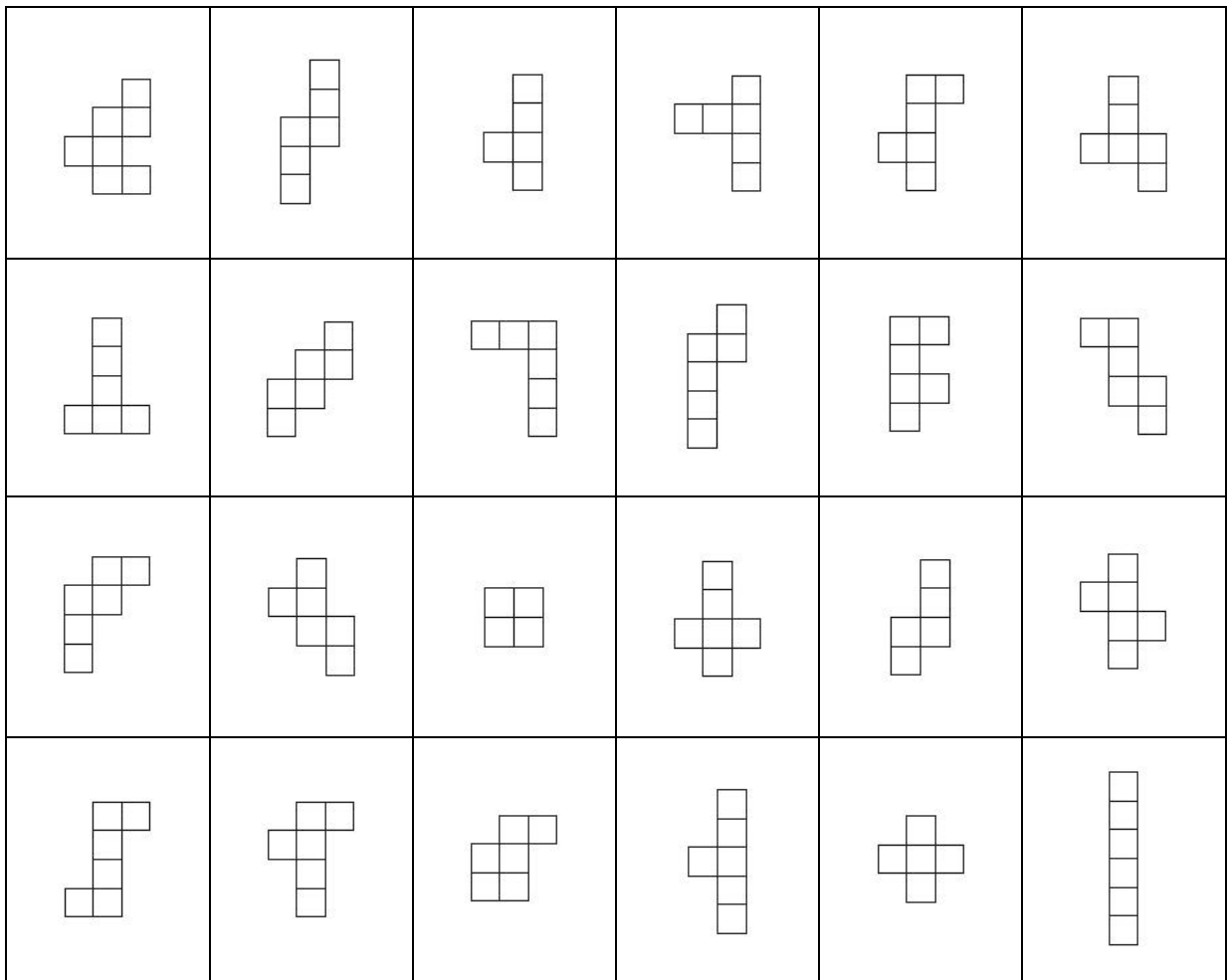
The objective of this activity is for students to realise that:

- (a) there are many possible nets for folding into a cube;
- (b) different nets use different amounts of paper; and
- (c) it is big business determining the net that gives the required cube for the least area of paper.

There are several options for implementing this activity. Students could do either or all of the following.

- Draw as many nets as possible that will fold together to make a cube. What is similar about the cube nets that make them possible? How many different nets are there that will make a cube?
- Select the nets from the table below that will fold to form a cube – these could be copied and enlarged to allow students to test their assumptions.
- **Challenge.** Design a net to create the most cube boxes possible from a sheet of card (i.e. find the net that is most space efficient). Extend their design by adding gluing flaps to their chosen net and designing cover art so that it all fits together when folded.

Note: this activity covers net \rightarrow 3D and 3D \rightarrow net; sometimes starting from the net and other times from the 3D shape. It also has a real-world problem in the challenge – producing cubes with the least wastage.



Drawing nets from 2D shapes and for 3D shapes

This activity allows students to explore the 3D shapes that can be made from the following combinations of 2D shapes. It moves on to get the students to visualise what 2D surfaces go where on the 3D shape.

1. Draw a net for the following shapes that would fold into a 3D shape:

- 2 congruent squares and 4 rectangles;
- 2 congruent triangles and 3 rectangles;
- 2 triangles and 3 trapeziums.

Note: Combinations can be given where there is:

- a surplus of 2D shapes where students have to determine which one to omit; or
 - one shape missing and students have to determine what the 2D shape would be.
2. **Problem solving:** A 3D shape (e.g. a cube) can be shown with patterns on each face – students have to find a net for the shape and then draw patterns on the net so that, when folded, it is the same as the shape.

Using Euler's formula

Students can use Euler's formula ($\text{surfaces} + \text{vertices} = \text{edges} + 2$) to design nets of possible 3D shapes.

- Provide students with a selection of surfaces, vertices, edges for 3D shapes as below.
- Ask the students to determine from which of the selections it is possible to make a 3D shape and from which it is not.

Surfaces	Vertices	Edges	Tick if possible
6	8	12	
4	6	8	
8	17	25	
12	10	20	
9	15	22	
14	16	30	

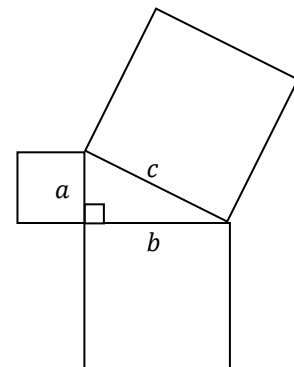
- Ask students to make nets and create designs on the net so that, when the nets are folded into 3D shapes, the designs match along the edges (design must be constructed to go around the shape and cross edges).
- Challenge.** Ask students to link to packaging challenge from earlier. Ask students to design cover art for package so that writing is the same way up all the way around and readable on top when viewed from the front.
- Investigation.** Finally, for the last activity, provide students with numbers of vertices, surfaces and edges – use Euler’s formula so that it is theoretically possible to make the 3D shapes (e.g. 4 vertices, 5 surfaces and 7 edges because $7 + 2 = 4 + 5$). Give students polydrons or construct-o-straws or another commercial construction kit to try to make the shapes. State that they may not be able to make them all; their job is to find which ones can be made. Ask them to work out why.

2.4.4 Pythagoras’ theorem

This subsection looks at Pythagoras’ theorem, a theorem that is crucial in trades – it is used to square foundations, walls and so on.

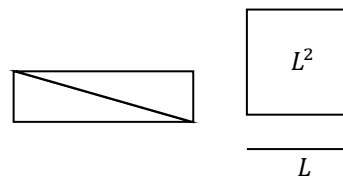
Discovering Pythagoras’ theorem

Pythagoras’ theorem states that a right-angle triangle’s sides have lengths so that $a^2 + b^2 = c^2$, where a , b and c are the triangle’s sides as on right. a and b are called the adjacent sides (next to the right angle) and c is the opposite side (opposite to the right angle).



There are famous right-angle triangle types: the 3-4-5 triangle, as $9 + 16 = 25$, and the 5-12-13 triangle, as $25 + 144 = 169$ which is 13×13 .

If one wants to square a foundation that is a rectangle, one ensures that the diagonal forms a triangle where the length of the diagonal obeys Pythagoras’ theorem in relation to the sides of the rectangle.



Using squared paper

Show that the square of a line of length L , that is, L^2 , is the area of a square made with the length of that line. This means that if squares are made on the sides of a right-angle triangle, as above, and the area of the largest square is equal to the sum of the areas of the other two squares, we have Pythagoras ($a^2 + b^2 = c^2$).

Draw three different right-angle triangles on squared paper so that the triangles’ corners are where lines cross and one of the adjacent sides is vertical and the other horizontal. Draw squares on each side of each of the triangles (as in diagram above).

Find the area of the three squares for each of the triangles and put them on a table as below:

Triangle	Area of square on side a (a^2)	Area of square on side b (b^2)	Area of square of side c (c^2)	Sum of areas for a and b ($a^2 + b^2$)
1				
2				
3				

Using the data encourage students to show that Pythagoras' theorem holds for these – to see that $c^2 = a^2 + b^2$.

Drawing methods

There are other ways to show Pythagoras' theorem – methods can be looked up online. Here is one drawing method based on rearranging four copies of the right-angled triangles.

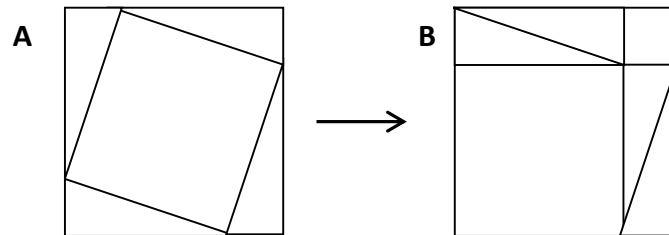


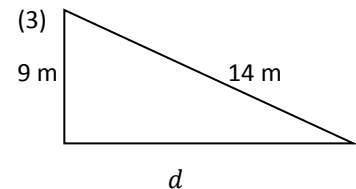
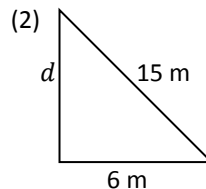
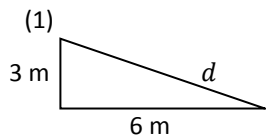
Diagram A has four triangles plus a large square; diagram B has the four triangles and two smaller squares. Since diagram A is the same size as diagram B, this means that the area of the large square (c^2) is the same size as the sum of the areas of the other two squares ($a^2 + b^2$). This means that Pythagoras' theorem holds.

Applying Pythagoras' theorem

Some applications are as follows.

1. If we have a right-angle triangle and know the measurements of two of the sides, we can use Pythagoras' theorem to find the length of the third side. The three ways are: $c^2 = a^2 + b^2$, $a^2 = c^2 - b^2$, and $b^2 = c^2 - a^2$.

Calculate d :

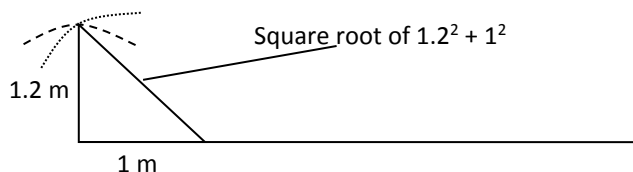


2. If we have two sides of a rectangle, we can work out what the diagonal length would be if the angle between the two sides is a right angle by using $c^2 = a^2 + b^2$. If we measure the diagonal and the length does follow Pythagoras' theorem, this means the angle is a right angle and the rectangle is "square".

Challenge. Mark out a side of a path on the floor of a room or in the yard (even better, a garden bed that needs squaring). The path has to be 1.2 m wide. Using pegs or blocks and measuring tape:

- (a) mark out the 1.2 m end of the path so it is square to the line; and
- (b) mark out the other side of the path so that it is parallel to the first side and square to the end.

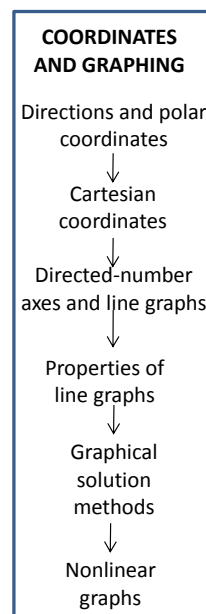
Hint: For the end:



Investigation. You mark out a rectangular concrete slab for a house. You measure the diagonals – one is longer than the other. What does this mean? Particularly for the squareness of the slab? How can it help us square the slab?

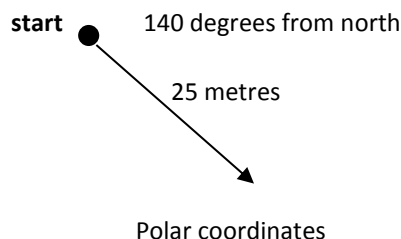
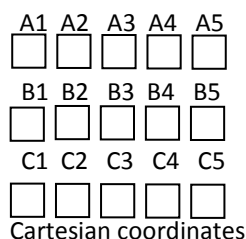
3 Coordinates and Graphing

The sequence for this chapter is shown on the right and covers directions and polar coordinates, Cartesian coordinates, directed-number axes and line graphs, properties of line graphs, graphical solution methods, and nonlinear graphs. Major concepts for coordinates and graphing include position, maps, and graphs which are briefly overviewed below.



Position

Initially, position can be defined in terms of everyday words such as “here”, “there”, “near”, “far”, “under”, “over”, “above”, “below”, etc. Then position is seen in terms of coordinates (see below): (a) polar – where directions and distances are given to define a position; and (b) Cartesian – where numbers (and letters) are used to more formally determine position in terms of ordered pairs of numbers and/or letters (e.g. “row C seat 4”, “two blocks south and one block west”, or an ordered pair of numbers).

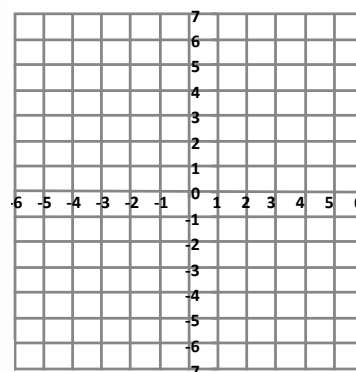
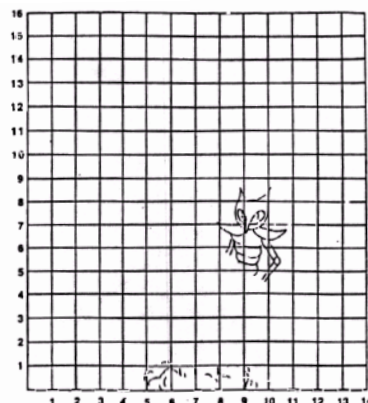


Maps

Position is extended to maps which can be based on (a) polar coordinates, where directions are shown from a starting point by directions and distances (e.g. hand-drawn or computer maps showing turns and distances, modern GPS systems which verbally/visually give directions); or (b) Cartesian coordinates, where position is in terms of two values or coordinates (e.g. theatres and street directories where the values are often given by a letter for the row and a number for the column, and latitude and longitude, where the values are both numbers).

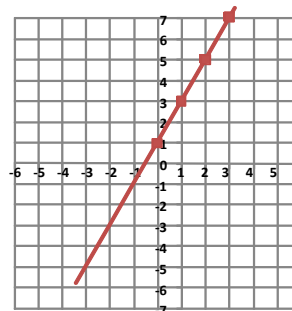
The special areas that need to be explored are direction in terms of: (a) north, east, west and south (i.e. compass bearings and smart phone directions), and the use of this knowledge with knowledge of map reading to enable learners to undertake orienteering and bushwalking activities; and (b) latitude and longitude as this is the basis of GPS systems that find position on the earth’s surface.

Another particular extension in later years is the use of negative numbers in coordinates. Initially coordinates are all positive (as in the first diagram on the right) and then there are four quadrants as the negative numbers are brought in (as in the second diagram on the right).



Graphs

Line and other graphs. If two points are plotted on axes, they can be joined by a straight line. This straight line reflects a sequence of points that relate linearly, in terms of equations of form, for example, $2x + 3$. Such lines, diagrammatically represented on the right, are defined by their slope (the rate determined by y increase divided by x increase or y increase across x increase of 1) and where they cut the y axis. The slope is usually represented by m and the y -intercept by c , thus the general equation of a straight line is $y = mx + c$. The line on the right has slope of 2 because y increases by 2 for each increase of x of 1, and cuts the x axis at 1. Thus the equation is $2x + 1$.



Nonlinear graphs. These are also formed from plotting points on axes, most noticeably the circle and the parabola.

3.1 Directions and polar coordinates

In this section, we provide an overview of activities that focus on **position from a polar perspective**. As will become evident, these require strong development of language. As well as this, we provide a RAMR cycle lesson that encapsulates the major learning from this section.

3.1.1 Overview of polar activities

Polar activities relate to direction and distance – they are like the GPS system, giving direction and distance.

Early teaching activities

1. **Play Polar “Mr Here”.** Obtain a small toy, hide the toy every day, provide students with instructions to find it that relate to direction (e.g. *above the table, away from the blackboard, to the right of the mathematics books*, and so on).
2. **Set up room to develop language.** Take students outside and photograph them doing polar position things “towards” something, “away from” something, “next to” something, and so on. Try to think of all the positional words you can (e.g. near, far, over, close, and so on). Take photographs of students doing these things, print them, attach words to them, and display around classroom.
3. **Giving and receiving directions.** Set up situations where students have to give directions to other students to walk through objects (e.g. turn left, walk ahead two steps, and so on). If safe, student following directions can be blindfolded but needs someone to walk with her/him.
4. **GPS game.** Use a mat to set up a town with shops, churches, and so on. Play the GPS game where one student acts as a car and the other as a GPS giving directions to travel from one place to the other.

Direction activities

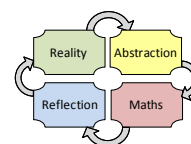
1. **Treasure hunts.** Set up situations where students have to find things given a starting position, a direction and a distance (e.g. *Go to post with red circle on it, walk towards the big tree, when you have travelled 15 stick lengths, look around for a rock, treasure will be under it*).
2. **Constructing directional maps.** Have students draw plans for walking between familiar things (e.g. *Draw a map from here to the Principal’s office*).
3. **Following directional maps.** Follow someone else’s plan for getting from a start to a finish.
4. **Computers/GPS systems.** Give students experience with maps that you get off the Internet to go to places. Give students experience following a GPS system for a car.

Orienteering activities

1. **Drop 20 cents.** Introduce pacing for distance in metres and compasses for direction in degrees from north and move on to orienteering-type activities: *Drop 20 cents on the ground. Go forward 10 m, turn 120 degrees using compass, repeat this twice more, how close are you to the 20 cent piece?*
2. **Starting point.** Hide some things (letters for a word, numerals for a large number, or words and phrases for the answer to a joke) in the yard. Give students a starting point and the angles or compass bearings and distances to these hiding places. Students follow these instructions to find the things.
3. **Shapes.** Give students shapes and compasses and ask them to make the shapes on the oval with stones or flags at the corners. Move to instructions to travel from home to school and between towns (and so on).
4. **Orienteering track.** Get students to follow an orienteering track (short distance, pacing and angles, not running). Give them a start position and then distance and angle to the next position and so on until they get to finish point. At each position, have something they write into their directions that showed they got this far. Hide this something so they have to look around for it so it is not visible as they walk up to the position.

3.1.2 Polar RAMR cycle: Make your own treasure map

This cycle sets the scene for a map from the reality of the students, abstracts ability to pace metres and find direction in terms of north and relates this to maps. It also mathematically prepares a treasure map for other students to follow and reflects reality to find situations where this type of map is used in our lives.



Reality

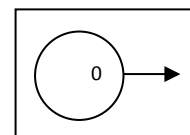
Discuss the following directions. Have they had to give directions to another? Can they draw a rough map in terms of direction and distance to get around the school or somewhere else in the local area?

Prepare a map or a list of directions for the students to follow. Look up on the Internet for road directions to drive from one place to the other in a car.

Abstraction

Body. *Set up a 10-metre distance between two lines and get students to pace this distance. Get them to calculate how many of their normal steps to 10 m. Also get them to practise walking distance in metres – measure some distances and get them to see how accurate they are in walking those distances.*

Use Silva compasses or smart phones. If not available, make up a rough compass from a circular protractor (as on right) on a cardboard rectangle. Always align arrow with north (pick something like the side of a building or a tree in the distance which is close to north). Then follow direction given by angle 0 to 360° degrees when arrow is pointing north.



Teach students to find and walk distance – give them some directions to follow, so they can become used to doing it.

*Practise finding things when distance and direction given. Set up a map for them to follow (give starting point and then distances and angles to follow). See old geometry book, **Space and Shape in the Primary School**, on the YDC website for information on this.*

Hand. *Translate the walking activity to the drawing of a map giving distances and angles and vice versa. Use a ruler (10 m = 1 cm) and a protractor to draw the maps.*

Mind. *Encourage the students to imagine walking around a map in their minds. Give them verbal directions from the classroom when they have their eyes shut and see if they can guess where they will end up.*

Mathematics

Language/symbols. Ensure that students have all the language associated with directions – north, south, east and west; near, far, above, below, over, under, left, right, and so on. Take photos of students doing this and put on board with the words. Ensure students can use straight lines and angles and that they use a ruler and a protractor.

Practice. Make up a treasure map for students to follow. When they seem able to do this, get groups of students to make up their own maps for other groups to follow.

Connections. Connect this work to angle and distance and the measurement of both.

Reflection

Validity. Ask students to make up directions for their neighbourhoods and draw maps.

Applications. Find maps in the world the students could interpret – maybe elders could provide directions.

Extension. Flexibility – show a variety of types of maps; reversing – do this by going from map to walking and vice versa; generalising – ask students to describe the features of maps – what gives what.

3.2 Cartesian coordinates

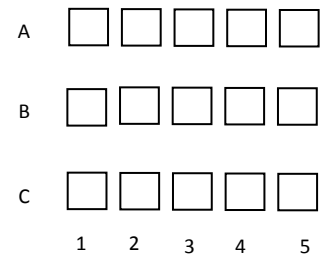
Cartesian coordinates relate to finding position on a grid, like a street directory. This section covers activities plus a RAMR lesson. The activities include games plus plotting and finding points with positive and negative numbers.

3.2.1 Overview of Cartesian activities

Early activities

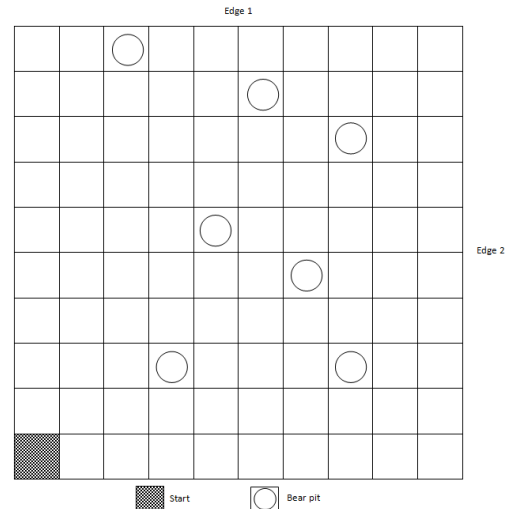
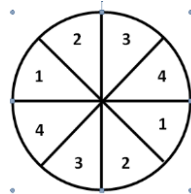
1. **Play Cartesian “Mr Here” or Treasure hunt.** Take a small toy and hide it every day, but provide students with instructions to find it that relate to the position of other things (e.g. next to the red box, in front of the whiteboard, and so on).
2. **Set up room.** Take students outside and photograph them being “above” something, “under” something, “in front of” something, “in the middle of” something, and so on. Try to think of all the positional words you can that relate to the fixed position of other objects in terms of being in line. Take photographs, attach words to them and display around classroom.
3. **Draw diagrams/maps of familiar areas you are in.** Ask students to draw a map of their classroom. Discuss what they produce and how it relates to the room (often not exact in terms of position in room but usually OK in terms of what it is next to – e.g. one student put all desks in a corner around the teacher’s desk leaving half the room with nothing in it).
4. **Draw diagrams/maps of familiar areas.** Ask students to draw a map of their house from memory and discuss what they produce.
5. **Construct Cartesian maps.** Draw maps that have enough features so that students can use them to find things (e.g. draw a map of the junior school).
6. **Follow Cartesian maps.** Follow someone else’s Cartesian map to find where something is.

- Set up room for coordinates.** Put chairs in rows and columns and labels on wall (e.g. A, B, C, and so on, for rows, and 1, 2, 3, and so on, for columns – as on right). Students are given coordinates by which to find their chair and table for the lesson. These arrangements can be changed as can whether letters or numbers are used. The coordinates can also be used to find things.
- Set up with Maths Mat.** Set up the Maths Mat with coordinates. Have students discover and plot coordinate points by standing on the appropriate square.

