YuMi Deadly Maths

Algebra

Prep to Year 9

QUT

YuMiDeadly

Growing community
through education

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians
of the lands in which the mathematics ideas for this resource were developed,
refined and presented in professional development sessions.

DEVELOPMENT OF THIS BOOK

This version of the YuMi Deadly Maths Algebra book is a modification and extension of a book developed as part of the Teaching Indigenous Mathematics Education (TIME) project funded by the Queensland Department of Education and Training from 2010–12. The YuMi Deadly Centre acknowledges the Department's role in the development of YuMi Deadly Maths and in funding the first version of this book.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at QUT which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Educational Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

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ABOUT YUMI DEADLY MATHS

From 2000–09, researchers who are now part of the YuMi Deadly Centre (**YDC**) collaborated with principals and teachers predominantly from Aboriginal and Torres Strait Islander schools and occasionally from low socio-economic status (SES) schools in a series of small projects to enhance student learning of mathematics. These projects tended to focus on a particular mathematics strand (e.g. whole-number numeration, operations, algebra, measurement) or on a particular part of schooling (e.g. middle school teachers, teacher aides, parents). They resulted in the development of specialist materials but not a complete mathematics program (these specialist materials can be accessed via the YDC website, http://ydc.qut.edu.au).

In October 2009, YDC received funding from the Queensland Department of Education and Training through the Indigenous Schooling Support Unit, Central-Southern Queensland, to develop a train-the-trainer project, called the **Teaching Indigenous Mathematics Education** or **TIME** project. The aim of the project was to enhance the capacity of schools in Central and Southern Queensland Indigenous and low SES communities to teach mathematics effectively to their students. The project focused on Years P to 3 in 2010, Years 4 to 7 in 2011 and Years 7 to 9 in 2012, covering all mathematics strands in the Australian Curriculum: Number and Algebra, Measurement and Geometry, and Probability and Statistics. The work of the TIME project across these three years enabled YDC to develop a cohesive mathematics pedagogical framework, **YuMi Deadly Maths**, that covers all strands of the *Australian Curriculum: Mathematics* and now underpins all YDC projects.

YuMi Deadly Maths (**YDM**) is designed to enhance mathematics learning outcomes, improve participation in higher mathematics subjects and tertiary courses, and improve employment and life chances. YDM is unique in its focus on creativity, structure and culture with regard to mathematics and on whole-of-school change with regard to implementation. It aims for the highest level of mathematics understanding and deep learning, through activity that engages students and involves teachers, parents and community. With a focus on big ideas, an emphasis on connecting mathematics topics, and a pedagogy that starts and finishes with students' reality, it is effective for all students. It works successfully in different schools/communities as it is not a scripted program and encourages teachers to take account of the particular needs of their students. Being a train-the-trainer model, it can also offer long-term sustainability for schools.

YDC believes that changing mathematics pedagogy will not improve mathematics learning unless accompanied by a whole-of-school program to challenge attendance and behaviour, encourage pride and self-belief, instil high expectations, and build local leadership and community involvement. YDC has been strongly influenced by the philosophy of the Stronger Smarter Institute (C. Sarra, 2003) which states that any school has the potential to rise to the challenge of successfully teaching their students. YDM is applicable to all schools and has extensive application to classrooms with high numbers of at-risk students. This is because the mathematics teaching and learning, school change and leadership, and contextualisation and cultural empowerment ideas that are advocated by YDC represent the best practice for **all** students.

YDM is now available direct to schools face-to-face and online. Individual schools can fund YDM in their own classrooms (contact ydc@qut.edu.au or 07 3138 0035). This Algebra resource is part of the provision of YDM direct to schools and is the fourth in a series of resources that fully describe the YDM approach and pedagogical framework for Prep to Year 9. It focuses on teaching algebra, namely (a) repeating and growing patterns; (b) change and functions; (c) equivalence and equations; and (d) principles of arithmetic and algebra. It covers introducing unknown and variable, solving for unknowns, linear relationships, line graphs and algebraic computation including substitution, expansion and factorisation. It overviews the mathematics and describes classroom activities for Prep to Year 9. Because YDM is largely implemented within an action-research model, the resources undergo amendment and refinement as a result of school-based training and trialling. The ideas in this resource will be refined into the future.