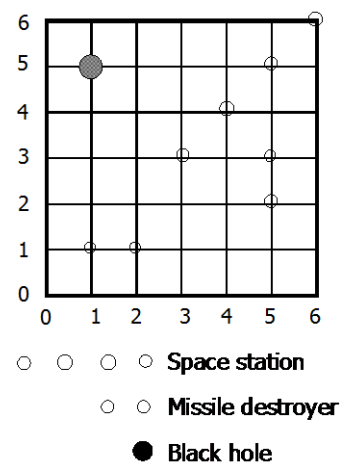


Coordinate games and activities

- Bear Pits.** This game can be played in small groups. You will need the playing board on right and a spinner with numbers 1, 2, 3, 4 as shown below (use a paper clip for the pointer); coloured marker per player. Each player starts at the point of origin (the shaded square) and spins the spinner twice – the first spin determines how many squares **across** you can move; the second spin determines how many squares **up** you can move. Place your marker on the region/square you landed on. If you land on a bear pit, you must go back to the start. The first player to reach either Edge 1 or Edge 2 is the winner.



- Seek and Destroy** (a variation of Battleships). This is a game for two players. The objective of the game is to seek out and destroy your opponent's space station and missile destroyers without falling into the black hole. If you fall into the black hole, the game is over. When you've had one game each, try to destroy your opponent's space station and missile destroyers in a set number of calls. (a) Each player marks off a 6 × 6 section of the square grid to be their war zone. Then sit back-to-back so that you can't see each other's grid. (b) On your grid, you are to place one space station (four points joined), two missile destroyers (two points joined per destroyer) and one black hole. The points can be horizontal, vertical or diagonal but they must form a straight line. An example of a "war zone" is shown on right. (c) One player begins the game by calling a pair of coordinates (e.g. 3, 2). On the war zone on right, this would be a miss and the other player gets a turn. If it had been a direct hit, then you would get another turn. It takes four hits to destroy the space station and two hits to destroy a missile destroyer.



What solution strategies did you develop? That is, did you guess each time or did you refine your guesses after the first two or three turns? How is the grid in *Seek and Destroy* different from the grid in *Bear Pits*? How are the two grids the same?

Plotting and finding points

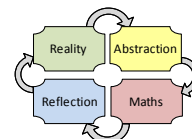
- Drawings.** Plot the points below on a grid. Join consecutive points (A, B, C etc.) with straight lines.
 A (6, 2); B (4, 2); C (4, 4); D (6, 6); E (6, 13); F (7, 15); G (8, 13); H (8, 6); I (10, 4); J (10, 2); K (8, 2); L (8, 1); M (6, 1), A (6, 2).

Make up your own shape; draw it on a grid with corners where two grid lines intersect. Translate points to coordinates and provide coordinates to other students to draw the resultant shape.

2. **Maps and directories.** Collect maps and directories. Set up activities where students have to use coordinates to find positions. Provide positions and ask the students for coordinates.
3. **Construct maps.** Start with a grid. Construct/create your own map for this grid (say towns and roads). Set questions from the map that other students could answer.

Construct a map of the school with a grid that covers the map. Ask students to find their way around this map by giving directions in terms of coordinates.

3.2.2 Cartesian RAMR cycle: Directed axes and flips-slides-turns



This activity covers activities above and then extends coordinates to flips, slides and turns.

Reality

Consider the reality of a mirror – student on one side and image on other. Consider a mirror along a wall. With people and images on each side; we can place the person with a + distance and the image with a – distance. Similarly many sports can be considered in this way; rugby league or soccer is 0 on the halfway line and positive/negative on each side; what about tennis?

Set up games which have a grid with two sides, + and –. People move on each side trying to stop one person getting through; only one person can move and one member of a team has to get behind the other team; players can move sideways but not back. It is like a game of chequers or chess but with the pieces seen in terms of + and – in relation to the halfway line.

Abstraction

Body

1. Use mat to set up a grid with positive and negative numbers on the x axis; use rope to break mat into two halves and place numbers on grid lines.
2. Place numbers along x and y axes. Plot points on either side with students. Position students at coordinates. Go both directions: teacher states position and student moves to position; student stands in a position and states coordinates.
3. Repeat this for four quadrants. Use rope to break the mat into four quadrants. Once again, put positive and negative numbers on axes and repeat process: teacher gives coordinates \rightarrow students stand in position AND teacher positions the students \rightarrow students state coordinates.
4. Plot points with students to make shapes including plotting straight lines. Discuss slope of lines – relate to equation of line. Do many examples.

Hand

1. When doing above with body, replace students with a disc in their position and copy this onto a drawing of the mat (use graph paper); once again, coordinate \rightarrow position and position \rightarrow coordinate.
2. Repeat all this for much larger coordinate situations – say 20×20 squares so quadrants all go from 0 to +10 or –10.

Mind

1. Imagine a coordinate system as large as needed and plot points on it in the mind.
2. Get students to shut eyes and draw out what they see with a finger in front of them.

Mathematics

Language/symbols

1. Use the ordered pairs $(-2, 5)$ to repeat the work above.
2. Ensure all language is understood (e.g. axes).

Practice

1. Ensure that there are practices where bodies and pens are used to plot and determine coordinate points.
2. Ensure that there are practices that reinforce the relation between symbols and position so it is well understood, particularly in four quadrants.

Reflection

Validation

1. Get students to look for coordinates in their lives – particularly with opposite sides where one side could be called negative.

Application (flips, slides and turns by coordinates)

1. Set up mat so there is a positive and negative side: one axis -5 to 5 and the other 0 to 6 .
2. Use the flip-slide-turn work from Chapter 4 (see 4.1) and the students' bodies on the mat to show flips, slides and turns. Use a rope or band to divide the grid into two; label the axes.
3. Have three or four students make simple 3- to 5-sided shapes one side (holding on to corners of a 3- to 5-sided shape) and have three or four other students (with help from class) on the other side making the shape using the three different changes – sliding across the centre line (so shape and image is same distance away from line on each side), reflecting about line, and rotating 180° .
4. Repeat this for another group of students and another shape but, this time, use the labels on the lines to give the coordinates of the first shape and have students say where students should go for the image by **stating the coordinate**.
5. Repeat the above again but getting the students to state the coordinates before the shape is made, and to state the coordinates of corners to show flips-slides-turns.
6. Copy the original shape and its slide, flip and turn onto graph paper – write in the coordinates of corners. Repeat this activity by only using paper – not the mat.
7. **Computers.** Repeat the above on computer. Place a grid on a computer, draw simple shapes, use the mouse and the software to slide shape along a line that starts and ends on coordinates, flip shape about a line that starts and ends on coordinates, and rotate shape 90° , 180° and 270° about a centre which is a coordinate. Write down the coordinates.
8. **Quadrants.** Repeat the above but for the mat/grid divided into four sections using x and y axes (the coordinates can now be negative). Make shapes in one quadrant (top right is normal). Use coordinates to slide the shape across axes, flip the shape about axes, and rotate the shape 90° .

Extension

1. **Flexibility.** Use a variety of grids and shapes.
2. **Reversing.** Do this in the coordinates $\leftarrow \rightarrow$ position changes (and see generalising below).
3. **Generalising.** Write down the starting and finishing coordinates, and **compare differences** in these coordinates for flips, slides and turns; use these comparison to suggest **coordinate rules** for these slides, flips and turn; then reverse and flip, slide and turn only with coordinate rules and use these to draw the flips, slides and turns and see that the rules hold.

3.3 Directed numbers, axes and line graphs

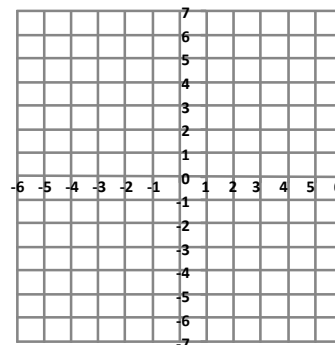
In this section we look at setting up axes, including with negative numbers, so we can plot line graphs. We plot line graphs and see that two points define a line graph. We show a RAMR lesson that focuses on slope and y-intercept to plot the line graph. This lesson leads into section 3.4, the properties of line graphs.

3.3.1 Plotting line graph activities

Plotting points

A line graph is a straight line. Therefore it only needs two points to define it. The following steps describe how to plot a graph.

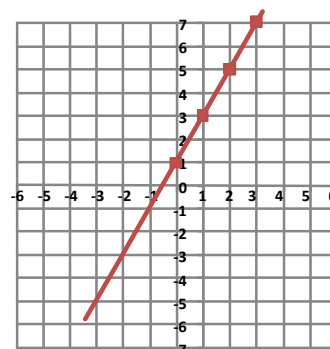
1. Draw a Cartesian coordinate system (as on right).



2. Determine points on this grid (source could be patterns or a function machine) as in table below:

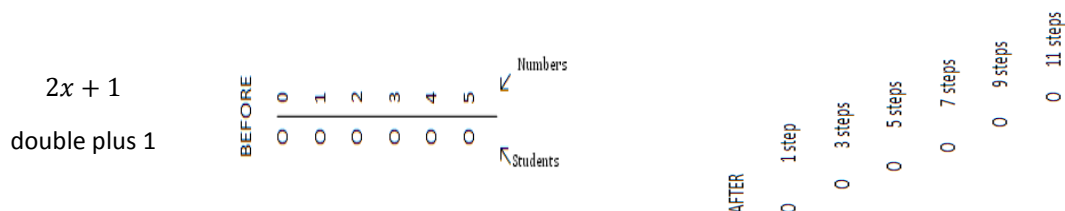
y	1	3	5	7	9	11
x	0	1	2	3	4	5

3. Plot these points and draw a line through them (as on right).
4. Since the points above have a fixed part of 1 and grow 2 for each change in x , the graph is a representation of $y = 2x + 1$.



Kinaesthetic to drawing activities

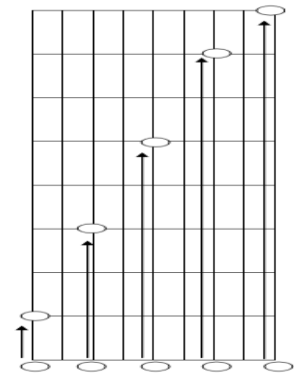
1. **Kinaesthetic activity.** Take students out to the school yard and line them up on a line. Number them in order from 0 to whatever. Take a linear equation (say, $2x + 1$) and state to each student that they have to double the number given to them, add one, and take that many steps (see diagrams below).



When they have finished, discuss that they have made the graph of linear equation $y = 2x + 1$ with their bodies. (*Note:* Train students to take the same-size steps.)

- Next, place students on a line (preferably on a large grid) so that they can make lines with negative slopes, that is, examples like $1 - 2x$. Here, of course, the students would have to start at the top of the grid, or along a line (not a wall) so they can step forward or back.

It is best if you can do this with actual students on a grid as shown on right (using a mat or a grid painted on the school yard). Students need to stand on intersections of lines so they imitate a graph.

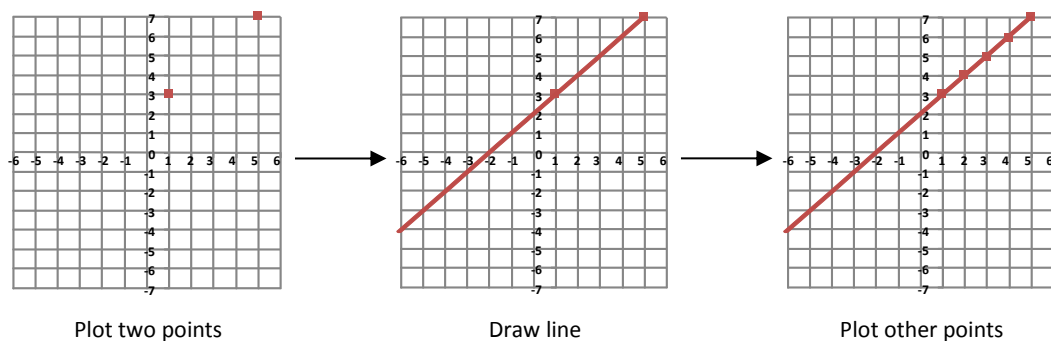


- Formalising kinaesthetic activity.** Get students to hold a rope so they can see that they have made a straight line. Then students can replace themselves with a token and draw a copy of the graph on graph paper to see what they have acted out. Finally, students should be encouraged to imagine the change in their mind (so completing body \rightarrow hand \rightarrow mind).

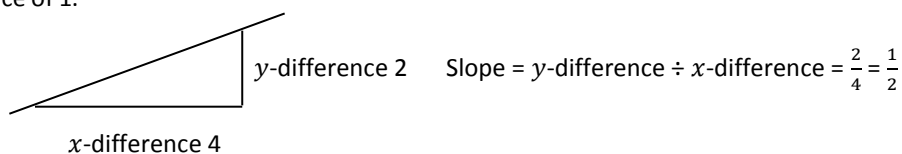
Defining a line graph

- Two points are all that is needed.** Relook at patterning activities from the YDM Algebra book. Use a table but plot only two points. Draw a line between them continuing on each side.

Plot extra points and show that they fit on the line and are not needed (as shown below). Repeat for different two points – show that the line is the same.



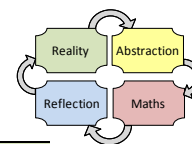
- Finding y-intercept.** Continue line until it cuts the y axis – this y value is the y-intercept.
- Finding slope through rate.** Pick two points – find the y-difference and divide by the x-difference – this gives slope which is a rate (i.e. y-difference per x-difference). Of course, we can simply find the y-difference for x-difference of 1.



Note: This is an unusual calling of rate in one way. Normally when multiplicatively comparing one thing against the other, rate is when attributes are different (e.g. km/h or \$/kg) while ratio is used when attributes are the same (e.g. concrete is 5:2, that is, 5 kg of sand to 2 kg of cement). This slope is distance per distance but they are of different axes.

- Fraction and negative slope.** Use fraction slopes and negative slopes to draw lines – practise a variety of these.
- Equation $y = mx + c$.** Construct a line graph from a pattern using a table of values. Find slope m and y-intercept c . Plot points for $y = mx + c$ and show the graph is the same. Repeat this for negative and fraction slopes.

3.3.2 Graphing RAMR cycle: Slope, y-intercept and line graphs



This cycle shows how slope and y-intercept define a line graph.

Reality

Discuss local situations which are linear relationships that can be represented by line graphs. Look at different patterns. For example:

- My boat can travel for 3 hours on its fuel tank before I need to refill, and 2 hours on a can of fuel, how long can I travel on 1, 2, 3 (and so on) cans of fuel? [$2x + 3$]
- I bought a \$3 ice-cream for everyone plus a \$5 chocolate. How much for 1 person, 2, 3, and so on? [$3x + 5$]
- I had \$13 and I paid \$2 per hour to be there, how much money did I have left or how much do I have to go into debt? [$-2x + 13$]
- X XOO XOOOO XOOOOOO and so on [$2x + 1$]

Act out these problems with materials representing the objects in the situation.

Abstraction

Body. Use students on grids representing different numbers to act out the line graph (see kinaesthetic activity in section 3.3.1 above). Copy the resulting line graphs.

Hand. Use results for different numbers to make a table and plot points using this table. Use the points to draw a line graph. Calculate the rate (m) and the y-intercept (c), extending the graph if necessary. Now represent the graph with an equation $y = mx + c$ putting in the values of m and c . Use the equation to plot the graph again and to see that it represents the line determined from the situation.

Look at patterns and redo the activity in section 2.4.6 of the YDM Algebra book on relationships between linear growing patterns and graphs for linear equations.

Prepare tables for all the above situations with all x values present and in order from 1. Look at the difference between the elements, and the value for 0 (may have to extrapolate that by extending the graph). Show that the slope is the difference and the y-intercept is the 0 value. Ensure students practise examples with negative and fraction slopes.

Mind. Imagine graphs with various slopes (fraction and negative) and y-intercepts. Draw the graphs in the air with fingers while eyes are shut. Be able to imagine all types.

Mathematics

Language/symbols and practice. Ensure words such as slope, etc. are understood, as is $y = mx + c$. Practise relationship between lines and $y = mx + c$, going from line graph \rightarrow equation and equation \rightarrow line graph.

Connections. Ensure connections are made between line graphs (m and c) and pattern rules, and also between line graphs and function machine rules.

Reflection

Validation/Application. Ensure that students can see the relationships in Mathematics above. Undertake applications as follows.

1. **Spaghetti bridge.** Materials: spaghetti, foam cup, paper clip for hook. Use spaghetti, one strand, between two desks. Suspend the cup with an opened out paper clip, hooked over the spaghetti. Add nails (or similar) one at a time to see when it collapses.

Repeat for two strands of spaghetti, then three, and so on. What will they hold? How many strands would you need to support a mobile phone (or similar).

Create a chart to record the results and graph the outcomes.

2. **Heart rates.** Take pulse before starting. Do step-up exercises for two minutes then take pulse again. Continue to record the pulse every two minutes until the heart rate has fallen back to normal. Record on a chart and plot the graph. Discuss the slow-down rate. (Note: Athletes drop back to normal very quickly).

Extension. Flexibility – do many forms of linear situations and for many slopes; reversing – make sure you go from situation to line graph and reverse and line graph to equation and reverse; generalising – make sure all can generalise a line graph to an equation.

3.4 Properties of line graphs

In this section, we look at some of the properties of line graphs: gradient (slope), distance between points, and midpoint between two points on a line. This section should be considered in relation to lines, angles, and Pythagoras' theorem. It should also be considered in relation to negative numbers in the YDM Number book and to the patterning chapter of the YDM Algebra book. It is important that information in this section is elicited as discoveries rather than told directly to students.

3.4.1 Gradient (slope) activities

1. **Relate patterns to linear equations.** Start with a pattern, e.g. oo, oo x, oo xx, oo xxx, oo xxxx, and so on. Determine fixed and growing parts, determine position rule $[n + 2]$ and plot graph. Rename position as x and pattern value as y , redraw graph and rewrite the rule ($y = x + 2$). This is the linear equation or function that the graph now represents. Repeat this activity if necessary.

2. **Extend line graphs to four quadrants.** Continue lines into negative coordinates.

3. **Reverse the situation.** Start with a linear equation, e.g. $y = 3x - 2$. Draw a table of x and y values as on right and fill in the table for $x = 0, 1, 2, 3$ and so forth. Draw the graph. Go backwards and forwards with other examples: graph \rightarrow linear equation (interpreting) and linear equation \rightarrow graph (plotting).

y				
x	0	1	2	3

4. **Look for a relationship.** Relationships between patterns and graphs of linear equations can be found by relating solutions to different patterns, looking for similarities and differences. Provide examples of patterns that have fixed growing parts (pattern rule is a linear equation) and are related in a way that enables relationships with regard to graphs and functions to be seen. In other words, look at patterns with changing growing parts and the same fixed part (e.g. patterns that lead to $x + 1, 2x + 1, 3x + 1$, and so on), and equations with the same growing part and changing fixed parts (e.g. patterns that lead to $2x, 2x + 1, 2x + 2$, and so on).

Study patterns and their pattern rules and identify information on a table as below.

PATTERN	COMPOSITION	POSITION RULE	GRAPH CHARACTERISTICS
(a)	Grows by 2, fixed part of 1	$2n + 1$	Slope 2, y -intercept 1

This leads to the **general rule for slope**: if the growing part is p , the pattern rule starts with pn and the graph has slope p . If the fixed part is c , this means that the **function for a linear equation** is as follows:

$f(x) = mx + c$ where m is the slope (and the growing part if from a linear growing pattern) and c is the y -intercept and the constant part for the zero term ($n = 0$).

3.4.2 Distance and midpoint activities

Once we have plotted line graphs, we can find distance along lines from one coordinate point to a second and halfway along a line from one coordinate to a second. The formulae are:

1. **Distance.** Distance between two coordinate points (x_1, y_1) and (x_2, y_2) is the square root of the sum of the x -difference squared and the y -difference squared:

$$\text{Distance} = \sqrt{(x\text{-difference})^2 + (y\text{-difference})^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

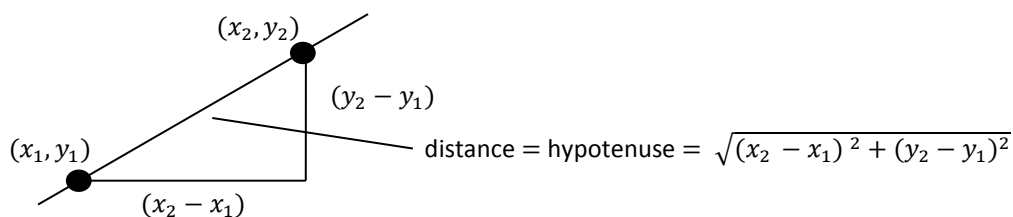
2. **Midpoint.** The midpoint between two coordinate points (x_1, y_1) and (x_2, y_2) is the x and y average:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

This subsection looks at some activities to find and to apply the formulae above.

Distance between two points

1. **Parallel to axes.** Find distances where lines are parallel to the axes – this is just the x -difference or the y -difference.
2. **Pythagoras.** Recap Pythagoras' theorem and use this with the line – this gives the formula.

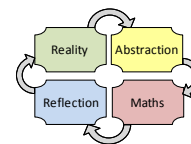


3. **Application.** Find various distances using Pythagoras' theorem for a variety of slopes – including fraction and negative.
4. **Problems.** Given midpoint and one point, find the other end of a line segment.

Midpoints

1. **Discovery for simple examples.** Draw a line between two points, use a ruler to halve it and write down coordinates of the midpoint. Discover the rule for halfway.
2. **Application.** Use a ruler to find halfway and check it is the midpoint using the formula.
3. **Extension.** Extend to finding one-third points and quarter points along lines. What is the general rule?

3.4.3 Properties of line graphs RAMR cycle: Distance and midpoints



This looks at finding distance and midpoint formulae for straight-line graphs.

Reality

Discuss the point of no return where a plane is better flying on than turning back. Discuss how we find these and how they are related to distance. Look at plotting travel on a grid like in radar – how do we know the point of no return?

Act this out – give two positions outside – students pace out distances – students re-pace to mark halfway. Try this on a grid.

Abstraction

Body. Start with mat and simple examples to find distances, half distances and midpoints.

Hand – Stage 1: Distance

The shortest distance between two points is a straight line and this line is a graph of a linear equation. This distance can be found by using Pythagoras' theorem as below.

1. Plot two points (say, $x = 1, y = 1$ and $x = 4, y = 7$) as on right. (Note: Should do this first on the mat or outside with the students' bodies.)
2. Draw a line between the points (and ongoing, as shown on right).
3. Make a right-angle triangle with the sides parallel to the x and y axes as shown on right.
4. The sides parallel to the x and y axes have lengths (here, $4 - 1 = 3$ for x and $7 - 1 = 6$ for y).
5. Using Pythagoras' theorem, $d^2 = 4^2 + 7^2$ therefore $d = \sqrt{16 + 49}$.
6. Repeat this for other points; ask students for pattern or generalisation – encourage or elicit that: (a) distance is calculated by Pythagoras' theorem; (b) it is based on the x -difference and y -difference; and (c) the actual distance is:

$$\sqrt{(x\text{-difference})^2 + (y\text{-difference})^2}$$

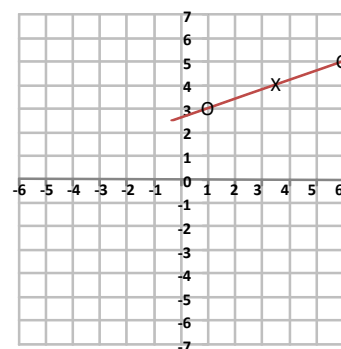
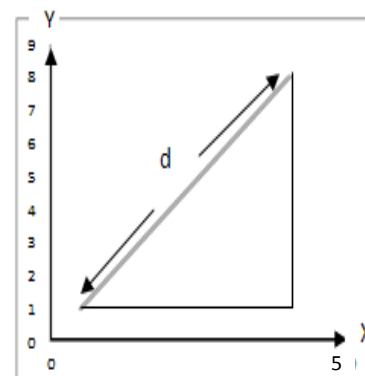
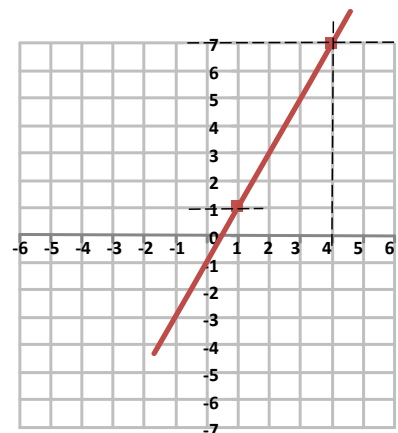
7. With advanced students, generalise so that if two points are (x_1, y_1) and (x_2, y_2) then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

8. Practise whatever generalisation students come up with.

Hand – Stage 2: Midpoints

1. Gather students around a large grid. Stand students on two points (see O's on graph on right). Join them with a rope (line).
2. Get a student to stand halfway (can take the rope and fold in half to find this point but must keep direction), as shown by X on right.
3. Discuss where this midpoint is in terms of coordinates; students should justify answers. For example, points are $(1, 3)$ and $(6, 5)$ – halfway is $(3\frac{1}{2}, 4)$ because this is halfway for x 's and for y 's.
4. Get students to draw this on a graph (place tokens on the grid to help). Repeat for other examples, some simpler and some harder.



Mathematics

Try to elicit a pattern, but do not insist on correct answers. Be happy with "the midpoint is the point halfway between the x 's and halfway between the y 's".

If looking for formality, start with easier examples such as (4, 2) and (8, 8). Discuss how to find halfway for x 's of 4 and 8. Show that this is 6 which is 2 more than 4 and 2 less than 8. Show that this can be found by the average of the two points, that is, adding the two points and dividing by 2 ($\frac{4+8}{2} = 6$).

Make sure the connection to Pythagoras' theorem is obvious for the distance calculation.

Reflection

Two things here: generalisations and applications. Flexibility is in the examples. Reversing has been covered throughout the activities by ensuring we go from midpoint to end points and end points to midpoint.

Generalisations

1. Generalise distance so that if two points are (x_1, y_1) and (x_2, y_2) then

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

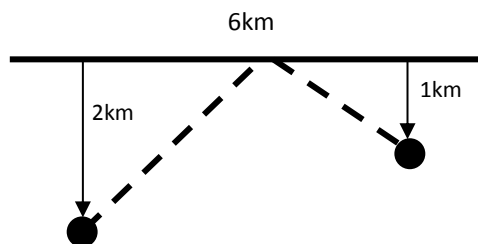
2. Try to get to the midpoint formula: that is, midpoint between (x_1, y_1) and (x_2, y_2) is

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

3. Practise the generalisation students come up with; try to get them to see it formally but the important thing is that they see it informally.

Applications

1. Give distance, direction (slope) and one end point – students have to find other end point.
2. Give midpoint and one end point – students have to find the other end point.
3. Find one-third and quarter points and reverse.
4. A freeway and two towns are positioned as on right. Where will the turn-off be placed so the new roads (dotted line) are as short as possible (and therefore the least expensive)?



3.5 Graphical solution methods

Equations such as $3x + 7 = 22$ and $4x + 3 = x + 9$ can be solved by backtracking ($3x + 7 = 22 \rightarrow x = \frac{(22-7)}{3} = 5$) and the balance rule ($4x + 3 = x + 9 \rightarrow 4x + 3 - 3 = x + 9 - 3 \rightarrow 4x = x + 6 \rightarrow 4x - x = x + 6 - x \rightarrow 3x = 6 \rightarrow \frac{3x}{3} = \frac{6}{3} \rightarrow x = 2$), as described in the YDM Algebra book. However, they can also be solved by plotting points and seeing where lines cross; for example, $3x + 7 = 22$ where $y = 3x + 7$ crosses at $y = 22$; $4x + 3 = x + 9$ where $y = 4x + 3$ crosses at $y = x + 9$.

This section will describe activities and provide a RAMR lesson for graphical solution methods.

3.5.1 Overview of graphical methods activities

There are two things to do – build expertise with the plotting approach, and use applications.

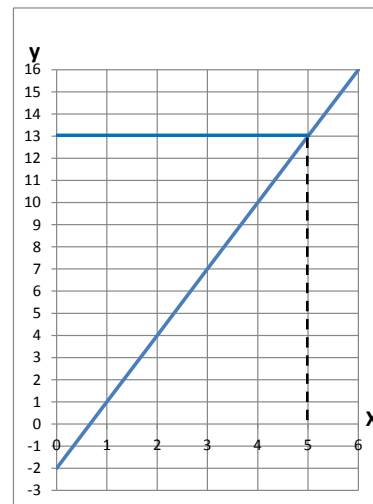
Plotting techniques

These have two parts: (a) determining what graphs have to be drawn; and (b) finding where they cross.

1. **Simple methods.** Consider $3x - 2 = 13$

(a) Determine the two graphs – this is based on the equation having two sides – the two graphs are the sides of the equation:
 $y = 3x - 2$ and $y = 13$.

(b) Draw the lines and see where the line graph $y = 3x - 2$ crosses 13 ($y = 13$) (see diagram on right). The line from $y = 13$ crosses the line for $y = 3x - 2$ at $x = 5$. Therefore, the solution to $3x - 2 = 13$ is $x = 5$.



Do a few examples and also solve the equation by backtracking to check that the same answers are calculated as from the graph.

2. **More complex methods.** Consider $3x - 2 = 2x + 2$

(a) Determine the two graphs – this again is based on the equation having two sides – the two graphs are the sides $y = 3x - 2$ and $y = 2x + 2$.

(b) Draw the lines and see where the two line graphs cross.

Do a few examples. Also solve them by the balance rule and check that the answers are the same.

3. **Reversing.** Give a point and find/draw two graphs that cross at that point.

Applications

For these, do not start with equations and line graphs; start with a problem which has to be translated to equations and line graphs.

1. **Knotted rope.** Provide several ropes of different thickness each with three knots in them. Task is to work out (without untying the knots) how long each rope is.

Clue: Tie more knots and graph the resulting length each time, so you are recording the effect of the knot.

(a) Make a table or chart:

No. of knots	3	4	5	
Length of rope	340	240	140	and so on

(b) Plot the points as a graph and work backwards to find the length of rope at zero knots i.e. the full length of the rope; length at zero knots will be the y -intercept.

Note: It can be useful to cut the ropes (different thicknesses) all at the same length (e.g. 960 cm) so you (the teacher) know the y -intercept (length of the rope before the rope is knotted) and so that there are different line graphs of different slope but with the same y -intercept. Also, all the graphs will have a negative slope.

(c) Discuss the implications of this - could you extend the graph in either direction? Why or why not?

2. **Cars.** Tom started 40 km closer than Fred. Tom travelled at 80 km/hour and Fred at 100 km/hour. When does Fred catch Tom?

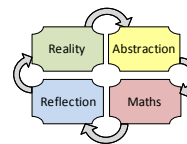
(a) Let x be number of hours that Fred drives. At this point, Fred has travelled $100x$ and Tom has travelled $80x + 40$.

(b) Plot these graphs and see where they cross.

(c) Check answer by substitution.

3.5.2 Graphical methods RAMR cycle: Solving two-sided equations

This cycle looks at how to solve problems with unknowns for linear situations by graphing and finding where graphs cross.



Reality

Problem. Start with problems that relate to students' lives, e.g. NRL or netball. For example: The "space shots" ball (a game where payers line up and shoot three points as fast as they can) Team A was losing 8 to 13 to Team B at half time, but scored at 3 points per minute in the second half while Team B only scored at 2 points per minute. Answer these questions:

- At what time in the second half did Team A pass a score of 26?
- At what time in the second half did Team A pass Team B?
- If halves are 10 minutes, what was the final score?

Students need to see that Team A's score in the second half is $3m+8$ and Team B's is $2m+13$ where m is number of minutes. Then (a) is answered by seeing when $3m+8 = 26$; (b) is answered by seeing when $3m+8 = 2m+13$; and (c) is answered by the scores at 10 minutes. Students need to show how this can be done by drawing line graphs.

Acting out. This could be solved by simply acting out the scores at each minute in the second half:

Minutes	0	1	2	3	4	5	6	7	8	9	10
Team A	8	11	14	17	20	23	26	29	32	35	38
Team B	13	15	17	19	21	23	25	27	29	31	33

Abstraction

Body–hand–mind. Act out the problem first, then with materials and pen and paper; check what is done, and then imagine what is done.

Part One: Simple equations

- Determine the two graphs.** Make up problems like in Reality above – e.g. You have \$5 and you earn \$8 per hour, how many hours to earn enough to buy a \$69 phone? Go through and show that the two equations are $y = 8x + 5$ and $y = 69$.

One way to do this is to make a table of hours (e.g. 0, 1, 2, 3 and so on) versus money (e.g. \$5, \$13, \$21, \$29 and so on). Look at the differences to obtain the growing part (which is 8) and fixed part (which is 5) and remember that this gives $8n + 5$ in patterns or $y = 8x + 5$ in graphs. The other side is the target and this means getting to \$69. Students need to show that such targets are a horizontal line, in this case at 69 or $y = 69$.

- Use line graphs to solve the problem.** Construct appropriate axes (y axis 0 to 70 and x axis 0 to 10). Draw the graphs. See where they cross. Then check to see that this is the correct time by calculating when you get to \$69 from the table.
- Redo these steps for two other problems – try "space shots" and "cars" above.

Part Two: Complex equations

- Determine the two graphs.** Make up more complex problems (extend the simple ones): You have \$5 and you earn \$8 per hour, Bob has \$20 but only earns \$5 per hour. When do you have the same money as him? How many hours longer does Bob have to work for his \$69 phone than you have to work?

Repeat the process of Part One – you will need to do chart/table for both you and Bob. The two equations are $y = 8x + 5$ and $y = 5x + 20$.

2. **Use line graphs to solve the problem.** Draw axes, draw graphs, and see where they cross. Check results. See when Bob gets to or over \$69 by seeing the point on the x axis where his graph cuts the line for $y = 69$.
3. Redo these steps for new examples (e.g. space shots and cars).

Mathematics

Symbols/language. Ensure students understand equations (e.g. that = means “same value as” not “where to put the answer”).

Practice. Separately practise Steps 1 and 2 above for the two parts, then practise both and all parts together. Make sure students can go: story \rightarrow equation \rightarrow graph **and** graph \rightarrow equation \rightarrow story.

Connections. Two crucial activities:

1. **Reversing.** Really push hard to ensure that students are able to go from real-world situations to equations to graphs and vice versa. **Ensure reversing is done** – get students to make up situations for equations and graphs that you give the students.
2. **Backtracking/Balance.** Draw strong **connections** between action of graphs and backtracking and balance as ways to solve equations for unknowns/problems of this type.

Reflection

Validity. Get students to find problems in their world.

Applications. We have done this across the cycle but maybe there is more to be done (e.g. rich tasks with more than one problem).

Extension. Flexibility – check all different situations for equations; reversing – very important as mentioned in Mathematics above (go from story \rightarrow equation \rightarrow graph **and** graph \rightarrow equation \rightarrow story); generalising – ensure students can begin to see where the unknown is being used in a general sense.

3.6 Nonlinear graphs

As stated throughout the YDM Algebra book, the focus up to Year 9 is on linear graphs. However, other graphs are possible and should be pre-empted in junior secondary years.

3.6.1 Plotting nonlinear graphs

Begin with a nonlinear expression such as $x^2 + 2$. Start to plot points where (x, y) is such that $y = x^2 + 2$.

1. Determine the nonlinear expression: $x^2 + 2$
2. Use it to relate x and y coordinates: $y = x^2 + 2$
3. Construct a table of points:

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	27	18	11	6	3	2	3	6	11	18	27

4. Plot the points on a graph.
5. Draw a smooth curve through the points.

Challenge: What happens if x and y are related by $x = y^2 + 2$? Draw it. Try plotting these two other examples: $x^2 - 1$ and $1 - 2x^3$.

3.6.2 Relationships in plotting nonlinear graphs and functions

The most common extension of linear graphs is quadratics. There are relationships between the equation and the shape of the graph. Can you find them? (Note: A similar activity to this is in section 2.5 of the YDM Algebra book.)

1. Plot and find the rules for the following quadratic examples for $x = -5$ to 5 (use technology if available):

(a) $y = x^2 - 1$ (b) $y = 2x^2 + 1$ (c) $y = 2 - 3x^2$ (d) $y = x^2 + 2x + 1$ (e) $y = 3x + 1 - 2x^2$

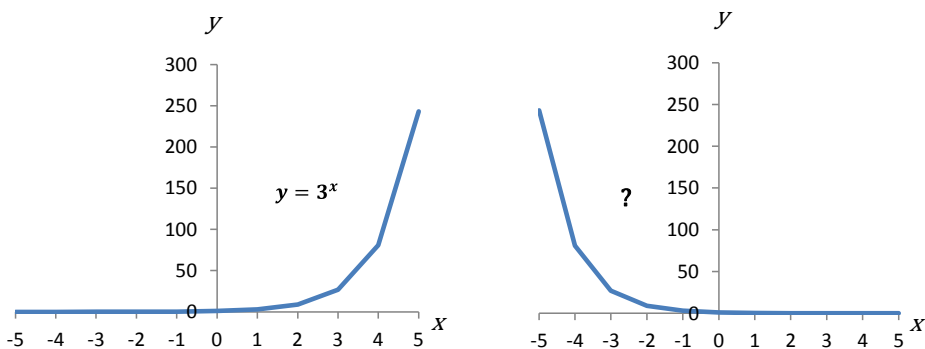
Look at the quadratic graphs and develop some relationships between the graphs, their shape and the coefficients of x^2 and the constant number. What would $ax^2 + bx + c$ look like?

2. The next level of equations or functions is cubics. These have an x^3 . Try plotting these three cubics:

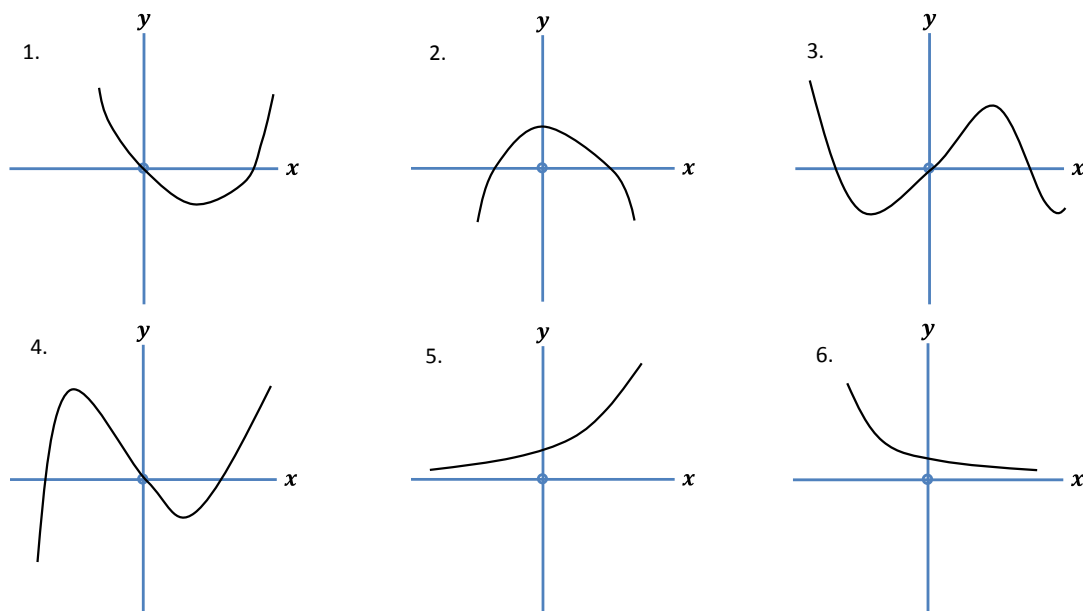
(a) $x^3 + 1$ (b) $1 - 2x^3$ (c) $\frac{1}{2}x^3 - 2x^2 + 3x + 2$

What patterns have you found between these forms of equation? What about the coefficient and the constant term?

3. Finally, there are the exponentials. Plot the graph of $y = 2^x$ for x from -5 to $+5$. What does the graph look like? How would we change the graph of 3^x below left to the symmetrical one on the right? (What is its equation?)



4. Reversing is always important in maths, so look at the six graphs below and say what characteristics they have (there could be two options). That is, are they squares, cubics, exponentials or logarithmic? What could their coefficients be?

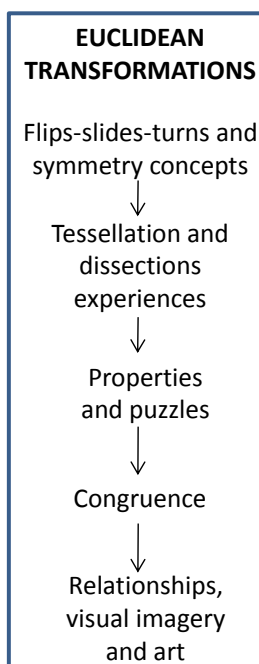


4 Euclidean Transformations

This section covers the changes that occur in the built environment when it operates without breakages, loss and other problems. That is, it covers changes that do not affect the size and shape of solid and plane shapes. Such changes are of three types – the flip (reflection), slide (translation) and turn (rotation). With regard to pictures (and plane shapes), flips relate to and are the defining concept behind line symmetry, while turns perform the same function for rotational symmetry. Lastly, flips, slides and turns have application in two other geometric ideas important to the built environment:

1. tessellations (e.g. tiling patterns) that provide the framework for much of the building in the built environment, as they focus on how a shape can repeat itself to cover space without gaps and overlaps (like bricks in a wall); and
2. dissections (e.g. jigsaw puzzles) that provide the framework for placing much of the built environment together, as they focus on how smaller shapes are joined to make a larger shape.

Flips, slides and turns, symmetry, tessellations and dissections are not the traditional geometry of the primary school, although parts are well known, but they are a collection of ideas that are full of interesting puzzles and activities. They also build visual imagery and relate to art. Thus, the teaching sequence for Euclidean transformations is as shown on the right.



This chapter is divided into five parts associated with each of the following five areas:

1. an overview of **flips, slides and turns**, looking at their meanings, properties and relationships (principles), and their use in shape puzzles and art (and looking at the materials that can teach them);
2. an overview of line and rotational **symmetry** (definitions, changes, constructions) and their use in classifications and art (and looking at the materials that can teach them);
3. an overview of **tessellations** (constructing tiling patterns, developing angle rules, using grids) and their application in solids, art and shape puzzles (and looking at ways to teach and materials that form puzzles);
4. an overview of simple and complex **dissections** (jigsaws and puzzles) and the way they are used to build visual imagery; and
5. a final section on integration and extension, covering congruence, relationships, visual imagery and art.

Note: Many more ideas can be found in the old geometry book, *Space and Shape in the Primary School*, in the Resources section of the YDC website.

4.1 Flips, slides and turns

Flip, slide and turn are the everyday names for three mathematical changes: reflection about a line, translation in a direction for a certain distance, and rotation around a centre point respectively (as shown in the diagrams below). One of the names, **flip**, can lead to misunderstanding in that the everyday meaning of the name (e.g. “flipping a car”) is not really a reflection but a turn or rotation in the vertical plane. This has to be clarified during teaching.



This section will cover the flip-slide-turn ideas that are useful in teaching early childhood, primary and junior secondary school years. These include the following ideas:

1. **Concepts and constructions.** Defining or giving meaning to each change, acting out/modelling changes.
2. **Properties.** Investigating each change and developing properties that differentiate each change.
3. **Art.** How slides, flips and turns can be used to develop designs and paintings.
4. **Relationships between flips, slides and turns.** A slide is equal to two flips where the flip lines are perpendicular to the direction of the slide and the distance between the lines is half the length of the slide. A turn is equal to two flips where the flip lines pass through the centre of the turn and the angle between the flip lines is half the angle of the turn.
5. **Congruence of shapes.** Two shapes are congruent if one has been changed to the other by a combination of flips, slides and turns only (as a consequence of this, congruent shapes are the same in size and shape and only different in orientation).

For ease of presentation, the activities will be given under headings that refer to the materials being used – body, tracing paper, Mira and computer.

4.1.1 Body/Students' actions

In this subsection, we look at how we can get students to act out with their bodies, and with toys and objects, the three changes.

Concepts–constructions

Slides are relatively straightforward. Get the students to strike a pose (should not be symmetrical), then walk in a direction (should not be directly forward – walk diagonally or backward or sideways) without turning their body or changing the pose. Students should then repeat this with toys and objects. Finally, draw an arrow and have objects moved along the direction and length of this arrow without changing orientation.

Turns are not quite as easy. At the beginning, they are straightforward – strike a non-symmetric pose and rotate body, and then rotate objects and toys similarly. The problem comes with rotating about a centre. There are two ways. The first way is to get a cart, sit a student in it and have the student strike a pose. The cart is wheeled around a centre with the student not changing their pose. As the cart turns, so will the student. The second way is to place one student as centre and connect to a second student by rope or string. The second student strikes a pose (non-symmetric) and then, with rope/string taut, walks around the centre student while maintaining orientation towards the centre. Maintaining orientation towards the centre will take some teaching – it means that the pose turns as the student turns – one way to do this is to have part of the pose (e.g. one arm) pointing at the centre (along the rope) and to ask the walking/turning student to keep turning his/her pose so that this part (e.g. the arm) stays pointed towards the centre (along the rope). The second way is more important as it requires the student to be active in determining what happens in a turn. Students should repeat these activities with objects and toys.

Flips could be the most difficult. Reflection can be practised by having two students sitting and/or standing facing each other. In turn, one of the students strikes poses (non-symmetric) or moves around and the other student acts as a mirror. Getting a student to experience the act of reflection on their own requires them to strike a non-symmetric pose, walk towards an imaginary mirror (e.g. a line on ground), and to imagine passing through the mirror to become the reflection. This would require the student turning around and facing the other way, and changing the pose so left becomes right and right becomes left. However, it is the important way for learning because, like the second *turn* way, it requires the student to be active in achieving the change. Again, students should repeat these activities with objects and toys.

Properties

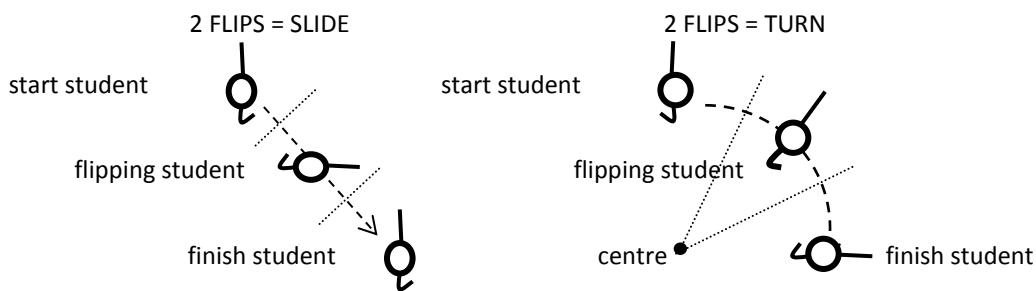
Determining the properties of the three changes can be helped by body movements or moving objects and toys. For instance, as they are acting out the changes students could be directed towards seeing that:

- for a slide, all parts of the body/object/toy move in parallel the same distance and direction;
- for a turn, all parts of the body/object/toy turn but the parts close to the centre stay close to the centre and vice versa; and
- for a **flip**, left and right interchange, with all parts near the flip line staying near the flip line and vice versa.

Relationships

It is possible to act out the relationships that *two flips make a slide* or *two flips make a turn*. To do this for the slide, get two students to strike the same pose (one in front of the other) and the front one to complete a slide (resulting in the slide being depicted by a start student and a finish student). Then get a third student to copy the pose, stand in front of the start and do two flips in the direction of the slide. The second flip will end with this student in exactly the same pose as the finish student for the slide so, if flip lines are the right distance apart (half the slide distance), it shows that two flips is the same as one slide.

This process can be repeated for the turn but here the flip lines will go through the centre and be half the angle of the turn. Both processes can be repeated with objects/toys.



4.1.2 Tracing paper

The best way to begin a more formal look at flips, slides and turns is through using tracing paper. This enables students to explore flips in relation to a flip line, slides in relation to a length and direction arrow, and turns in relation to turn arrows.

Concepts–constructions

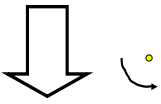
Small-sized copies of possible worksheets and instructions are below.

FLIP

To construct a flip: (a) copy shape and line onto tracing paper, (b) fold tracing paper along line (fold away), (c) copy shape through onto other side of tracing paper – draw with a dotted line, (d) open tracing paper and place on top of drawing and dotted line, (e) press hard and copy back onto original page (shading back of drawing of shape with pencil will help), and (f) draw in flipped shape as a dotted line.

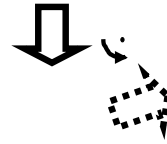
SLIDE

To construct a slide: (a) copy shape onto tracing paper and place a dot at start of arrow, (b) holding original page fixed, move tracing paper so that dot slides along arrow to end without turning the tracing paper, (c) press hard and copy shape back onto original page, and (d) draw in slid shape as a dotted line.



TURN

To construct a turn: (a) copy shape and centre dot onto tracing paper and draw a dotted line from the centre to the start of arrow, (b) holding original page fixed and placing pencil point on centre, turn the tracing paper so that line slides along arrow to end without moving the pencil point, (c) press hard and copy shape back onto original page, and (d) draw in turned shape as a dotted line.



Properties

The use of tracing paper allows more formal ways of identifying properties. For example, (a) corresponding points in start and finish shapes joined by straight lines show that change is parallel and differing lengths (i.e. near to flip line stays near to flip line after change and vice versa) for flips (as well as left to right and vice versa); (b) the same method shows that change is parallel and same length for slides; and (c) corresponding points in start and finish shapes joined by curves show that change is circular and the same angle for turns.