YDM underlies three projects available to schools: YDM Teacher Development Training (TDT) in the YDM pedagogy; YDM AIM training in remedial pedagogy to accelerate learning; and YDM MITI training in enrichment and extension pedagogy to build deep learning of powerful maths and increase participation in Years 11 and 12 advanced maths subjects and tertiary entrance.

Contents

1   *Purpose and Overview*

YuMi Deadly Mathematics (YDM) is based around (a) big ideas, connections and sequencing; and (b) the Reality–Abstraction–Mathematics–Reflection (RAMR) model. It endeavours to achieve three goals: (a) reveal the structure of mathematics; (b) show how the symbols of mathematics tell stories about our everyday world; and (c) provide students with knowledge that they can access in real-world situations to help solve problems. YDM argues that the power of mathematics is based on **connections and big ideas**. For arithmetic (number and operations), these come from algebra and lead to algebra. The best way to learn mathematics is by using these connections and big ideas; that is, using algebraic thinking to generalise arithmetic. This needs to be developed early so that students can better understand number and operations and be better prepared for formal algebra.

Algebra is the generalisation of arithmetic – it is the parts of arithmetic that hold for any number. Therefore, algebra is composed of big ideas. Thus, for YDM, this book is very important and shows how number and operations grow into structures that are very powerful and portable in their understanding of the world. However, the algebra of this book is not the algebra of x's and y's (although we do get to these symbols); it is interesting and motivating arithmetic that enables the development of big ideas and prepares for the x's and y's. It is activities that not only teach the arithmetic that should be taught each year in an engaging way but also pre-empt and prepare for secondary algebra.

The purpose of this book is to (a) reinforce arithmetic ideas from the YDM Number and Operations books; (b) extend the arithmetic ideas to generalities that make up algebra; and (c) develop understandings that are powerful and portable and can lead to high status professions. This book is designed to teach but also enrich the Australian Mathematics Curriculum. This bolstering does not result in new work, just a better way of developing teaching that reflects big ideas and, as we will argue below, is more in harmony with Aboriginal, Torres Strait Islander and low SES students.

The book provides sequences of algebra activities to enable YDM to be effective in developing holistic algebraic thinking. However, this also results in a lot of interesting and motivating activities that assist learning of necessary number and operations work in the early years and, at the same time, pre-empt later algebra work. As will be discussed below, this leads to a more effective and powerful way to teach mathematics.

The book is also based on the important distinction that, in arithmetic, 3 + 4 is the process and 7 is the product (or answer) while, for algebra, x + 4 is both process and product – there is only the process. This means that arithmetic processes – concepts, strategies and principles – rather than arithmetic calculations are needed for algebra.

This chapter looks at the nature of algebra and the nature of the most effective teaching of algebra. It covers connections and big ideas (section 1.1), sequencing (section 1.2), teaching and cultural implications (section 1.3), and overview of the book (section 1.4).

1.1   Connections and big ideas

This section overviews the role algebra plays in the structure of mathematics, describing its second level and holistic nature, how it is connected to the other strands within the structure of mathematics, and how it is based on a series of big ideas or principles that recur across Years P to 9.

YDM believes that, through connections and big ideas, mathematics can be taught so that it is **accessible as well as available** (see YDM Overview book); that is, learnt as a rich schema containing knowledge of when and why as well as how. The connected nature of rich schema means that it has knowledge as a structure of connected nodes, which facilitates recall (it is easier to remember a structure than a collection of individual pieces of information) and problem solving (content knowledge that solves problems is usually peripheral, along a

connection from the content on which the problem is based). As described in the Overview book, the reality and mathematics components of the RAMR cycle are built, in part, around connections and generalisation to big ideas.