Art

A design can be copied onto tracing paper and then the tracings used to reflect the design and turn the design to make a larger one, as below, which is better than the original. The reflections/rotations on the left are symmetry designs and on the right are frieze patterns. (Note: slides can also be used.)

design	reflect
reflect	reflect

design	rotate
rotate	rotate

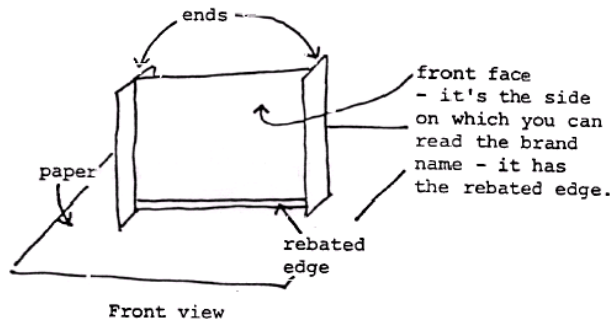
design	reflect	reflect	reflect	reflect	reflect
design	rotate	rotate	rotate	rotate	rotate

Relationships

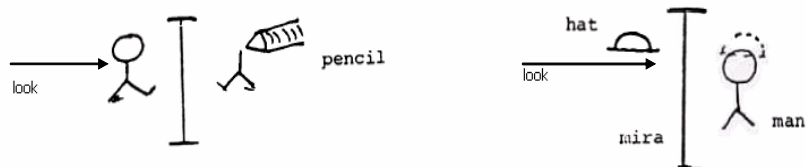
The relationships diagrams from 4.1.1 can be used as a basis for a formal investigation of relationships in that tracing paper can be used to do the two flips that equal a slide and the two flips that equal a turn.

4.1.3 Mira

A Mira is a red plastic mirror as shown on right. Its strength is that the red plastic allows you to look through the Mira to what is behind and, at the same time, to see the reflection of what is in front of the Mira. This means that a Mira can be used to superimpose and to draw reflections.

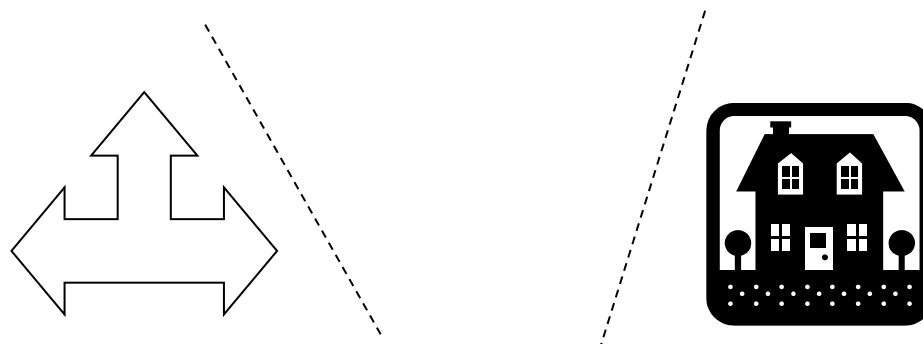


The diagram below on right shows how to use the Mira to draw reflections and superimpose a shape. Its purpose is to develop **spatial visualisation** (the ability to mentally manipulate, twist, rotate, reflect, slide or invert shapes), an understanding of the role of angles in reflection, and an understanding of the relationship between reflections and symmetry.



Concepts–constructions

The Mira is excellent for reflections. The Mira is placed on the flip line (with bevelled edge down and towards the student) and looking from the shape through the Mira, the student can draw the flipped shape in dotted line form as a copy of the original shape on the other side of the Mira. Small copies of two worksheets are shown below.



Worksheets for use with Mira mirrors

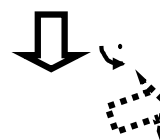
Properties, art and relationships

Because the Mira can only do flips or reflections, it is restricted in what it can do. However, it does flips really well. Everything that tracing paper can do with respect to flips, the Mira can do (and better). Therefore, it can be used to: (a) find properties of flips in the same way as was described for tracing paper (4.1.2); and (b) construct reflection-only artistic designs (symmetry designs and frieze patterns). With the drawings of the slide and turn from the tracing paper, the Mira is very effective in showing that:

- **two flips equal one slide** – with flip lines perpendicular to the direction, and half the distance of the length of the slide apart; and
- **two flips equal one turn** – with the two flip lines meeting at and running through the centre and the angle where the flip lines meet being half the angle for the turn.

Finding centre

If students are given a turn as shown on the right, the relationship above can be used to find the angle and centre of the turn. To do this, find two pairs of corresponding points on the figure and the rotation. Use a Mira to, in turn, mark in the Mira line that takes a point to its corresponding point. Where the two Mira lines meet will be the centre and the angle of turn will be double the angle between the Mira lines.



4.1.4 Computers

Concepts–constructions

Computers are excellent at doing slides. A shape or picture moved by click and drag will always slide and never change orientation. They will also easily do turns and flips. Thus, students should experience these early.

However, to flip around a line or to turn around a centre needs more work:

- (a) to flip around a line, group the line and shape, copy it, flip the copy, drag the flipped shape and line until the original line and flipped line are directly over each other; and
- (b) to turn around a centre, group the shape and centre, copy it, turn the copy the required amount, and drag the turned shape until the two centres overlap.

Properties, art and relationships

The computer is effective here because its accurate depiction of flips, slides and turns enables properties and relationships to be easily and accurately explored. And, in art, it is very effective because complicated designs can be grouped and then turned and flipped with ease.

4.1.5 Congruence

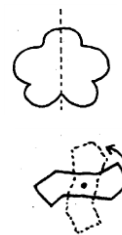
The best way to teach and think of congruent shapes is that congruent shapes are when one shape changes to the other by flips, slides and turns **only**. Properties can then be discovered. Activities are as follows.

1. **Constructing congruence.** Draw a shape and copy it from cardboard; starting from the original shape, flip, slide and turn the copy and draw what you produce; repeat this many times trying to get something that looks different – everything that you draw will be congruent to the original shape.
2. **Determining congruence.** You have two shapes – copy one of them; flip, slide and turn the copy to see if you can make the other shape; if yes, they are congruent and, if not, they are not congruent.
3. **Properties of congruence.** Construct two congruent shapes, measure sides and angles; discover properties (corresponding sides are the same length and corresponding angles are the same angle).

4.2 Symmetry

This section covers line symmetry (where a shape can be folded in half and have both sides match) and rotational symmetry (where a copy of a shape can be turned on top of itself and match in a part turn). It consists of the following ideas.

1. **Concepts.** Defining line and rotational symmetries, determining if shapes have these symmetries and the number of them they have. (*Note:* If a shape only matches at 360° it has no rotational symmetry but, if it matches at a part turn, the 360° is counted – thus rotational symmetries go from 0 to 2; it is not possible to have one rotational symmetry).
2. **Constructions–changes.** The first part of this is the reverse of the above in that instead of determining the number of line and rotational symmetries, students are required to construct shapes with given symmetries; the second part is investigating what kind of simple changes on a shape change the number of symmetries.
3. **Art.** This is using symmetries for art; however, this is the same as the art generated by flips, slides and turns so it will not be repeated here (the symmetric designs and frieze patterns).
4. **Classifications.** This is using numbers of lines and rotations of symmetry to define shapes; for example, with regard to symmetry, a square has four lines and four rotations, a rectangle has two lines (not diagonals) and two rotations, a rhombus has two lines (diagonals) and two rotations, a parallelogram has no lines and two rotations, an isosceles trapezium has one line and no rotations, an equilateral triangle has three lines and three rotations, and an isosceles triangle has one line and no rotations.
5. **Properties.** If a shape has more than two lines of symmetry, its number of line symmetries equal its number of rotations of symmetry, and the angle between consecutive lines is half the angle between consecutive turns.



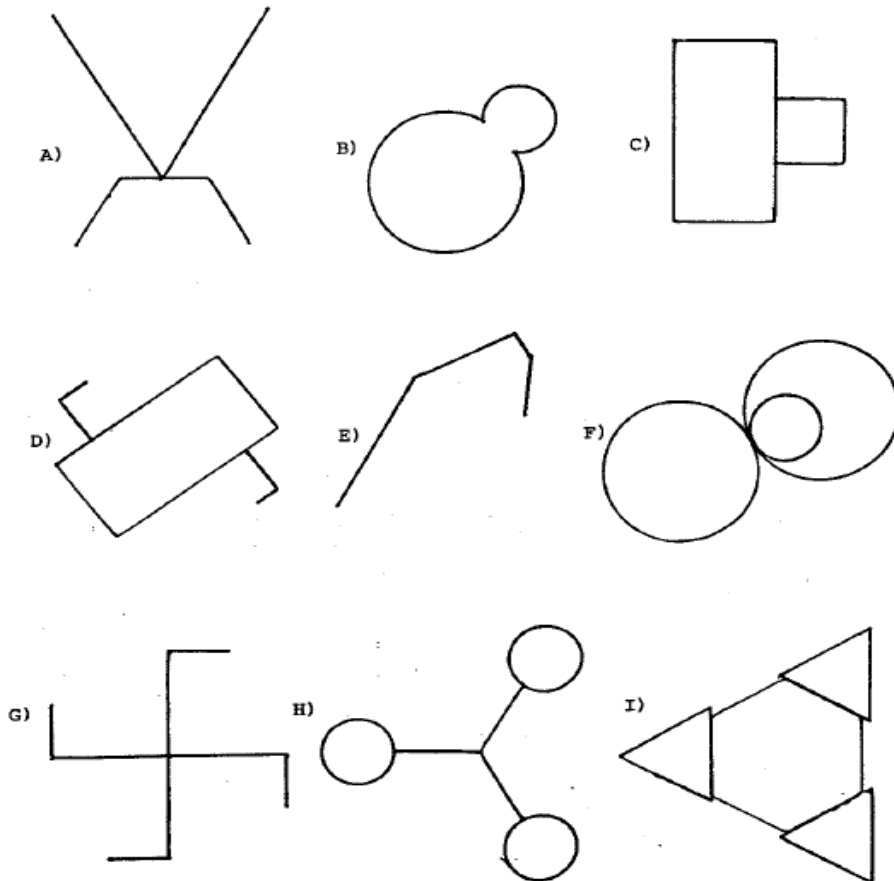
The following subsections provide some teaching ideas for symmetries under headings related to materials, notably, tracing paper and Mira. They focus on ideas (1) and (2) above. Idea (3) has been covered in flips, slides and turns. Idea (4) is suitable for Year 3 at the earliest.

4.2.1 Tracing paper

Concepts

Copying shapes onto tracing paper is a powerful way to test for line and rotational symmetry. The following are small copies of work cards that show the progression across the primary years – from identifying symmetries, to counting symmetries, to investigating symmetry change, to relating symmetries.

CARD 1.



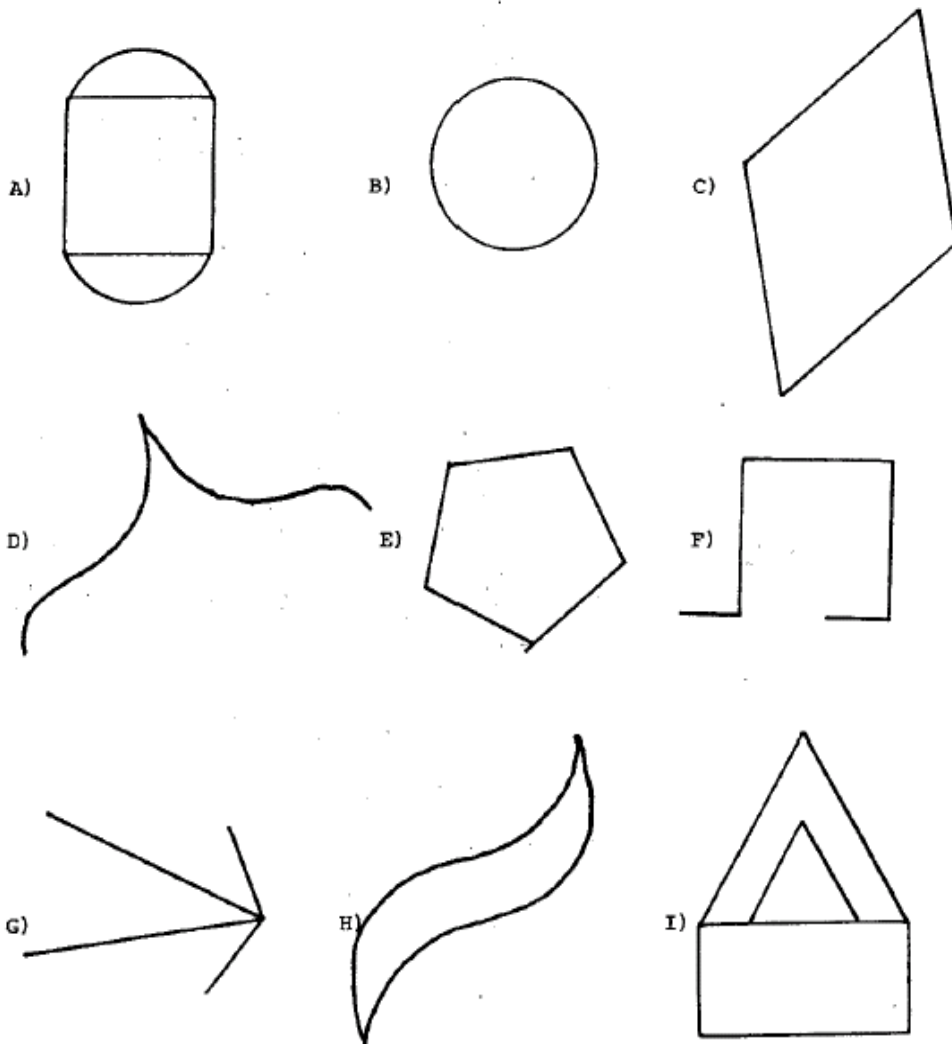
* Which designs match themselves by folding along a line?

* Which match by turning a tracing part of a full turn?

* Do any not match either way?

Note: The language is important here and needs to be developed – particularly the words “match”, “not match”, “folding along a line”, “part turn”, “full turn” and “either way”.

CARD 2.

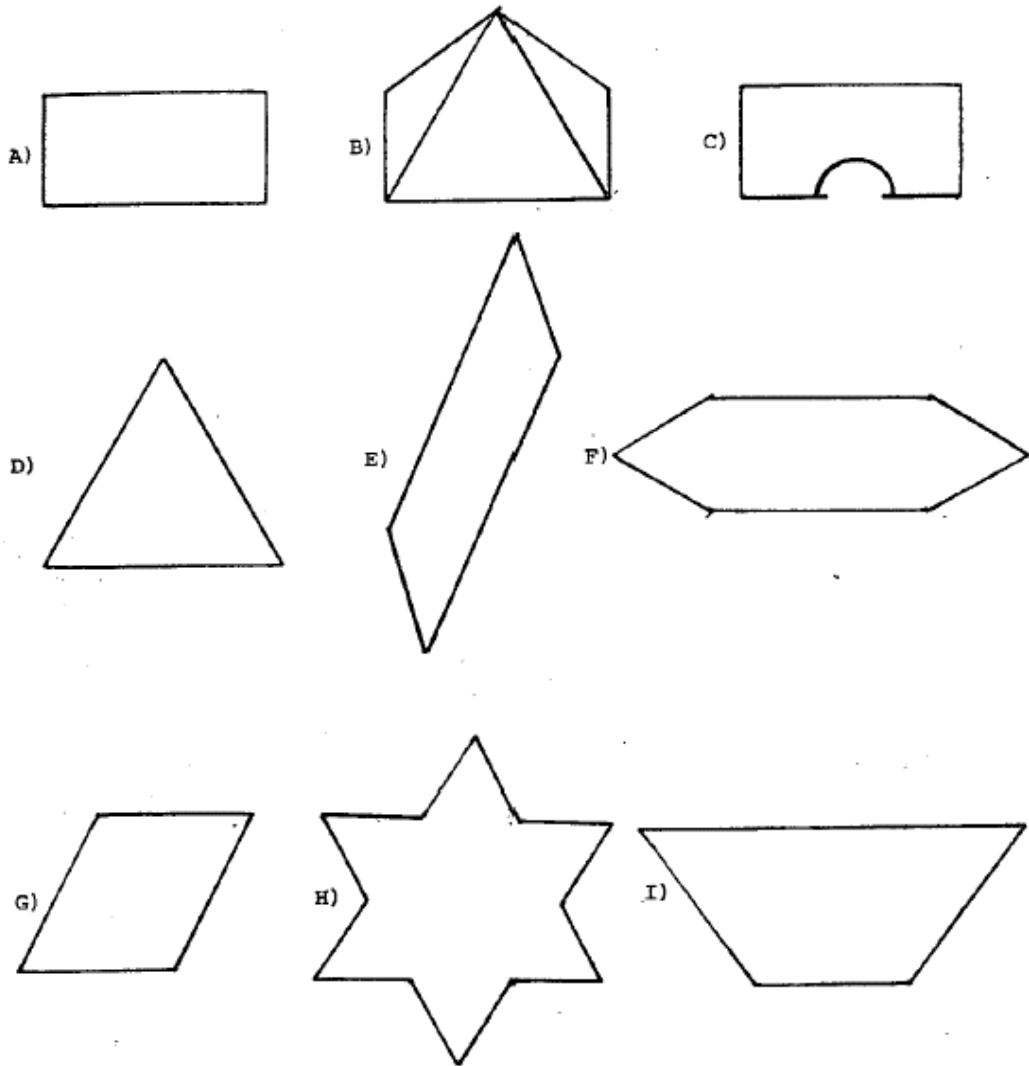


- * Which designs have line symmetry?
- * Which designs have rotational symmetry?

- * How many lines of symmetry do they have?
- * How many rotations of symmetry do they have?

Note: The number of lines of symmetry is the number of different ways a shape can be folded in half. The number of rotations is the number of different part-turn matches plus the 360° turn. The 360° turn is counted if there is a match on a part turn but not counted when there is no part-turn match – so the number of rotations goes from 0 to 2 – there is never one rotation of symmetry.

CARD 3



**Which shapes can be turned part-way around and look just as they did before?*

** Do some do both?*

** What is different?*

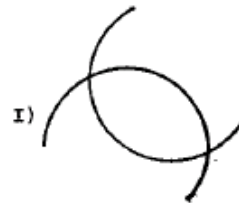
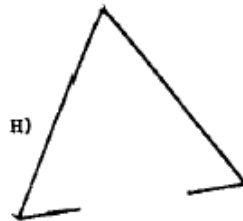
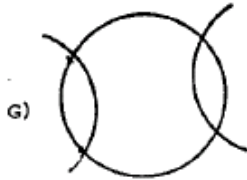
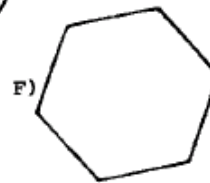
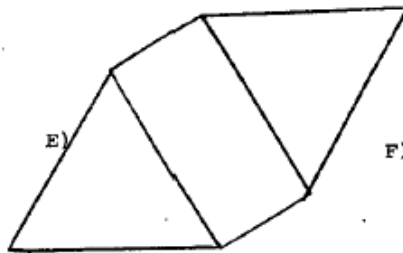
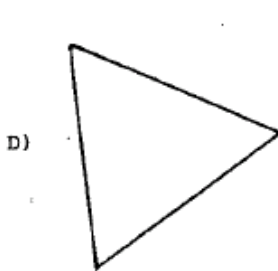
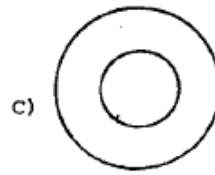
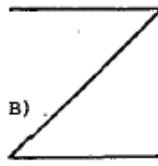
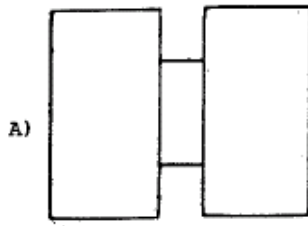
** Which can be folded so their parts match?*

** What is alike about shapes A and C?*

** What do you notice about shapes G and I?*

Note: The key questions here are “what is alike about shapes A and C?” and “what is different?” These point towards an important symmetry activity which is understanding how small changes in a shape can affect symmetries.

CARD 4



**For each design, find all lines of symmetry. Now, before you test it, do you think it has turning symmetry?*

Test it.

Were you right?

**What do you notice about the number of lines of symmetry a design has and the number of part-turns it makes?*

Note: If students have trouble seeing the pattern, get them to find numbers of line and rotational symmetries for several different shapes and put them on a table with headings as shown below. It is fairly easy to see that the following pattern emerges: (a) a shape can have no lines of symmetry but any number of rotations; (b) a shape can have one line of symmetry and no rotations; and (c) if there are two or more lines of symmetry, the number of rotations equals the number of lines (and the angle between the lines is half the angle between rotations).

Shape	Number of lines of symmetry	Number of rotations of symmetry

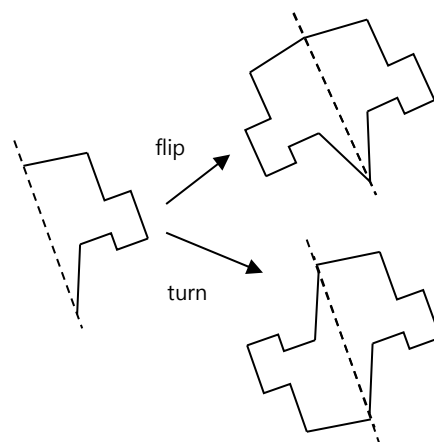
Constructions—changes

We will take each of these in turn.

Constructions

Reversing the shape → symmetries teaching process, and including symmetries → shape activities, is important and enhances learning. Some examples of how to do this are as follows.

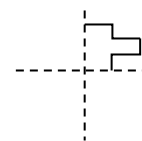
1. **One line, two rotations.** Draw a dotted line and a design on one side of the line as shown on right. For line symmetry, copy the design and the dotted line onto tracing paper, fold paper and retrace, and the result is a shape with one line of symmetry on the tracing paper. For rotational symmetry, copy the design and the line onto tracing paper as before but, this time, turn the tracing paper on top of the design and dotted line so that the dotted lines exactly match/cover each other, retrace the shape, and the result is a shape with two rotations of symmetry.



2. **Three rotations.** Construct three dotted lines at 120° to each other and draw a design in one of the three sections as on the right. Copy dotted lines and design on tracing paper, rotate 120° until dotted lines are aligned exactly and recopy design, and rotate a further 120° until dotted lines are aligned exactly again and copy design a third time. The result is a shape with three rotations of symmetry.



3. **Two lines, three rotations.** Repeat step 1 above but for the situation where there are two dotted lines crossing each other. For two lines of symmetry, copy design and dotted lines and then reflect four times. For four rotations, copy design and dotted lines and rotate four times. (*Note:* If for rotation the lines of the designs do not meet, join them along the dotted lines.)

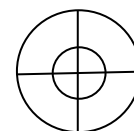


4. **More lines, rotations.** We can continue in this way noting that: (a) dotted lines which go from a centre outward in 2, 4, 6, 8, and so on directions give shapes with half the number of lines of symmetry and equal the number of rotations of symmetry; and (b) dotted lines that go from a centre outward in 3, 5, 7, and so on directions give shapes with equal the number of rotations of symmetry.

Changes

Learning how to change symmetries of shapes/designs with simple additions or deletions enhances understanding of symmetries. Some examples are as follows.

1. Find the symmetries of the shape on right (3 lines, 3 rotations) and change to a design with one line and no rotations by adding one line (how would you do this with a deletion of a line segment or part of a line segment?).
2. Find the symmetries of the design on right (4 lines, 4 rotations) and change to a design with one line and no rotations by deleting one line segment, then change to a design with two lines and two rotations with the deletion of a further line segment (how would you do this by adding lines?).



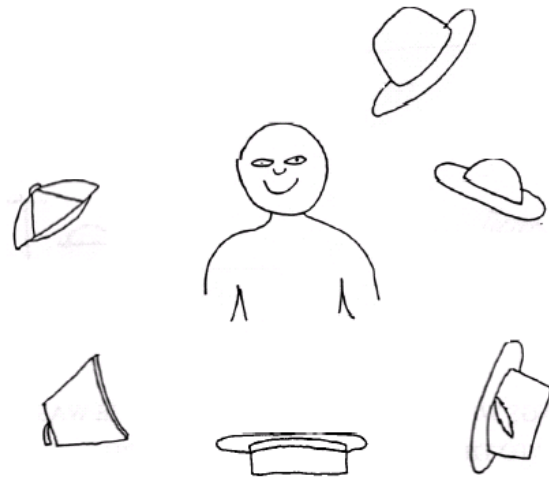
4.2.2 Mira

The Mira mirror will only do line symmetry but it does this well as in the following sequence.

Learning how to use a Mira

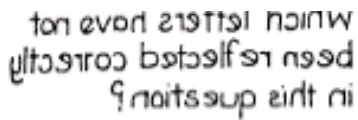
Which hat fits best on the man?

For each hat in turn, student places Mira between hat and man (with Mira perpendicular to an imaginary line between hat and man and bevelled edge down and facing student) and adjusts Mira until hat is on the man (or man is under the hat – depends on what way student is looking into the Mira).



After trying each hat, student judges which is best.

Use Mira to read the reflected writing below.

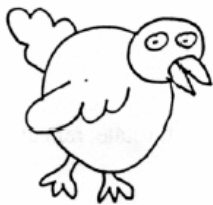


Use Mira to determine what is wrong with the reflection of the clock face on right.



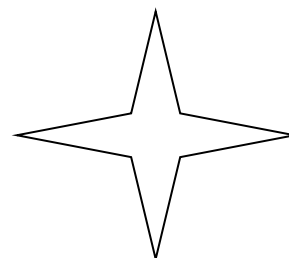
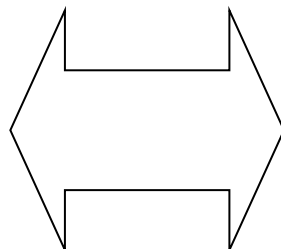
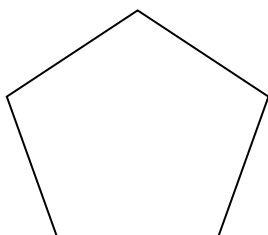
Practising flips

Student places Mira on the dotted lines and reflects the shapes about the dotted lines and draws what he/she sees when looking through the Mira away from the pictures of the bird and dog.



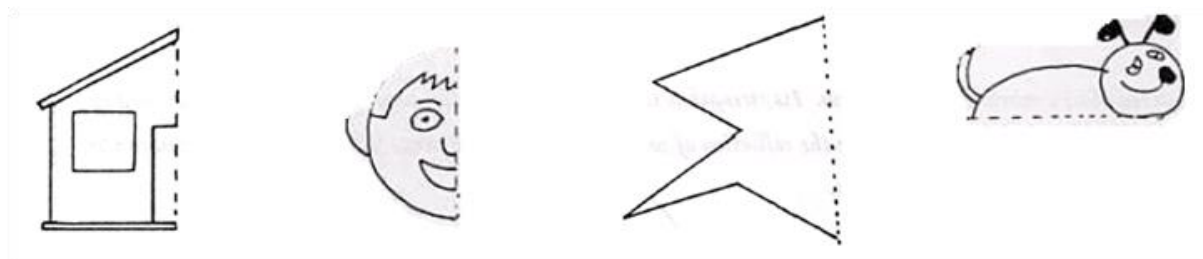
Drawing in lines of symmetry

Student places Mira on shapes/designs below and finds all lines of symmetry. If bevelled edge is facing student and is down, line can be drawn along this edge and be below the reflection plane of the Mira.



Completing line symmetric figures

Student places Mira along the lines, and then draws the other half of the figures. If the result makes sense, the resulting figure is line symmetric. Does the result always make sense? Why or why not?

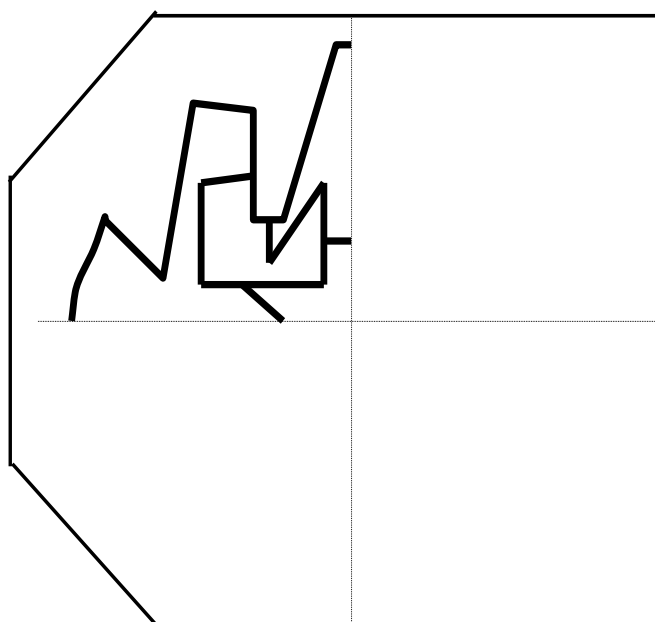


Constructions—changes

The Mira is very effective in constructing line symmetric figures and in checking if changes proposed to figures will achieve the desired line symmetric changes. It cannot help with regard to rotational symmetry.

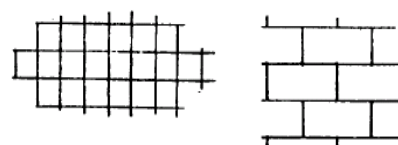
An example of construction is the “shield” on right – students are given a shield shape and two dotted lines crossing. They put a design in one of the quadrants (as on right) and use the Mira to flip it three times into the other quadrants. The resulting design has two lines of symmetry and if the artistry is good the image will look effective as a shield.

(Note: Can use tracing paper to redo shield as rotations which may look better. Try both from the same starting picture and see which looks best.)

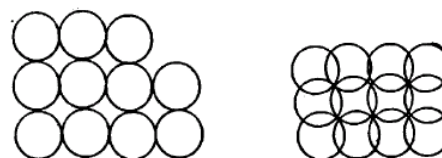


4.3 Tessellations

In a society that puts shapes together to build and cover and that packs shapes together to carry them around, shapes that fit and pack well are important. Shapes that fit together without gaps or overlaps are called **tessellations**. Squares and rectangles tessellate, as shown in the figures on right.



Circles do not tessellate, as shown in the figures on right. When attempting to pack circles, there will always be either a gap or an overlap.



However, circles can be the starting point for shapes that do tessellate. For example, the shapes tessellate into the pattern at right. Here circles have had curved pieces taken out of opposite sides so that they fit together. As well, special shapes can be made that fit together with the circles so that both shapes together tessellate.



Tessellations are useful in developing **spatial visualisation**, the ability to mentally manipulate – flip, slide and turn – shapes (important for NAPLAN), and an informal knowledge of the interior angles of polygons. In this subsection, tessellations cover the following.

1. **Concept–constructions.** What is a tessellation and how to construct them.
2. **Art.** How tessellations can be used for art.
3. **Grids and puzzles.** How tessellating shapes can form grids and puzzles (e.g. pentominoes).
4. **Properties.** These range from general properties such as *a tessellation pattern has to show evidence to convince the observer that it is a pattern that is ongoing in all directions* to more formal properties such as *single shapes and groups of shapes only tessellate if their interior angles are factors of a complete turn (360°) either on their own, in combination or in addition.*
5. **Solid tessellations.** 3D shapes that tessellate.

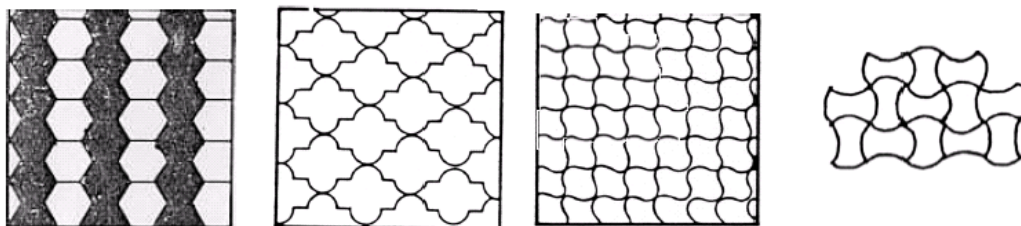
Some tessellation activities are given in the following subsections.

4.3.1 Concepts–constructions

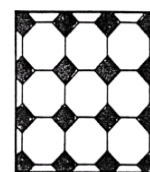
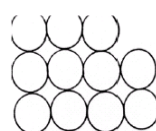
The idea is to build a concept of tessellations through constructing them. The sequence is as follows:

- (a) one-shape tessellations – which shapes tessellate and which do not and why;
- (b) two or more shape tessellations – which combinations tessellate and why and what non-tessellating shapes (as single shapes) can now tessellate with another shape; and
- (c) building shapes from combining tessellating shapes and seeing which of these tessellate.

These shapes tessellate by themselves.



Some shapes do not tessellate by themselves, but some tessellate with another shape. Even for the circles, a shape can be found to tessellate with it. And circular pieces can be taken from and added to other shapes so they tessellate. For example:



- (a) construct a tessellation for the left shape of the two on the right (based on a rectangle and two semicircles);
- (b) construct a second tessellation for the right shape of the two; and
- (c) make another tessellating shape out of semicircles and rectangles.



Some other examples:

- (a) investigate and find a shape that will tessellate with a circle and draw the tessellation;
- (b) draw a double tessellation when one shape is a semicircle; and
- (c) can you think of any more double tessellations that include a circle or part?

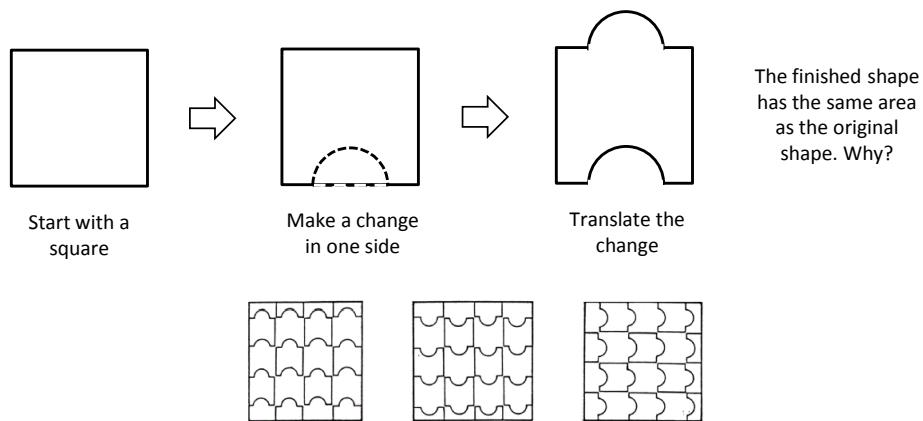
4.3.2 Tessellations, art and fabric design

Tessellating shapes are the basis of two art forms – Escher-style art and fabric design. We will look at both.

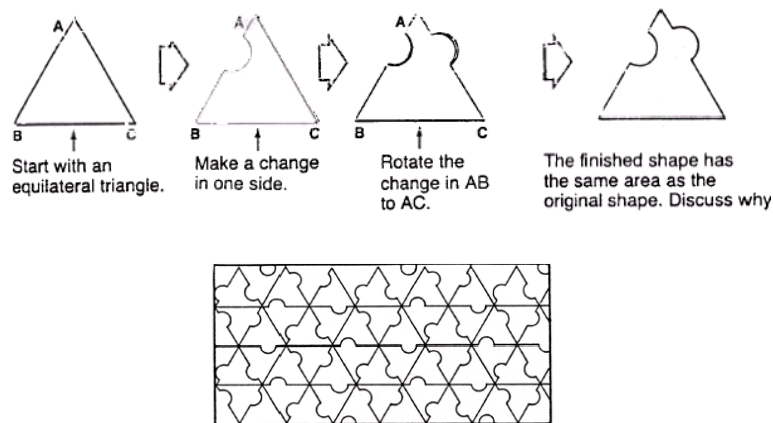
Escher-style art

This is based on doing something to one side of a tessellating shape and undoing it to the other. This can be done by cutting a tessellating shape out of cardboard and then cutting a piece from one side and then adding it to the other with sticky tape. After one or more attempts at this, the resulting changed shape is copied onto two colours of paper and turned into a pattern of shapes as in the examples below. Note the use of flipping, sliding and turning. Note also that these lead to the actual art of Escher – look this up on the Internet and learn how it is done.

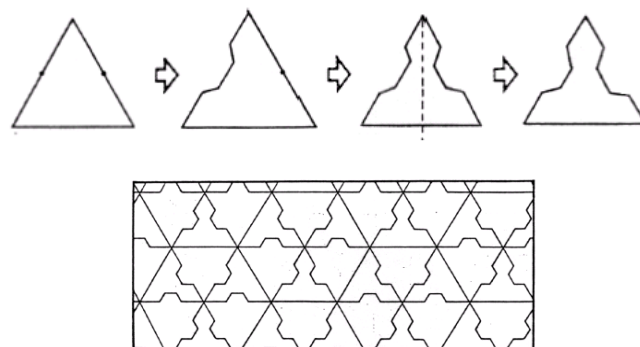
1. Changing shapes by **sliding** (i.e. translating):



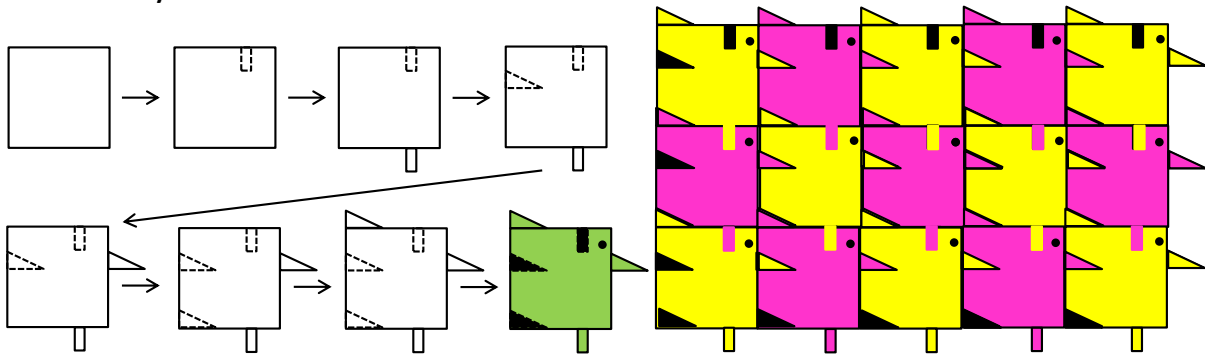
2. Changing shapes by **rotation** – note that if you use an isosceles triangle, then you will have to rotate and flip every alternate shape in order to make the shape tessellate:



3. Changing shapes by **reflection**:

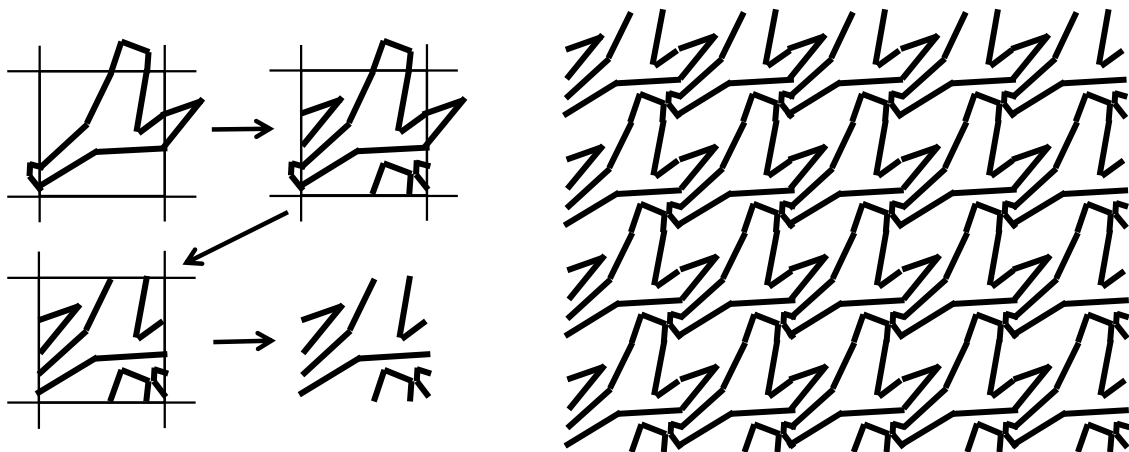


4. Escher-style art:



Fabric design

In this method, a tessellating shape is chosen and a design drawn in it. If the design goes outside the shape, this outside part is redrawn inside the shape on the opposite side. Then the shape with design is repeated to make the fabric (the repeat can be flipped, slid or turned) as follows:



4.3.3 Grids, puzzles and properties

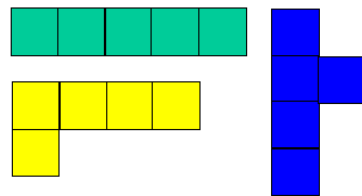
Grids and puzzles

When a shape tessellates, it can form graph paper. When it is in the form of a grid or graph paper, then shapes can be made from it – containing two, three, four, five, and so on shapes. The different shapes form sets that are the basis of shape puzzles – the two that are known best are **hexiamonds** and **pentominoes** – there are also **hexagonal animals** and **McMahon 3-colour triangles and rectangles**. Look these up on the Internet – they are also in the old geometry book on the website. Some activities are given on right and below.

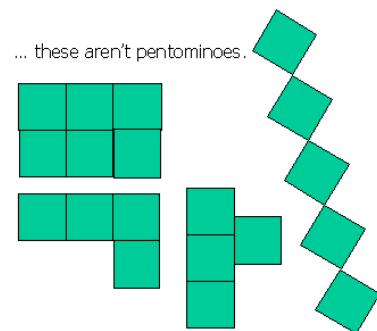
Activity 1: Pentominoes

A pentomino is a shape formed by joining 5 squares at their edges.

These are pentominoes ...



... these aren't pentominoes.



There are 12 different pentominoes; 3 have been given. Try to make the other 9.

Remember, if you can rotate or flip one of the pentominoes so that it is the same as another one, then it is NOT a different pentomino.

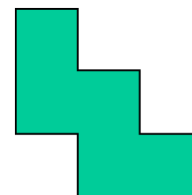
GOOD LUCK!

Activity 2: More on pentominoes

The pentominoes are 12 different shapes that can be made with 12 squares. They are shapes which look like these letters: N, Z, P, T, U or C, L, Y, X, F, W, I, V – but be careful, some of the pentominoes are not exactly like their letters – they look a little strange.

These 12 shapes can form other shapes as puzzles – some examples are below:

- (a) C, N and P; C, Y and P; and C, P and V can combine to form 3×5 rectangles;
- (b) Y, L, W and P; P, N, Y and C; and V, T, W and P can form 4×5 rectangles; and
- (c) all 12 pentominoes can form 4×15 , 5×12 , and 6×10 rectangles, and a large W as on the right.



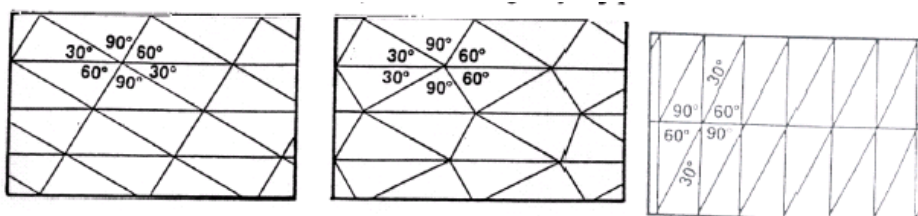
Activity 3: Grids

Make up or obtain some grid paper based on equilateral triangles (called isometric grid paper). Draw “islands” composed of three, four and five triangles (islands is a general name for any shaped grid – as these are equilateral triangles, their real names are triamonds, tetriamonds and pentiamonds).

- (a) How many different islands are there for three, four and five triangles? (*Note: If they equal each other through flips, slides and turns, they are not different.*)
- (b) Do all these islands tessellate?
- (c) Can you make puzzles using them? (*Note: Construct puzzles by taking shapes and putting together to make a shape – draw around shapes to make the puzzle for those shapes.*)

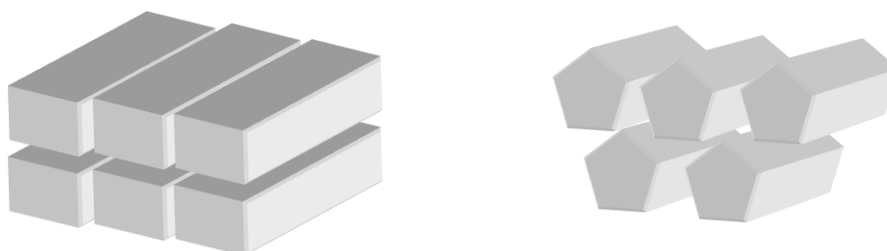
Properties of tessellating shapes

There are two properties. First, a general one, that the pattern must convince it is going in all directions forever and is not just a pretty, confined design. Second, in order to tessellate polygons, all the interior angles that touch a point must total 360° .

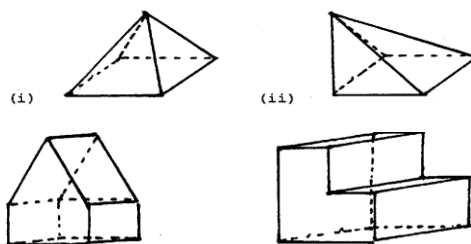


4.3.4 Solid tessellations

These are solid objects that can be piled or packed together so they can fill up a truck or room – they are 3D shapes that pack together without gaps or overlaps. Solids that tessellate tend to have tessellating faces and flat surfaces. Thus, square, rectangular, triangular, and hexagonal prisms tessellate while other polygon-based pyramids, such as pentagonal and octagonal prisms, do not.



This holds as long as the vertex is above the base (e.g. on right, (i) tessellates and (ii) does not).



It is also possible to mix a second (or third) solid shape to produce a tessellation. For example, triangular and square prisms will tessellate. Solid shapes which are the combination of tessellating solids will also tessellate (e.g. the solids on right tessellate).

To teach this idea, collect the following: prisms (e.g. erasers, various shapes of dowelling cut in sections, various types of small packets); pyramids (e.g. tetrapaks); cylinders (e.g. circular dowelling cut into sections, cans and bottle tops); spheres (balls); and other solids (e.g. L-shaped dowelling sections, small wood and plastic houses). Then do the following:

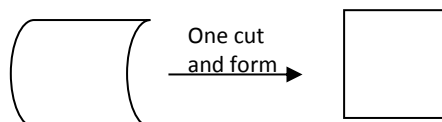
- stack examples of the following solids to determine whether they tessellate: cube, rectangular prism, triangular prism, square pyramid, tetrahedron, sphere, cone and cylinder; and
- visit a supermarket and consider articles you find on the shelves and how they relate to solid tessellations.

Packaging in our society is a compromise between strength, cost, appearance and packing (tessellation). The best shape to pack (or tessellate) is a rectangular prism (the common box). It is also cheap to make because it can be folded from a net. However, it is weak as corners and edges are points of weakness. The strongest surface is a curve like a sphere or cylinder with semi-spherical ends but this does not tessellate well. Also the sphere is the package where the least material (surface area) encloses the most contents inside (volume). Square prisms are the best in this regard when we consider prisms.

4.4 Dissections

Dissections are an extension of jigsaw puzzles. There are two types:

- Simple.** Pieces of the puzzle are given and have to be joined to form the shape – an example is the circle on right where the pieces are cut out and mixed up and the student has to reassemble it.
- Complex.** Two shapes are given and the first shape has to be cut up so that its pieces can form the second shape – an example of a complex dissection is given on right.



Dissections cover the following:

- Concepts–constructions.** Activities which introduce students to dissections (the two types) and enable students to construct dissections.
- Puzzles.** Activities that allow students to experience the many types of puzzles that have emerged from dissections (e.g. tangrams, egg puzzle and Soma cube).
- Visual imagery.** Activities to practise finding shapes that match other shapes or fit into spaces left in a design.

4.4.1 Concepts–constructions

Find examples of jigsaw-type simple dissections and complex dissections. These can also be constructed as follows.

Simple

Hand out grid paper (e.g. squares, triangles, rectangles); cut out the biggest shape that you can by following the grid lines (e.g. a rectangle, a diamond, a hexagon); cut this into five or six pieces along the grid lines; use the

original grid paper to make a template; provide the pieces to a student to re-form into the shape (note that students can make up these puzzles for each other).

Complex

Take a simple shape (e.g. square, circle, triangle, t-shape) and cut it into two or three (or more) pieces using straight cuts; re-form these pieces into a different shape (stick together with tape) and trace around this; the final shape becomes the starting shape and the first shape the final shape. A copy of both is given to a student with the instruction to cut the first shape into two or three or more parts by straight cuts and form the second shape.

4.4.2 Puzzles

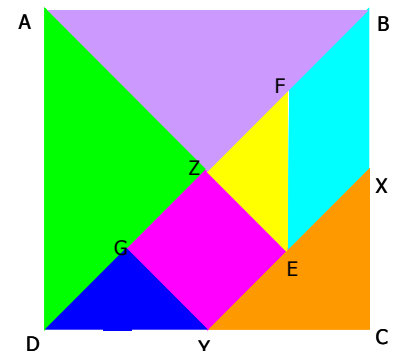
There are many puzzles that come from dissections.

Tangrams

The tangram is one of the most common 2D dissection puzzles – some examples are given below.

Tangram is the name given to a puzzle that consists of a square that is cut into five different-sized triangles, a small square and a rhombus. (The diagram below shows how the pieces are produced.) These pieces can be arranged to form over 300 different figures. Tangrams are Chinese in origin and appear to be about 4000 years old. The name was probably derived from the Chinese word *t'an* which means *extend* or the Cantonese word *t'ang* which means *Chinese* and then the European term *gram* was added. The word *tangram* seems to have been coined between 1847 and 1864.