1.1.1 Algebra as abstraction of abstraction (or generalisation of generalisation)

Abstraction is a process by which a **generality** is determined from particular examples. In Western mathematics, an important abstraction is number. By experiencing, for instance, many examples of two items (e.g. 2 eyes, 2 hands, 2 chairs, 2 children, and so on), learners generalise the language "two" and the symbol "2" as representing the "twoness" that is common to the examples. In a similar way, learners gradually build understanding of the language and symbols of all numbers.

When numbers and their names and symbols are new to learners, meaning lies with the items. For example, 2 + 3 is thought of as 2 items and 3 items. Counting all the items gives the solution 5 items. Thus 2 + 3 = 5 is thought of as 2 items joining 3 items to make 5 items. The focus of thinking is on the items. However, over time as more and more experience is gained, it becomes less necessary to think of items when we use numbers. After a while, 2 + 3 can be considered as equal to 5 without having to think of 2, 3 and 5 as specific items. The thinking simply happens on the symbols 2, 3 and 5. That is, the numbers become the focus or "objects" of thought; not the items that underlie them. At this point, the activity with real-world items has been **abstracted to numbers and arithmetic**.

However, abstraction does not stop with number. After a further time, learners start to see that sometimes things are the same regardless of the size and type of the numbers. An example of this are "turnarounds" (what is mathematically called the commutative principle); that is, for any number, addition is the same regardless of the order in which numbers are added (e.g. 2 + 3 = 3 + 2; 656 + 172 = 172 + 656; $3\frac{1}{4} + 2\frac{2}{5} = 2\frac{2}{5} + 3\frac{1}{4}$, and so on). For this principle, letters such as x and y can be introduced as symbols for variables (i.e. to stand for "any number") and used to represent the principle, that is, $x + y = y + x$. (*Note*: The commutative principle can be extended to more than two numbers and to algebra, and it only holds for addition and multiplication.)

Similarly to numbers, when variables and their names and symbols (letters) are new to learners, meaning lies with the numbers that the variables could represent. For example, $2x + 3$ is thought of as two multiplied by "any number" plus 3. Solving $2x + 3 = 11$ means thinking like "I have a number, I multiply it by two, add 3 and end up at 11; to solve it, I subtract the 3 from 11 (get 8) and divide the 8 by 2 (get 4), so x = 4". The focus of thinking is on the numbers. However, over time as more experience is gained, it becomes less necessary to think of variables as numbers. The thinking simply focuses on the letters (e.g. $2x + 3x = 5x$ without thinking of x as a number). Thus, the variables become the focus or the "object" of thought. At this point, the numbers and arithmetic have been **abstracted to variables and algebra**. Overall, what this means is that the development from the real-world items to variables and algebra involves two steps: (1) abstraction from items to numbers and arithmetic, and (2) abstraction from numbers and arithmetic to variables and algebra. That is, algebra is an **abstraction of an abstraction** (see figure below).

| Step 1: Items | Step 2: Numbers and arithmetic | Step 3: Variables and algebra |
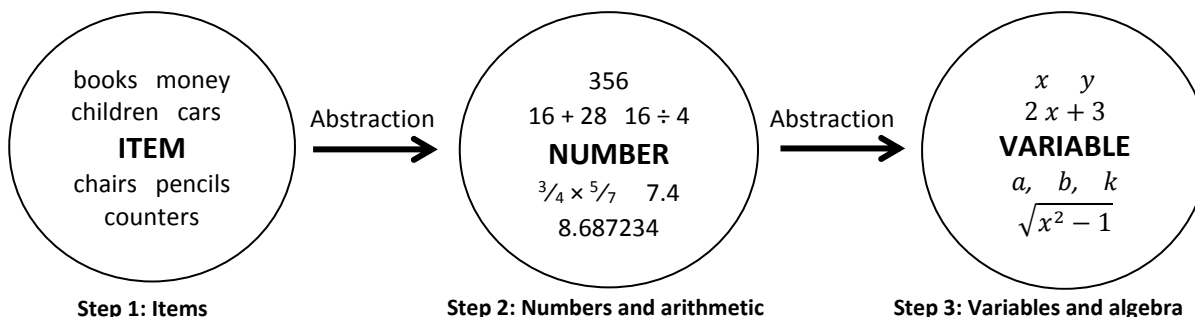
Interestingly, the process of abstraction involves gain and loss. Power is gained – we end up with knowledge that is much more portable and applies to a wider set of situations (i.e. the knowledge can be used in many situations). However, meaning is lost – the knowledge is highly symbolic and relationship back to the items it initially came