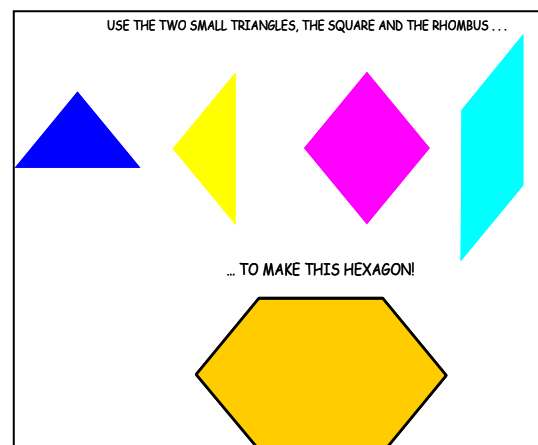
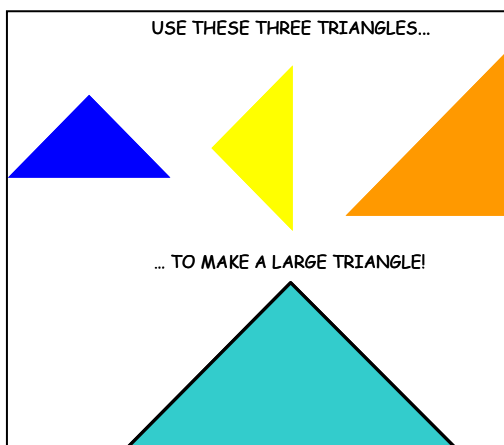
To make the tangram pieces, a square (8 cm × 8 cm) is dissected as on right: draw the diagonal BD; find the midpoints of BC and DC; label these X and Y respectively and draw the line XY; find the midpoint of the diagonal BD, label it Z and draw the line AZ and extend it to meet XY at E; from E, draw a line parallel to BC, meeting BD at F; and from Y, draw a line perpendicular to BD, meeting it at G.

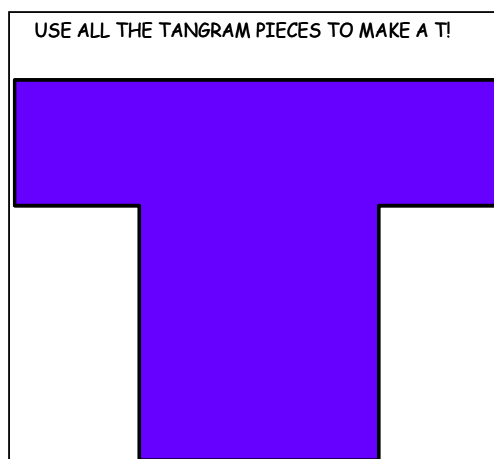
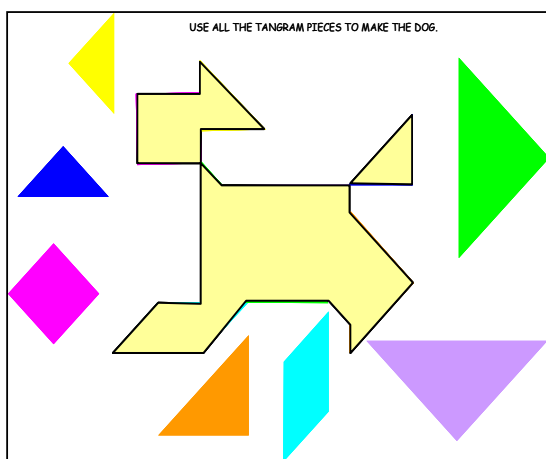


Tangram activities, sequences for puzzles, and other puzzle types are given below.

Activities

Give pieces and a shape to construct out of the pieces. Four examples are below.





Other puzzles

Two other common 2D puzzle types are:

- egg puzzles – an egg shape cut into pieces that can be re-formed for many shapes; and
- McMahon 4-colour triangles and 3-colour squares – triangles and squares divided into different-coloured sections that form 24 pieces can be re-formed to make puzzles where the final puzzle has the same colour on edge all around and two shapes are only fitted together if they join at the same colour.

3D puzzles are very important as these act as preparation for our 3D world – architecture, brain surgery, etc. Two examples are:

- Soma cubes – 7 pieces that make a $3 \times 3 \times 3$ cube and many other shapes; and
- wooden cube pentominoes – 12 3D pentomino pieces that can be used for 3D as well as 2D puzzles.

Sequencing of puzzles

Puzzles should initially be presented in a sequence from easy to hard, taking into account the following:

- number of pieces: small number \rightarrow large number;
- complexity of shape and clues: clues \rightarrow no clues;
- type of movement: only turning \rightarrow turning and flipping;
- shape outline: full outline \rightarrow small copy of outline \rightarrow no outline;
- presentation: subtasks \rightarrow full task only; and
- organisation of tasks: similar \rightarrow dissimilar (all mixed up).

4.4.3 Visual imagery

These activities are important in the NAPLAN testing. They ask students, for example, to determine which shape is the same as another or to identify the shape that will fit into a space. Some examples are as follows.

- Ask students to work out the 12 pentominoes and the 12 hexiamonds on square and triangular (isometric) graph paper.
- Give students a shape (e.g. a key) and then ask which of four shapes is the same as it except for orientation and position. (*Note:* Can allow younger or less experienced students to cut out a copy of the shape and to try to fit it with the other shapes by actually flipping, sliding and turning the copy).
- Ask students to fill in a gap in a puzzle by choosing the appropriate shape. End games of puzzles are good for many things, e.g. give the half-finished puzzle and ask for the piece that will complete the puzzle.

4.5 Integration and extension to properties, congruence and art

In the later years, we start to look across ideas for larger relationships. Some of these and extension activities are given in this section. They include congruence and the relationships between flips, slides and turns, and line and rotational symmetry, as well as ideas on using this geometry in art. Many ideas that could be in this section have already been covered in sections 4.1 to 4.4.

4.5.1 Congruence

As mentioned in subsection 4.1.5, the best way to teach and think of congruent shapes is that congruent shapes are when one shape changes to the other by flips, slides and turns **only**. Properties can then be discovered. Activities are as follows.

1. **Constructing congruence.** Draw a shape and copy it from cardboard; starting from the original shape, flip, slide and turn the copy and draw what you produce; repeat this many times, trying to get something that looks different – everything that you draw will be congruent to the original shape.
2. **Determining congruence.** You have two shapes – copy one of them; flip, slide and turn the copy to see if you can make the other shape; if yes, they are congruent and, if not, they are not congruent.
3. **Finding properties of congruence.** Construct two congruent shapes by flips, slides and turns – measure sides and angles to discover the properties.

The properties are:

- corresponding sides are the same length; and
- corresponding angles are the same angle.

Congruent shapes can look very different because they have been flipped and turned. Taking an equilateral grid (isometric) and looking at all the different shapes that are made from six triangles (called hexiamonds) is an effective way to practise determining which shapes are congruent.

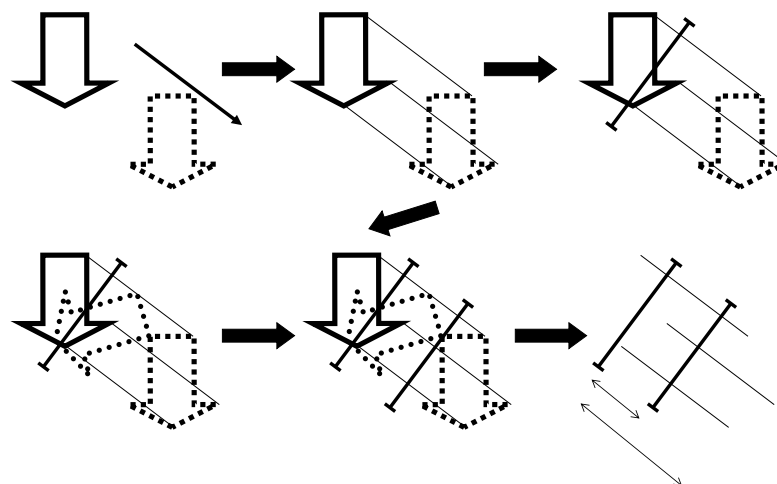
Congruence has, in the past, formed a strong component of NAPLAN testing. For instance, determining what shape will fit into what place is a congruence activity.

4.5.2 Relating flips and line symmetry to slides-turns and rotation symmetries

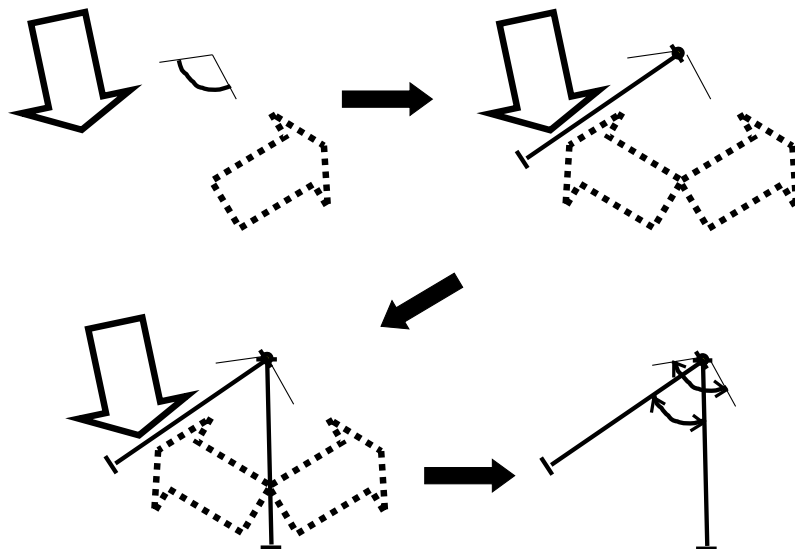
Using our bodies, we can show that a slide or turn can be completed by two flips; this can be done more formally using tracing paper to do the slides and turns and a Mira to do the flips. The diagrams below illustrate this.

Relationships

Slide and two flips:



Turn and two flips:



The relationships are as follows:

- (a) **two flips equal one slide** – with flip lines perpendicular to the direction, and half the distance of the length of the slide apart; and
- (b) **two flips equal one turn** – with the two flip lines meeting at and running through the centre and the angle where the flip lines meet being half the angle for the turn.

If the number of line and rotational symmetries are placed on a table as below, then students can discover a relationship between line and rotational symmetry:

Shape	Number of lines of symmetry	Number of rotations of symmetry

It is fairly easy to see that the following pattern emerges:

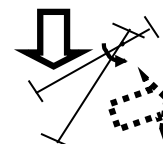
- (a) a shape can have no lines of symmetry but any number of rotations;
- (b) a shape can have one line of symmetry and no rotations; and
- (c) if there are two or more lines, the number of rotations equals the number of lines (and the angle between the lines is half the angle between rotations).

As is evident, the relationships regarding flips-slides-turns and line and rotational symmetry are similar, particularly in terms of half the angle.

Extension

Finding centre

If students are given a turn as on the right, the relationships above can be used to find the centre and the angle of the turn.



1. To find the centre, determine two pairs of corresponding points on the original and final shapes. Use a Mira to, in turn, mark in the Mira line that takes each set of corresponding points to each other. Then where the two Mira lines meet will be the centre.
2. To find the angle of turn, determine one set of corresponding points between original and final shape. Use the Mira to flip the original shape so that these corresponding points align (draw in the Mira line). Then find one more set of corresponding points on the flipped shape and the final shape and repeat the above (draw

in the Mira line) – this will take the flipped shape to the final shape. The two Mira lines cross at the centre and the angle between the lines is half the angle of turn.

Relating flips to congruence

Take any two congruent shapes where the original shape has been flipped, slid and turned to make the final shape. Using the Mira, it can be shown that three flips will get from the original to the final shape. To do this start with two corresponding points on the original and final shape and flip so that these two points are brought together. Then find a second set of corresponding points between flip 1 and final shape and flip again so these points are aligned. Then a third set of corresponding points between flip 2 and final will take the flip 2 shape to the final shape.

4.5.3 Artistic patterns

A lot of understanding of Euclidean geometry can come from art. Some examples are as follows.

1. Escher-style drawings that tessellate and fabric design from a tessellating design are examples that challenge students. Work on these has been summarised in section 4.3 (and more detail is given in the old geometry book, *Space and Shape in the Primary School*, in the Resources section of the YDC website).
2. Constructing shape puzzles (the reverse of solving them) is also challenging, particularly for the type 2 dissections shape → pieces → another shape.
3. Another example of artistic design through mathematics is **Modulo Art**. In this, students:
 - (a) develop a pattern of numbers in a square grid;
 - (b) provide each number with a design that shares a common base with other designs for other numbers;
 - (c) replace the numbers with designs; and
 - (d) slide, flip and turn the design pattern to make a larger design.

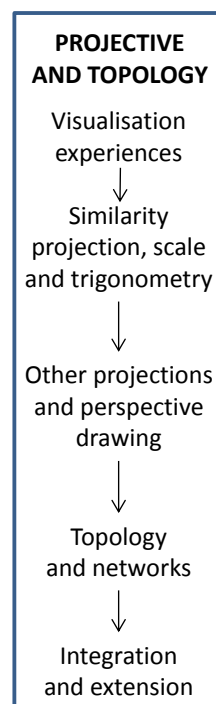
Note: A workbook of Modulo Art activities for a variety of designs is on the YDC website under Resources – Student Learning – Geometry.

5 Projective and Topology

Projective and topology are the transformational geometries that cover similarity, trigonometry, perspective and networks. Along with the previous chapter on Euclidean transformations, they cover the important change perspectives of geometry. Euclidean is given more space than projective and topology in this book because it is given more space in the Australian Mathematics Curriculum and because Euclidean transformations deal with the built environment which is important for so many vocations, particularly engineering, architecture, and the mining, engineering and construction trades, and thus needs emphasis.

This chapter will deal with projective and topology topics as shown on the right. However, at times it is hard to differentiate some topics from Euclidean. For example, the first subsection, Visualisation, deals with perception and mental rotation, ideas that were also canvassed in Chapter 4. In more detail, this chapter covers the following:

1. **Visualisation.** This is the ability to perceive the world, that is, to understand that there is perspective and direction in terms of how something is perceived and that there is a need for mental rotation – the ability to rotate a figure in the mind and to rotate self around the figure.
2. **Similarity projection.** This is the change of an overhead or film projector (or a TV) or the change in divergent light where the object and its shadow are kept parallel. It leads to similarity, scale and trigonometry; it is the change where straightness and ratio of sides remain the same.
3. **Other projections.** These are the affine projection of sunlight where the parallel rays keep shadows parallel and straightness unchanged, and the perspective projection of divergent light where only straightness remains unchanged, while angle and parallel change. It leads to perspective drawings.
4. **Topology and networks.** This is where we study the change of living things. Straightness and length change, but inside remains inside and closed remains closed – twisting, bending and stretching are acceptable but not joining and punching holes. It leads to many games and puzzles including networks, which are particularly important for the modern day.
5. **Integration and extension.** This is where we acknowledge that sections of the above lead to the major concepts like similarity and trigonometry, particularly when they are integrated with other topics. In particular this section looks at interactions between flips, slides and turns and Cartesian coordinates, similarity and trigonometry (plus applications of trigonometry), and topology and Euler's theorems for networks.



5.1 Visualisation

Visualisation has two parts: (a) perception of shapes, figures and collections of objects (and the built environment) from different directions and distances; and (b) mental rotation in the mind of objects, figures and collections or mental rotation in the mind of self around these things. We will look at these two geometric ideas under three headings: informal drawings, visual imagery activities, and formal geometric drawing.

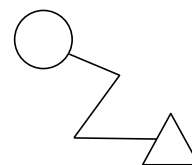
5.1.1 Informal drawings

These are activities in the early years to build understandings of **perception** and **mental rotation**.

Perception

Get students to draw what they see in many situations, and then to draw what they would imagine they would see from those situations, and finally to begin to choose from alternatives. Some ideas are provided below.

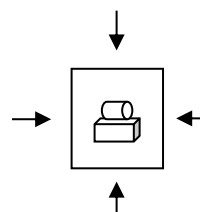
1. Put two identical objects out to be drawn so that one is close to students and one is far from students, and ask students to draw what they see. Put a large object well away and a smaller object close to the students and repeat. Reverse the position of the objects. Discuss what faraway objects look like in relation to those that are close.
2. Build a small town out of blocks. Place it on a table. Draw it as it is. Back-light it and draw it as framed in front of the light from different directions. Discuss how things change in light, at dusk and at night.
3. Photograph things from different positions and ask students where the photographs were taken or which photograph was taken from what position. Reverse this – ask the students to predict what a photograph will look like, take the photograph and compare to the predictions.
4. Introduce language of position and perception, set up divided situations where students are on opposite sides and cannot see what the other students have, have built or have drawn. Ask students on the side that has the information to use words to describe what they have to allow the other students to identify the object, make the figure, or draw the design.
5. A divided activity can be done simply by placing students back-to-back (one facing forward and one facing back) and showing a design (as on right) to the students facing forward. These students use language to describe what they see and to direct other students to draw it.



Mental rotation

Start by getting students to draw things from different directions – front, back, right, left, above, and below. Some ideas are provided below.

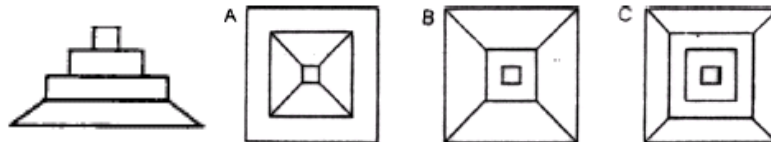
1. Put an object or toy on a table; divide students into four groups and place one group on each of the four sides of the table (sides of table labelled front, back, left, right). Hand students paper with four areas marked and labelled (front, back, left, right). Give students time to do drawing from their position, then every group moves around 90° and draws again. This is repeated until all four sides are drawn. Students discuss drawings and how close they were to the real object.
2. Repeat (1) but this time turn the object/toy 90° and do not move the students.
3. Repeat (1) and (2) but this time, add in drawings from above and below (good to have a glass table).
4. Repeat (3) but use photography – students predict first and construct front, back, left, right, top and bottom collages from pictures.
5. Build in to (3) above activities where students have to predict what something will look like from various directions, determine direction of photograph when given a photograph of an object/construction, and choose from options as to which photograph is from a given direction.
6. Reverse all the above and build solid objects/constructions from 2D pictures/drawings taken from different directions (i.e. teach views → object as well as object → views).



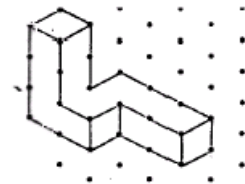
5.1.2 Visual imagery activities

The purpose of these activities is to (a) develop **spatial visualisation** (the ability to mentally manipulate, twist, rotate, reflect, slide or invert shapes); (b) develop **spatial orientation** (the ability to picture the arrangement of a set of shapes in relation to each other or to some other object); (c) develop the ability to remain unconfused by the **changing orientation** of a shape or set of shapes; and (d) develop the ability to determine **spatial relations** in which the body orientation of the observer is an essential part of the problem. Materials are dotted paper, pencil, MAB ones, and plain paper; processes are constructing and thinking flexibly and visually; and problem-solving strategies are patterning and modelling.

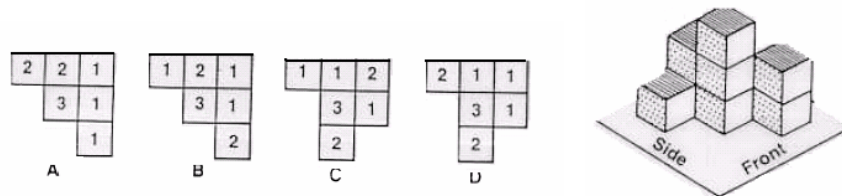
1. Which is the correct bird's-eye view of the building on the left?



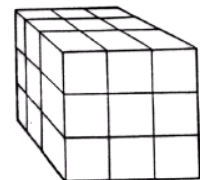
2. Copy the given shape on the dotted paper provided.



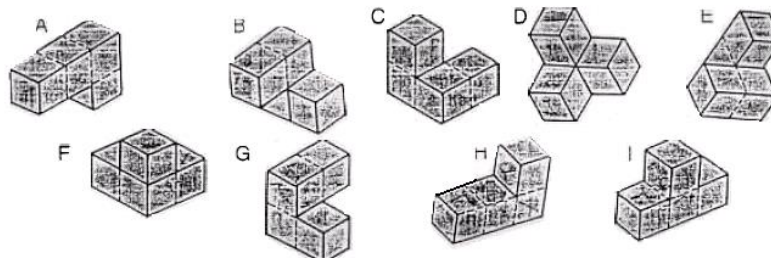
3. Which plan below is the base plan for the building shown below? ('1' means 1 storey, '2' means 2 storeys, etc.).



4. How many blocks would you need to make the large cube on the right? Imagine that you've glued all the blocks together and then dropped the cube into a tin of red paint. If you then separated all the individual blocks, how many of them would have: (a) all 6 faces painted, (b) 5 faces painted, (c) 4 faces painted, (d) 3 faces painted, (e) 2 faces painted, (f) 1 face painted, and (g) 0 faces painted?



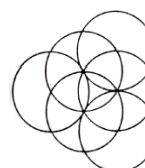
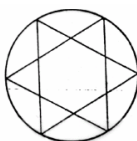
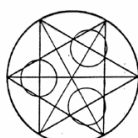
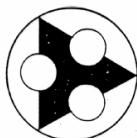
5. There are four pairs of matching blocks below. See if you can find them.



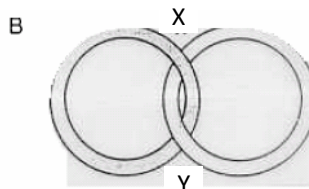
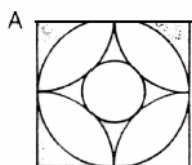
5.1.3 Formal geometric drawing

For these activities, objectives, processes and problem-solving strategies are the same as for visual imagery; materials are pencil, compasses, plain paper, and coloured pencils.

1. Construct and colour as many of the shapes on right and below and as you can.



2. Using your compasses, draw larger versions of the shapes below and then colour them.



- If you were to draw the diagonals of the square in A above, where do you think they would intersect? Find out if your guess was correct.
- If you were to draw a line through points X and Y in B above, would it be perpendicular to the line joining the centre points? Find out.
- Find three examples of concentric circles in the shapes above.
- Add another link to the chain in B above.

5.2 Similarity projection

This section focuses on the similarity projection – a projection that enlarges or reduces a shape without changing the shape or orientation – in other words, two shapes where one is an enlargement of the other. This means that angles stay the same and lengths are in the same ratio. It leads to three important mathematical ideas:

- similar shapes**, which are where one shape enlarges the other but can also be considered in terms of angles equal and lengths of sides in the same ratio;
- scale**, which is the ratio of the sides in similarity and, as an example, how a small plan can show a large construction; and
- trigonometry**, which uses similarities between right-angle triangles to allow distances to be calculated from other sides of shapes.

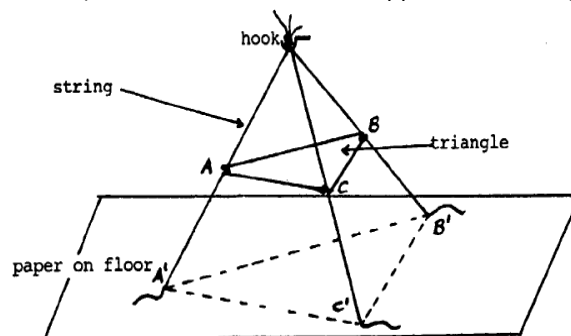
5.2.1 Similar shapes

The idea is to develop similarity as enlargement and then discover the properties.

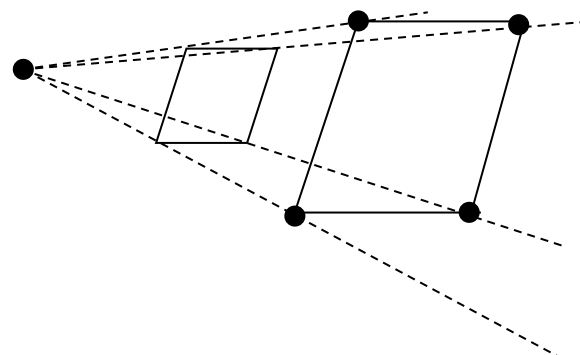
Initial activities

1. **Shadows.** Use a diverging light (e.g. projector, candle, or torch) to cast shadows – look at how things get larger but shape does not change if things are kept parallel. (It is useful to show what happens when things do not remain parallel but we will leave this to next subsection.)

2. **String projections.** Hold a shape parallel to a large paper sheet (with notches at corner of shape); put a hook in the roof and pull down a string through each corner – this marks the corner of the enlarged shape. Draw the enlarged shape and compare to original. In later years can compare angles and sides and show the properties of similarity.



3. **Animation.** Draw a shape and a dot outside it (or can be inside). Draw dotted lines from the dot through the corners. Measure distance from dot to corner, then measure on the same length and put a dot. In this way, the shape can be doubled to a similar shape as on right (a triple size similar shape requires measuring double distance past the corner, and so on for larger similar shapes).



This “animation” method allows students to experience similarity as an enlargement. In later years, the similar shapes can be measured for the properties of similar shapes.

4. **Maps.** Need to look at maps as a similarity projection downward of the land onto flat paper and thus understand where there are differences between maps and reality.
5. **Computers.** Computers can up-size and down-size a shape easily using the size function as long as the “lock aspect ratio” is ticked so everything changes in proportion. Note that an approximate similarity up-size and down-size can be achieved by dragging a corner of a figure (when clicked) towards or away from the opposite corner. This similarity change can be compared with size changes when “lock aspect ratio” has not been ticked or when the clicked shape is dragged by a point that is not a corner point (see next subsection).

Properties

After meaning has been introduced, activity can move on to properties. To do this, construct two shapes where one is an enlargement of the other and measure angles and length of corresponding sides. Divide lengths on the enlargement by corresponding lengths on the original, and compare ratios. Compare size of corresponding angles. Make tables of data. These will show that the ratio of sides and size of angles remains constant for similar shapes (i.e. same shape, different size).

5.2.2 Scale

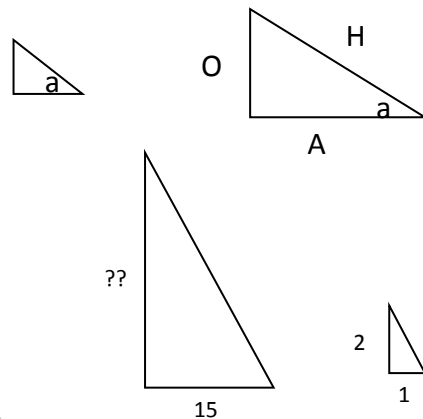
Similar shapes are enlargements and the size of enlargement is the common ratio. Thus we can turn around or reverse similarity and use it for enlargement. So suppose we wish to enlarge something (a shape or pattern) to double its size. We simply have to construct a similar shape whose side ratio is 2. Some ways of doing this are as follows.

1. **Grids.** Place a square grid over original shape, then draw a double-size grid. Mark on the double grid similarly to the original grid in terms of where the shape cuts grid lines and then draw in the missing lines. The result will be double the size of the original.
2. **Use animation.** Simply have the lengths from point to corners equal to the lengths from corners to dots which are used for the second shape.
3. **Use measurement.** Measure all lines and angles on the original shape. Pick a line on the original shape, measure a line out for second shape which is 2× as long, then construct the second shape double size by keeping angles same size and doubling length of all sides/lines.

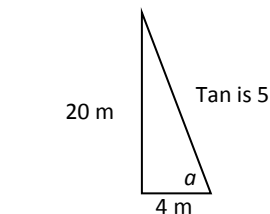
5.2.3 Trigonometry

This can be introduced or at least pre-empted in Years 4–7. For trigonometry, focus on right-angle triangles. The only way to change their shape is to change one angle (not the right angle). All similar shapes to a given right triangle then have lines in the same ratio and this ratio is determined by the angle (change the angle and the shape is no longer similar and ratios change).

Thus, if we have two different-sized right-angle triangles with the same angle (see on right), they are similar and their sides are in ratio. If H is opposite the right angle, O is opposite angle a and A is adjacent to angle a , then we know the O sides are in the same ratio as the A sides and as the H sides.



This can be used to work out long lengths from short lengths. If the shadow of a 2 m post is 1 m long and the tree's shadow is 15 m long as on right, then the fact that shadows are cast at the same angle means similar shapes and the tree is 30 m high. If corresponding sides are in ratio then ratios between pairs of corresponding sides are also the same. Thus we don't need two sides to get the third, we just need the angle a , one side and the side ratio for that angle.



Thus mathematics has determined three ratios for each angle a :

- $\tan a = O/A$,
- $\text{sine } a = O/H$, and
- $\text{cosine } a = A/H$.

Thus, if $\tan a$ is 5 and side A is 4 m, the height will be 20 m as on right.

This method can be used to measure heights of mountains when at sea, heights of objects above a position, and distances by observation from two points.

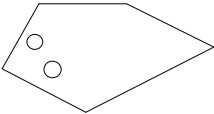
5.3 Other projections

This section deals with the two other projections – the **affine** projection and the **divergent or perspective** projection. The affine projection is shadows in parallel light from the sun – straightness and parallelness stay the same, length and angles do not. The divergent projection is shadows in divergent light with any position for the shape in relation to the screen – straightness stays the same, parallelness, length and angles do not. This activity leads on to perspective drawings.

Comparing the three types of projections is useful. The three projective geometries are concerned with straightness but not length. Because the size of the angles and lines may change, the shape may change. Some activities are below. These can be simplified for younger students.

5.3.1 Shadows

These activities develop an understanding of projective geometry and its transformations, and investigate the properties of shadows and the effect of different lights. The materials are: butcher's paper, texta pens, cardboard, scissors, projector light, whiteboard, whiteboard markers, ruler, plain paper. The directions are as follows.

1. **Parallel light activities.** Go out into the sunlight and make your shadow move; make it jump – make it as long as possible and then as wide as possible – trace your partner's shadow (on butcher's paper). Get into a group and move around without bumping into each other's shadow – try to catch each other's shadow.
2. **Divergent light activities.** Set up a slide projector light to shine on the whiteboard. Cut out a simple 5-sided shape from cardboard which has a sharp point (acute angle) and a right angle, two holes, and pair of parallel sides (like the one shown on right). Cast shadows with this shape onto a screen – move the shape and the screen around.
3. **Comparing parallel and divergent.** If you take the 5-sided shape and cast shadows with it in sunlight, you can compare differences with divergent light. The big difference is that sunlight keeps parallel sides parallel.
4. **Comparing similarity and perspective projections.** Compare the similarity and perspective projection activities as follows.
 - (a) *Similarity.* Keep the shape and screen vertical and cast shadows of your shape on the screen. Make and trace (on the whiteboard) some interesting shadows. What do you notice about the shape and its shadow? Repeat the activity with shape and screen at the same oblique – what do you notice? Can you change the number of sides, the number of holes, and the parallelness of the sides?
 - (b) *Perspective.* Repeat the activity above but place the shape and screen in the following orientations: shape vertical and screen oblique, and shape oblique and screen vertical. What do you notice – is it different to what happened under similarity? Now move shape and screen around – is there still a difference?

Answer the following questions:

- Can you make a shadow that is bigger than the shape, or smaller?
- Can you make a shadow in which the acute angle of the shape becomes a right angle or an obtuse angle in the shadow?
- Can you make a shadow that has more or less than two holes?
- Can you make a shadow that has more or less than five sides?
- Can you make a shadow where the parallel lines are no longer parallel?
- What does all this say about the differences between similarity and perspective projections?

The results will show that similarity projections keep straight lines straight, parallel lines parallel, two holes as two holes, angles the same and sides in ratio, while perspective only keeps straightness and holes the same. Angles, ratio of sides and parallelness change for divergent light.

5.3.2 Other materials/activities

String projections. Repeat the string projection in section 5.2.1 but this time do not keep shape and paper parallel (i.e. turn the shape). What happens? How is it different to similarity?

Computers. Repeat the similarity activity but do not lock aspect ratio or be careful to drag away from or towards the opposite corner. What happens? How is it different to similarity? Is there any change that is not possible? (Number of sides? Straightness?)

Photography. Take pictures of buildings and other objects from strange angles (e.g. looking up). How do the pictures look different to reality? What does this say about our eyes and how we see the world?

5.3.3 Perspective drawing

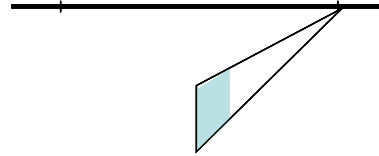
The activities below show a simple two-vanishing-point perspective drawing technique that may work in Prep to Year 3, plus the normal three-vanishing-point drawings.

Two vanishing points (drawing a cube in perspective)

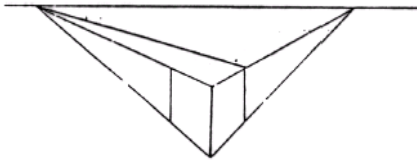
Step 1: Draw a horizontal line and mark two vanishing points (points of perspective) on it. Draw the front edge of the cube.



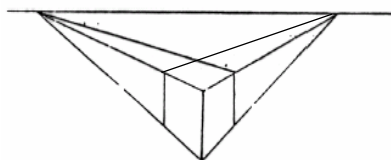
Step 2: Draw lines as shown towards one vanishing point. draw a vertical line to complete one face of the cube.



Step 3: Draw the edges of the face towards the other vanishing point. Draw a vertical line to complete the second face of the cube.

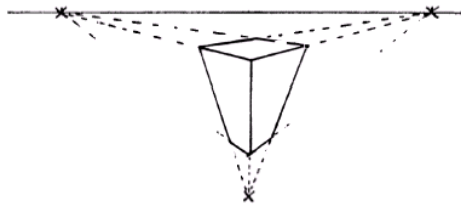


Step 4: Complete the diagram by drawing a line to the first vanishing point. Draw the edges of the cube with a firm line.

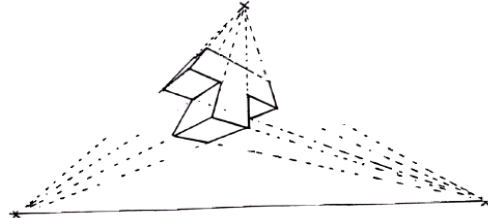


Three vanishing points

For experts, a third point of perspective can be added vertically below the other two . . .



. . . or vertically above the other two points of perspective.




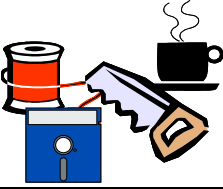
Try to draw in perspective each of the shapes shown above. Draw an N in 2-point perspective. Draw an F in 3-point perspective looking from below or above.

5.4 Topology and networks

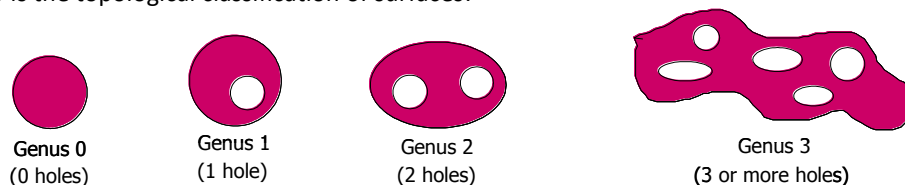
Topology is the final transformation and deals with natural growth and change. Topology changes straightness, length, angle and shape because it allows twisting, bending, deforming, and stretching but does not change inside-outside, closed-open or order along a line because it does not allow punching holes or joining together. It leads to the study of networks.

5.4.1 Topological classifications

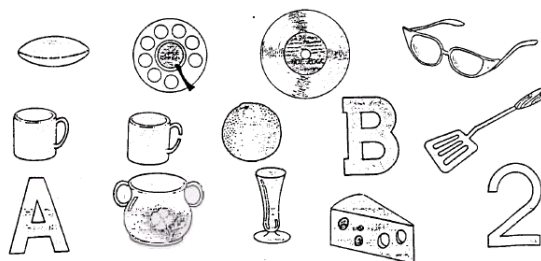
The purpose is to develop the understanding that there is more than one type of geometry; and to develop the notion that topological shapes are classified differently from Euclidean shapes. The materials are plasticine or play dough, balloons, and texta colours. The processes are twisting, tearing, rolling, and distorting, and the problem-solving strategy is modelling.

A surface with no holes	A surface with one hole (a broken surface)
	
<p>Topologists would say that these shapes are the same because all the shapes could be made from the same amount of plasticine which had an unbroken surface. No plasticine would be added to or taken away from the original piece of plasticine nor would the surface be broken.</p>	<p>Topologists would say that these shapes are the same because all the shapes could be made from the same amount of plasticine which had a broken surface. No plasticine would be added to or taken away from the original piece of plasticine. As well, the original piece of plasticine would have no joins in its surface.</p>

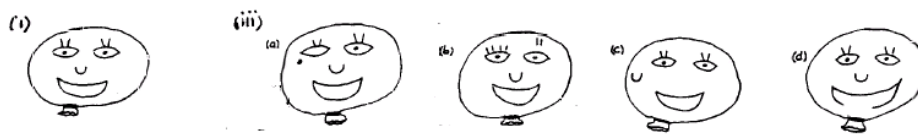
Shown below is the topological classification of surfaces:



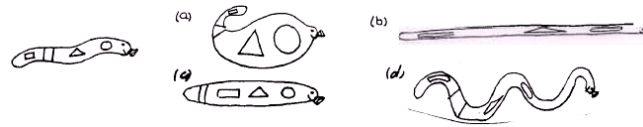
1. Start with a lump of plasticine or play dough with no holes in its surface. Make a sphere and a cube. Did you change the amount of plasticine you started with? Did you put a hole through the plasticine? Are the shapes topologically the same? What *genus* would these shapes be classified as?
2. Start with a lump of plasticine and put a hole right through it (use a pencil) like a donut. Make this donut shape into a cup shape (do not make any joins). Does each shape have just one hole? Did you change the amount of plasticine you started with? Are the shapes topologically the same? What *genus* would these shapes be classified as?
3. Construct each of the everyday shapes below and then classify each shape topologically. Which shapes are topologically the same?



4. Make, then classify topologically, the digits 0–9, then all the letters in your name (do they belong to one particular genus?). Is it possible to have a given name in which all the letters are: genus 0? What about genus 1? Genus 2? Genus 3?
5. Topological shapes are not usually concerned with straightness or length. Try the following activities. (a) Draw a face like (i) below on a round balloon. Blow up the balloon. Bend and distort the balloon. Draw the faces that you see. Which of the faces below can you make? Why can't you make the others? Can you make the faces bigger? Smaller?



6. Draw the snake below on a long balloon. Blow it up, then bend and distort the balloon. Can you make the following shapes?



5.4.2 Topological oddities

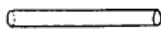
The maze of mirrors

Although you and your image look very different, topologically they are the same.



A bottle with no insides

A German mathematician named Felix Klein (1849–1925) devised a bottle that has an outside but no inside. Nobody will ever see an actual Klein bottle because it can never be made. The Klein bottle exists only in a topologist's imagination.



Start with a tube.



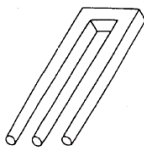
Flare out one end of the tube.



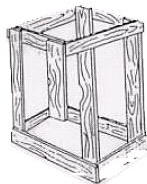
Stretch the neck so that it goes "inside" to meet the base.

Tricks

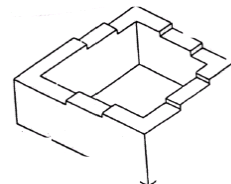
Tricky wickets



Baffling box



Stairs to nowhere



5.4.3 Möbius strip

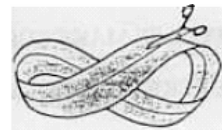
The purpose is to explore topological shapes and changes. Materials are paper, scissors, glue or sticky tape, and coloured pencils.

1. Make a Möbius strip: cut out a flat strip of paper about 50 cm long, twist the paper (once) and then join the two ends to make a closed ring.



2. Try to colour one side of the strip red and the other side green (use any two different colours). What do you notice?
3. Try to draw a line along the centre of the strip continuing until you come back to the same point from which you started. Are you convinced that this strip has only one side?
4. Predict what you think will happen if you cut along the middle of the strip as shown. Validate by cutting. (Cut along the line you have already drawn.) Were you surprised by what happened?

5. Make another Möbius strip. This time draw a line that is one third of the width. Continue to draw the line the same distance from the edge until you come back to the same point from which you started. Cut along the line you have just drawn. Do you have a chain of Mobius strips?
6. Make a Möbius strip where one end is twisted twice before it is glued to the other end. Repeat activities above. Make Mobius strips that have three or four twists and repeat activities above again. Can you discover a pattern emerging?

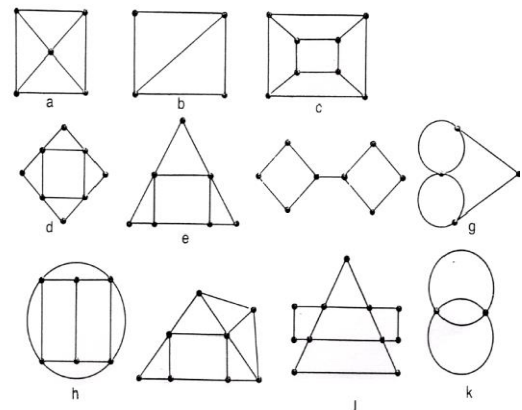


5.4.4 Networks

The purpose is to investigate network theory and to discover the mathematics underlying network theory. Materials are a sheet of shapes called networks. Networks are like maps – lines connecting nodes. Problem-solving strategy is looking for patterns; and thinking to be used is visual and flexible.

1. **Traversability.** The problem here is to determine which networks can be traversed (that is, traced without crossing or retracing a line or lifting your pencil from the paper). You can pass through any point (vertex) more than once. Some networks to practise on are on the right.

Euler developed a rule for when a network was traversable. He did it in relation to the town of Königsberg which was built on an island and two sides of a river with many bridges. The interest for the townspeople was whether you could walk around Königsberg and cross each bridge only once (see section 5.5 for more on Euler).



This can also be seen as the road painter problem in that the painter wants to be always painting and not wasting time travelling a road that has been painted.

2. **Bell telephone problem.** This is a problem for networks that came from running out telephone lines. To save money, the telephone company does not want redundant wiring – it wants networks where the least length of cable is used to reach every household (so only one line into each node – if possible). This is almost the inverse of traversability (which needs at least two ways in/out of each node/household). What would such an efficient network look like?
3. **Salesperson problem (one-visit networks).** This is the problem of salespeople who travel around all the nodes (towns or businesses) and wish to do this in the least time (and using the least petrol). What is the most efficient network/map for them to follow? Is it when they can visit all nodes with only one visit – so only two ways into each node (going in and coming out)? What would such a network look like?
4. **Investigation.** What is the “six degrees of separation” based on the game involving the actor Kevin Bacon? How has it affected networks?

5.5 Integration and extension across topics

This section has significant integration and extension. We cover the following: using coordinate systems to determine flips, slides and turns; properties of similar shapes leading to trigonometry applications; topology and Euler; and relationships among transformational geometries (Euclidean, the three projective geometries and topology). *Note:* This section will be reconsidered for the next edition of this book – if you have any good activities please contact YDC.

5.5.1 Cartesian coordinates and flips-slides-turns

We have studied flips, slides and turns and Cartesian coordinates – now we put them together.

1. Use students' bodies on a grid (or mat) – use a rope or band to divide the grid into two. Have 3-4 students stand at intersections on one side (acting as corners of a 3- or 4-sided shape) and have 3-4 other students (with help from class) stand on the other side for three different changes – the 3-4 people having been slid to the other side (so shape is same distance away from line on each side), reflected about line, and rotated 180°.
2. Repeat this for another 3-4 students but this time label the lines with numbers and students have to say where students go by stating coordinates.
3. Repeat the above but use elastic bands to make a shape. Coordinates of corners will be stated to show flips-slides-turns.
4. Repeat the above but for the mat/grid divided into four sections using x and y axes (the coordinates can now be negative). Make shapes in one quadrant (top right is normal). Use coordinates to slide the shape across axes, flip the shape about axes and rotate the shape 90°.
5. Place a grid on a computer, draw simple shapes, use the mouse and the software to slide shape along a line that starts and ends on coordinates, flip shape about a line that starts and ends on coordinates, and rotate shape 90°, 180° and 270° about a centre which is a coordinate. Write down the starting and finishing coordinates, and compare differences in these coordinates with the extent of the flip, slide and turn.
6. Use this comparison to suggest coordinate rules for these slides, flips and turns.
7. **Investigation.** Use coordinates and grids to: (a) slide a simple shape along an arrow (arrow gives direction and distance – start and end of arrow have to be a coordinate position); (b) flip a shape about a line (line starts and ends on coordinates); and (c) rotate a shape about a centre (which is a coordinate) angles of 90°, 180° and 270°.
8. Compare the start and finish coordinates – use these to propose coordinate rules for slides, flips and turns.

5.5.2 Similar shapes and trigonometry

Similar shapes are ones where the larger shape is a blow up or enlargement of the smaller (where the enlargement is regular, that is, the smaller and larger shape are parallel as the divergent light is used to cast the shadow).

Properties of similar shapes

The activity below can first be done with two similar shapes made by animation where the increase in the animation is double or triple. It will then be seen that the similar shapes have equal angles and sides doubled and tripled.

1. Use an enlargement situation to make two similar shapes as in section 5.2.1.
2. Measure the angles and the side lengths of the smaller and larger shape. Calculate the ratios of the sides.
3. Place these on a table as below. Repeat this for three examples.

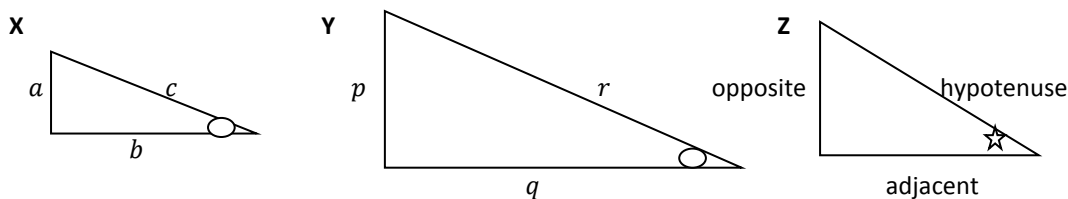
Shape sets	Smaller shape		Larger shape		Ratios of side lengths
	Angles	Side lengths	Angles	Side lengths	
1					
2					
3					

4. Look at results; what is the same? (angles, ratio of side lengths). Check this with another example.

The results show that the properties of similar shapes and same shape but different size are that corresponding angles are equal, and corresponding sides are increased by the same ratio. Interestingly, if all angles are equal, then sides are always in ratio and shapes are similar – this has led to a plethora of ways shapes can be similar (e.g. triangles with two sides in ratio and one angle equal are similar).

Applications in trigonometry

Right-angle triangles (right triangles) form an interesting subclass of similar shapes. Because one angle is 90° , determination of the angle at one other corner determines all angles. This means that all right triangles of the same first angle are similar (e.g. a right triangle with an angle of 30° is similar to all other right triangles with an angle of 30° no matter what the size). (Note: We are using division although it is ratio – they are the same.)



The two right triangles X and Y above are similar – they have the same angles. Thus p/a , q/b and r/c are in proportion, that is, they have the same ratio ($p/a = q/b = r/c$). Because of the nature of proportion, this means that sides are in ratio within a triangle, that is, $p/q = a/b$, $a/c = p/r$ and $b/c = q/r$. It also means that these sides are in the same ratio for any right triangle with the same angle as X and Y.

Therefore, if we know the angle of a right triangle, we know the three proportions that have been designated with special names as below. For triangle Z with angle ☆ above:

- Longest side is called hypotenuse, side next to angle is called adjacent, and side opposite to angle is called opposite; and
- Sine ☆ = opposite/hypotenuse, cosine ☆ = adjacent/hypotenuse, tangent ☆ = opposite/adjacent.

Activities

1. Right triangles have the same ratios for the same angle.

- Construct right triangles of different size with the same angle. Measure sides. Calculate the sine, cosine and tangent values. Are they the same? Check with another example.
- Construct a right triangle. Take the lengths of the three sides. Multiply them by the same number, or increase them by the same ratio. Put the lengths back together to form a triangle. Check that this is a right triangle similar to the starting triangle. Repeat this as necessary. Computers do this activity well.

2. Practising using trigonometry. Do many of these.

- Take triangles that are similar – have all lengths in first triangle known and only one in the second triangle known. Use trigonometry to calculate the remaining sides.
- Use a calculator that is able to calculate sines, cosines and tangents for all angles. Take a right triangle with angle and one side known. Use trigonometry to find the other sides.
- Reverse the above. Take a triangle with two sides known. Use calculator in reverse to find the angle and state which angle it is.

3. Finding height of an object by similar triangles and trigonometry. Consider the object and the ground as a right angle. Use an inclinometer to find angle to top of object from a known distance from the object. Use angle and known distance to find the height. The old geometry book *Space and Shape in the Primary Classroom* on the YDC website has a section called “8 ways to measure the height of a tree”. Four similarity methods from this section are as follows.

- (a) Use shadows to find height.
- (b) Use a mirror on the ground to find height.
- (c) Use a 45° angle to find height.
- (d) Use the ship method to find height of a mountain.

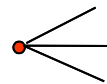
4. **Finding distance when measuring it directly** (e.g. across a ravine). The old geometry book has another section on surveying – try some of the ravine methods.

5.5.3 Topology and Euler

Euler developed a rule for determining whether a shape was traversable. It was based on classifying a vertex or corner as *odd* or *even* depending on how many paths lead from it (as on right).

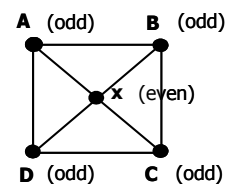


There are 2 paths leading from this vertex. 2 is an even number so this vertex is even.



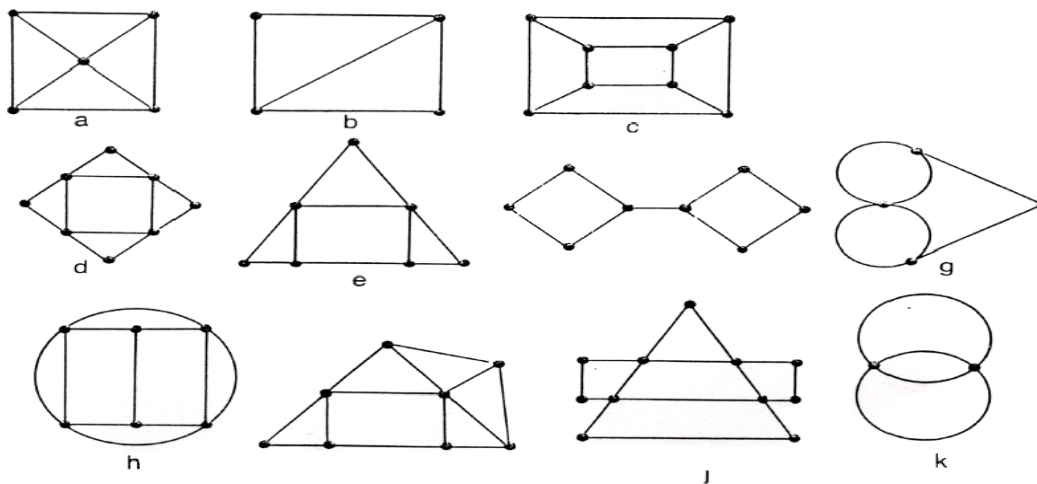
There are 3 paths leading from this vertex. 3 is an odd number so this vertex is odd.

Knowing this simple fact, we can classify all the vertices in the shapes shown below. The first example *a* is shown on the diagram on right. Thus we need to do the following.



1. Classify all of the vertices in the shapes provided below. (Write O for odd and E for even.)
2. Complete a table as shown below – the first two examples, *a* and *b*, have been done for you.

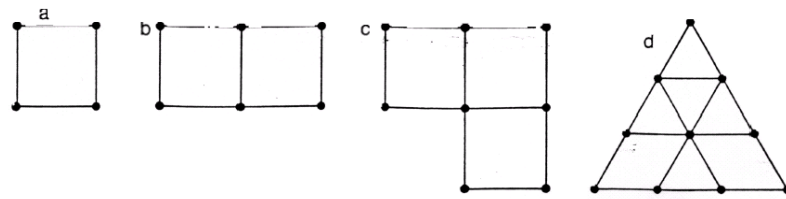
Network	Traversable (Yes/No)	Number of even vertices	Number of odd vertices
	No	1	4
	Yes	2	2



3. When finished, answer this question: Can you traverse a network if it has: (a) 0 odd vertices? (b) 2 odd vertices? (c) more than 2 odd vertices? Check by making up some more networks.

(Note: Euler's rule states that a network is traversable if the number of odd vertices is 0 or 2. For 0 odd vertices, one can start anywhere; for 2 odd vertices one has to start at an odd vertex.)

4. Now that you know Euler's Law, determine which of the shapes below are traversable and then validate by tracing.



5.5.4 Relationships among transformational geometries

An **investigation** is to experience all the different transformational geometries (there are more activities in the old geometry book, *Space and Shape in the Primary Classroom*, on the YDC website):

- A: Euclidean – flips, slides and turns;
- B: Similarity projection – casting shadows in divergent light with screen and shape parallel;
- C: Affine projection – casting shadows in sunlight with screen and shape at any angle;
- D: Divergent projection – casting shadows in divergent light with screen and shape at any angle; and
- E: Topology – can twist, bend and stretch but not join together, break apart or punch a hole.

If you do, you can come to see relationships among these transformational geometries and sequences in their progression as follows.

1. Dienes believed all these geometries should be taught. YDC agrees because:
 - A is how we handle the human-made world (construction, tiling, art, etc.);
 - B is how we project on film, etc.;
 - C is how sunlight works;
 - D is how our eyes work; and
 - E is how the natural world grows and changes.
2. Piaget believed that students should learn these geometries in the order E to A.
3. The things that do not change (invariance) are different in the geometries and specify a sequence opposite to Piaget:
 - A – straightness, parallel and length do not change;
 - B – straightness and parallel do not change and length is in ratio;
 - C – straightness and parallel do not change but length does;
 - D – straightness does not change but parallel and length does;
 - E – all of straightness, parallel and length change (but inside–outside stays inside–outside and closed–open stays closed–open while order on a line remains unchanged).
4. More can be found with deeper investigation.

6 Teaching Framework for Geometry

The teaching framework organises the content for geometry into a framework of four topics: shape, coordinates and graphing, Euclidean transformations, and projective and topology. Each of these topics is partitioned into sub-topics. Each sub-topic is described and any concepts or strategies used in the teaching framework are listed. They are also related to big ideas. Topics and sub-topics are chosen so as to represent ideas that recur across all year levels. The resulting framework is given in Table 1. This overall framework can be compared to the Australian Curriculum to produce year-level frameworks.

Table 1. Framework for teaching geometry

TOPIC	SUB-TOPIC	DESCRIPTION AND CONCEPTS/STRATEGIES/WAYS	BIG IDEAS
Shape	3D shape	Concepts (names and properties); constructions; relationships (between shape types); cross-sections; Euler's formula (between faces, edges and vertices)	Interpretation vs construction; Euler's formula
	2D shape	Concepts (names and properties); constructions (including drawings); relationships (between shape types); rigidity (how to make shapes rigid); Pythagoras' theorem (relating sides)	Interpretation vs construction; length, diagonal and rigidity relationships; Pythagoras' theorem
	Line and angle	Concepts (names and properties); constructions (including drawings) – interpretation vs construction; properties (relationships between line and angle); identity and inverse; shape formulae (angles and diagonals, rigidity)	Interpretation vs construction; identity and inverse; angle formulae; line and angle relationships; length, diagonal, angle and rigidity relationships
Coordinates and graphing	Coordinates (position)	Language (position language); polar (coordinates and mapping); Cartesian (coordinates and mapping); latitude and longitude	Interpretation vs construction; concepts of polar and Cartesian
	Graphs	Directed numbers, axes and plotting line graphs; quadrants; slope (rise over run); properties of line graphs in relation to equation; flip/slide/turn Cartesian properties; distance and midpoint activities; nonlinear graphs	Properties of line graphs in relation to equation; flip/slide/turn Cartesian properties
	Graphical methods	Solving equations for unknowns using graphical methods; applications of methods	
Euclidean transformations	Flips, slides and turns	Concepts–constructions (definitions and acting out/modelling changes) – interpretation vs construction; change vs relationship; properties; art (flips, slides and turns in art); relationships (between flips and slides and turns); congruence; flip properties	Change vs relationship; interpretation vs construction; transformational invariance; line symmetry/reflection relationships
	Symmetry	Concepts (definitions, properties and numbers of each type); constructions–changes (symmetric figures); art (symmetry and art); shape classification; relationships (between line and rotational symmetry)	Interpretation vs construction; line symmetry/reflection relationships
	Tessellations	Concepts–constructions (definition, patterns for one/more shapes, making tessellating patterns); art (tessellation and art); grids–puzzles (tessellation grid activity and shape puzzles); properties (angle properties for forming tessellations); solid tessellations; part vs whole; angle formulae	Interpretation vs construction; part vs whole; angle sum formulae
	Dissections	Concepts–constructions (definitions for simple and complex dissections, solving dissections – part vs whole); shape puzzles (2D and 3D and levels of difficulty); visual imagery	Interpretation vs construction; part vs whole

TOPIC	SUB-TOPIC	DESCRIPTION AND CONCEPTS/STRATEGIES/WAYS	BIG IDEAS
Projective and topology	Visualisation	Perception (of shapes from different directions and distances); mental rotation (rotating self or object)	Mental rotation
	Similarity, scale, and trigonometry	Similarity projection; similar shapes (and properties); scale (drawings and plans); trigonometry and applications	Change vs relationship; interpretation vs construction; transformational invariance; trigonometry rules
	Projections and perspective	Projections (and shadows); perspective (drawings)	Change vs relationship; interpretation vs construction; transformational invariance
	Topology and networks	Topology (change and puzzles); networks (Euler's formula); applications to network problems; transformational invariance	Change vs relationship; interpretation vs construction; Euler's formula; transformational invariance

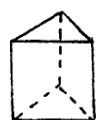
Appendix A: Major Geometric Ideas

The major geometric ideas are given under the four chapter headings.

Shape

Three-dimensional shape (solids)

This is a most crucial area as we live, work and play in a three-dimensional world. Initially we can use solids (bricks, Lego, etc.) to construct things. We can sort and classify solids in the environment by their properties — corners/no corners, rolls/does not roll, etc. We can study their surfaces and edges (leading into the study of line and two-dimensional shape). Names of solid shapes are determined by the type of surface and edge a shape has, whether they are flat, curved, or straight. We particularly focus on five types of solid (see below) and also solids which have flat surfaces and straight edges called polyhedra.



Triangular Prism



Square Based Pyramid



Cylinder



Cone



Sphere

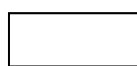
Two-dimensional shape (plane shapes)

This is another crucial area because it is how we represent our three-dimensional world (e.g. in seeing, drawing, video, film, etc.). Two-dimensional shape can be developed in two ways: (1) from study of the surfaces of the 3D solids, and (2) from the sub-concepts of line and angle. For example, the triangle can be considered as the end of a roof or as three straight lines (and three angles) which form a closed and simple (lines do not cross) boundary around a region.

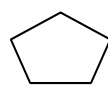
Two-dimensional shapes are classified by their boundaries. When these are straight lines, they are called polygons. Polygons have special names based on the number of sides and whether these sides are parallel, equal in length or meet at special angles (see examples below). When these are curved lines, special attention is given to circles. Shapes bound an area of a plane called a region. The circular region is called a disc. A section of a disc is a sector.



Triangle



Rectangle



Pentagon



Hexagon



Circle

Two-dimensional shapes have properties important in vocational education and training, for example, using diagonals to square foundations, using triangles to make shapes strong and rigid, and using Pythagoras' theorem and tangents for right-angle triangles.

Line and angle

This is associated with plane shapes. It relates to the types of lines and angles that classify shapes and to the diagonal and angle properties for plane shapes. Angles are acute, right, obtuse and reflex, and external and internal, while lines are equal, parallel, bisected, perpendicular, and diagonal.

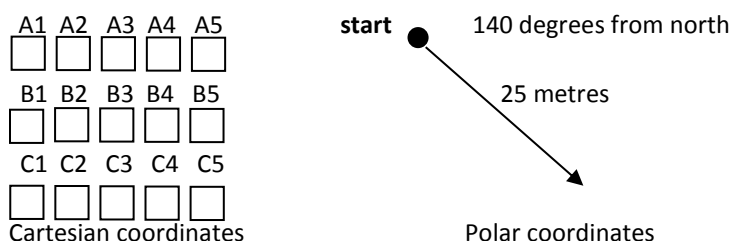


Coordinates and Graphing

Early position

Initially, position can be defined in terms of everyday words such as “here”, “there”, “near”, “far”, “under”, “over”, “above”, “below”, etc. This can be followed by instructions and drawings showing how to go from one place to another or where to find something.

Then position is seen in terms of coordinates (see below): (a) polar – where directions and distances are given to define a position; and (b) Cartesian – where numbers (and letters) are used to more formally determine position in terms of ordered pairs of numbers and/or letters (e.g. “row C seat 4”, “two blocks south and one block west”, or an ordered pair of numbers).



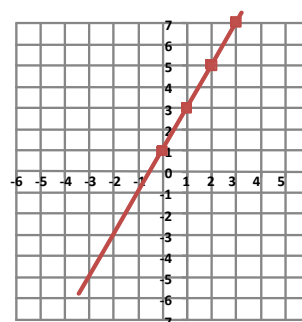
Coordinates and maps

Position is now extended to maps which can be (a) in polar form, where directions are shown from a starting point by directions and distances (e.g. hand-drawn or computer maps showing turns and distances, modern GPS systems which verbally/visually give directions); or (b) in Cartesian form, where position is in terms of coordinates (e.g. road maps, street directories, modern GPS systems that show position in terms of latitude and longitude).

A special area that needs to be explored is direction in terms of north, east, south and west (i.e. compass bearings), and the use of this knowledge with map reading to enable learners to undertake orienteering and bushwalking activities. Later in school, this is extended to include latitude and longitude.

Line graphs and properties

Line graphs are plots on x and y axes of linear relationships between y and x where the power of the variable x is 1, such as $y = 2x + 1$. Plotted onto x and y axes which incorporate negative numbers, they form a straight line, as for example $y = 2x + 1$ on the right.



The properties of line graphs are:

- (a) they are plots of linear relationships like $y = mx + c$ where the slope = m (the x coefficient) and the y -intercept (where line cuts the y axis) = c ;
- (b) they only need two points (a, b) and (c, d) to be drawn;
- (c) the slope $m = \frac{(d-b)}{(c-a)}$;
- (d) the midpoint between the two points = $(\frac{c-a}{2}, \frac{b-d}{2})$; and
- (e) the distance between the two points = $\sqrt{(c-a)^2 + (d-b)^2}$ (this uses Pythagoras’ theorem).

Line graphs and linear relationships relate to growing patterns and function machines – if the pattern or change rule is found by multiplying or dividing and adding or subtracting, then the numbers give the graph (e.g. $\times 3$ and $- 2$ is $3x - 2$, and $\div 4$ and $+ 1$ is $\frac{x}{4} + 1$).

Graphical solutions

Line graphs can be used to find solutions for unknowns in linear equations. To do this, plot both sides of the equation (get two line graphs) and the unknown will be the x -value of the point where the two lines cross. For example, $2x + 1 = 6 - 3x$ can be solved by plotting $y = 2x + 1$ and $6 - 3x$ on the same graph. These two lines cross at point (1, 3), so $x = 1$ is the answer. *Note:* Remember if we have $2x + 1 = 7$ we can still plot $y = 2x + 1$ and $y = 7$ (a horizontal line) and these will cross at point (3, 7) making unknown $x = 3$.

Nonlinear graphs

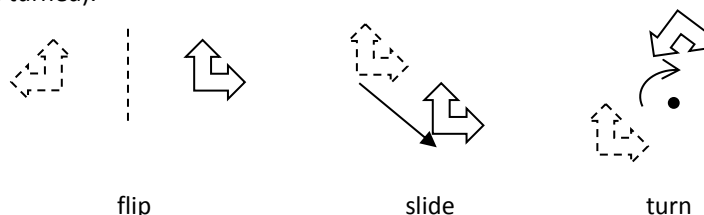
Nonlinear graphs are all other graphs except line graphs. This means graphs where the power of x is not 1 such as quadratics ($y = 2x^2 + 3x + 1$), cubics ($y = x^3 - x^2 + 2x - 3$), and exponentials ($y = 2^x$). They also include circles ($x^2 + y^2 = 1$). They can be plotted on axes with negative numbers but need to have a smooth curve through the points.

Each of the nonlinear graph types have specific shapes – quadratics are parabolas, cubics are like flat S's, and so on. Exponentials are flat to rapidly changing while logarithmic are the opposite.

Euclidean Transformations

Flips, slides and turns

A flip is a reflection (a line of reflection can be given around which the shape is flipped), a slide is a translation (a movement in one direction without change of orientation), and a turn is a rotation (a centre point can be given about which the shape is turned).

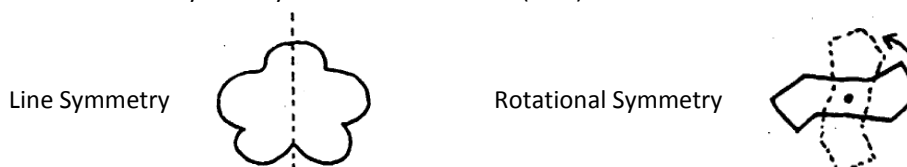


Flips, slides and turns are the only changes allowed in Euclidean transformations. Thus Euclidean transformations are the changes of our human-made world. They are changes which will not affect the object being moved.

Slides and turns are related to flips – a slide can be achieved by two flips where the lines of reflection are perpendicular to the slide and the distance between lines of reflection is half the length of the slide, and a turn can be achieved by two flips where the lines of reflection pass through the centre of the turn and the angle between lines of reflection is half the angle of the turn.

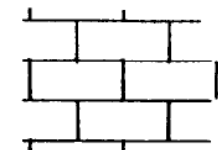
Symmetry

Shapes can be studied for both line and rotational symmetry and for the relationship between these two symmetries (for more than two lines of symmetry, the number of lines equals the number of rotations of symmetry and the angle between lines of symmetry is half the angle between consecutive rotations). Line symmetry means that the shape can be folded in half along a line; rotational symmetry means that the shape looks the same after a part turn (see below). Shapes can be classified by symmetry (e.g. square has four lines/four rotations of symmetry, while rectangle only two lines/two rotations). Line symmetry can be related to reflection (flip); while rotational symmetry is related to rotation (turn).

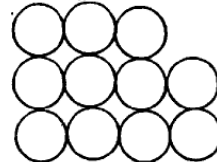


Tessellations

When a shape is such that many copies of it can be joined like tiles to cover a space without overlapping or leaving gaps, the shape is said to tessellate (see below). Tessellating shapes have angles that combine to give 360° or divide into 360° . Tessellating shapes give rise to different graph papers. Fabric designs result from tessellating patterns, as do Escher-style drawings. Shape puzzles come from tessellating shapes (e.g. pentominoes come from squares).



Rectangles tessellate



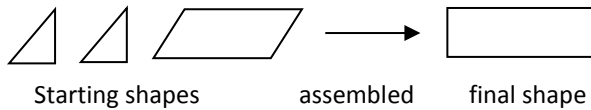
Circles do not tessellate

Tessellations can be introduced through tiling and shapes that tile without gaps or overlaps identified. Tessellations of more than one type of shape can be explored as can three-dimensional or solid tessellations (i.e. containers that pack well). Finally, tessellations can be used to identify shapes which form the basis of shape puzzles (e.g. pentominoes).

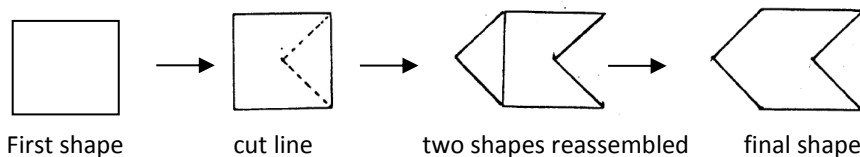
Dissections

Dissections emerge from jigsaw puzzles – they are puzzles where pieces (normally all different shapes) have to be put together to make a larger shape. There are two types of dissection: (a) *simple* – where pieces (starting shapes) are given and have to be put together to form a final shape; and (b) *complex* – where a first shape is given which has to be cut into pieces and reassembled to form a second shape. Dissections also include families of puzzles where one set of pieces is used to make many other shapes (e.g. tangrams, Soma cubes).

Simple dissection:

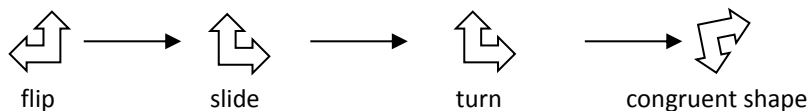


Complex dissection:



Congruence

Shapes are congruent if one can be changed to the second by flips, slides and turns. This means that congruent shapes are the same size and shape (they have the same angles and length of sides); they differ only by orientation.



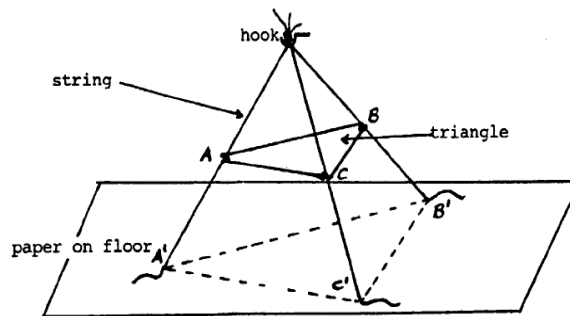
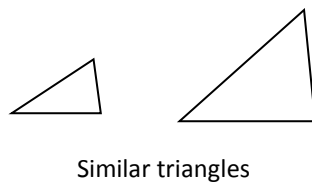
Projective and Topology

Visualisation

This is a collection of abilities that enable shapes (and reality) to be visualised and understood. They include abilities to understand how things look different from different positions and directions, and to mentally rotate self around, above and below a shape and to mentally rotate a shape. They lead into design where front, side, rear and top drawings are used to plan a building, and into computer simulations (including computer games) where buildings are shown in rotation or by the simulation moving through the building. They end with computer-based virtual replications of life/action in modern cartooning and special effects.

Similarity, scale and trigonometry

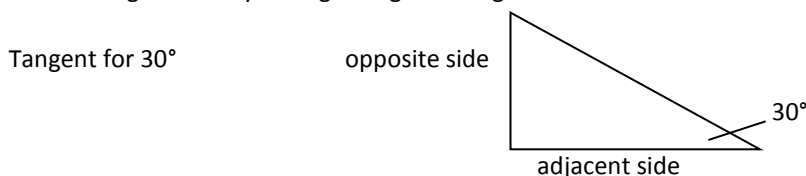
Two shapes are similar if one is an enlargement of the other where the shape and the enlargement are parallel. This is called a similarity projection, that is, same shape but different size (see below). Similar shapes are like enlargements in photography or the action of light through a film. This can be done with light or string and the resulting similar shapes can be studied for the properties of equal angles and lengths of sides in same ratio (proportion).



An example of a similarity projection using string

Thus, similarity is the basis of scale and scale drawings. A drawing and a scale is given. The final product has to be the same shape – same angles with the sides all increased or decreased by the scale. For example, a scale of 1 cm = 5 m would mean that a 3 cm wall would be 15 m in reality. Knowledge of coordinates, shapes, bearings and distance comes together with the ability to prepare and draw scale drawings of homes, schools, parks, suburbs, and so on.

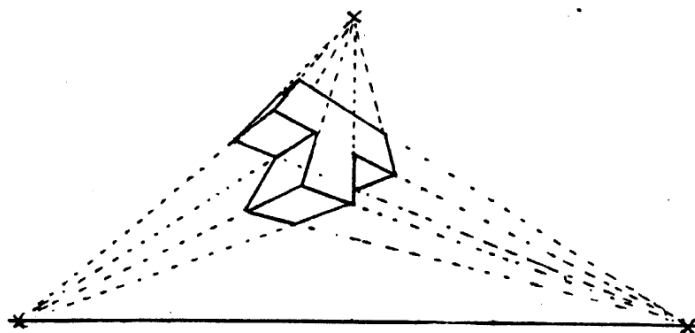
Similarity also underlies trigonometry because trigonometry is based on similar right-angled triangles – all right-angled triangles with one angle, say 30° , are similar so the same ratio of opposite side over adjacent side holds for all of them. This allows the tangent (the ratio of length of the adjacent side to the length of the opposite side) to work out lengths for any 30° right-angled triangle.



Projections and perspective

There are three projections; the first of these (which we will call the perspective projection) is from diverging light (e.g. casting shadows from a torch or candle). This form of projection is how we see the world – it is how reality forms pictures on the back of our retina. To study it is to understand the difference between the world and what our eyes see (and also what cameras see). Studying the changes that are possible between a shape and the shadows cast by it as it is turned (where the shape is not parallel to the screen for casting shadows) provides an understanding of this projection. Euclidean transformations leave length and straightness unchanged and,

therefore, angle unchanged; similarity projections, because shape and enlargement are parallel, leave straightness unchanged and keep sides in ratio, which means angle is unchanged. Perspective projections leave straightness unchanged but not length which means that angle is changed. This can be seen in the use of vanishing points to produce drawings that take account of perspective.

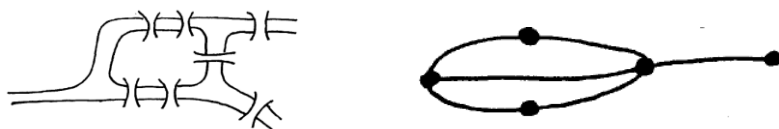


A perspective drawing of a block T-shaped balloon as it would be seen from below

There is another projection, the affine projection, which is equivalent to looking at shadows cast in sunlight which is parallel. Such shadows keep straightness unchanged but not length and angle, but they also keep parallelness unchanged. This can be seen in the differences of shadows of two flagpoles cast in torchlight and sunlight.

Topology and networks

The final form of transformation useful in Prep to Year 9 mathematics is topology. This is the change produced by living growing things – it allows for both straightness and length to change (it allows bending, stretching, and twisting but not cutting, tearing, punching holes and joining). It is the basis of such puzzles as the Möbius strip and has interesting activities based around open–closed and inside–outside. One important application of topology is networks, a particular type of map that connects centres (or points) with lines (or arcs). Networks are ubiquitous in modern society – they can be towns and roads, suburbs and bridges, telegraph lines, power grids, wiring in a house, postal or broadband services, or mobile phone connections. A network divides space into regions. Travelling along real networks can be modelled by tracing over drawings of networks. This enables many interesting travel problems to be studied by learners at their desks.



Network diagram examples



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Faculty of Education
School of Curriculum
S Block, Room 404
Victoria Park Road
KELVIN GROVE QLD 4059

CRICOS No. 00213J

Phone: +61 7 3138 0035

Fax: +61 7 3138 3985

Email: ydc@qut.edu.au

Website: <http://ydc.qut.edu.au>