from becomes more difficult (i.e. the knowledge is in a form that appears disconnected from the real world). Therefore, it is important in both arithmetic and algebra to continually connect to the real world (as in the RAMR cycle), to show the role of symbols in mathematics, and to understand mathematics as a language. The Mathematics as Story Telling (MAST) approach is excellent for this (see **Appendix A**).

1.1.2   Generalisation, big ideas and holistic teaching

Recapping, the abstraction from arithmetic to algebra is an abstraction from particular activities represented by numbers and operations (i.e. arithmetic) to generalised activities represented by variables and operations (i.e. algebra). These generalised activities are interesting in that, to hold for all numbers, they must reflect structural things in arithmetic. In fact, they reflect what is called the underlying structure of arithmetic. This means, at its most powerful level, algebra reflects the **"big ideas" in arithmetic** – ideas that hold for whole numbers, fractions, and measures as well as variables.

These big ideas are always present in what we do in arithmetic but are often undeveloped. A particular example may help in illustrating this.

> **Example**: The new mental computation approaches to computation are recommending that addition tasks such as 25 + 48 should be done by a strategy called compensation; that is, 25 + 48 is calculated by changing one of the numbers to something easy to add and then compensating for this change on the other number. Because 50 is easy to add, we could change 48 to 50 by adding 2 and compensate by changing 25 to 23 by subtracting 2. In this way, the addition can be easily calculated (i.e. 25 + 48 = 23 + 50 = 73).

> Most teachers stop here; they teach the strategy then support students to use it on other examples. To build big ideas, they need to go further. The important question that should be followed up is, "why does this work?".

> The reason is that 23 = 25 − 2 and 50 = 48 + 2, so we are adding and subtracting 2. Since −2 and +2 are opposites or **inverses**, this is the same as adding 0, the **identity** (that which does not change anything). This means that the value of 25 + 48 does not change when it becomes 23 + 50 because all we are doing is adding 0. Putting in all the steps, what we have done is:

Start $\longrightarrow$ Finish

$$25 + 48 = 25 + 48 + 0 = 25 + 48 - 2 + 2 = 25 - 2 + 48 + 2 = 23 + 50 = 73$$

> However, the big idea behind compensation is more than the −2+2 in this example. The big idea is that a first thing always equals a second thing as long as all we do is add 0 or something equivalent to 0. Thus to work out something complicated, all we have to do is find something the same as zero which changes it to something simple. This is an idea that can help us right across all mathematics (that is why it is called a **big idea**). For example,

PROBLEM	ZERO	WORKING
238 + 387	− 13 + 13	238 + 387 = 225 + 400 = 625
4.8 + 2.7 + 3.6	+ 0.2 + 0.3 − 0.5	4.8 + 2.7 + 3.6 = 5 + 3 + 3.1 = 11.1
3h 48m + 2h 37m	+ 12m − 12m	3h 48m + 2h 37m = 4h + 2h 25m = 6h 25m
$a^2 - b^2$	− ab + ab	$a^2 - ab + ab - b^2 = a(a-b) + b(a-b) = (a+b)(a-b)$

This example shows a general method for teaching difficult additions by changing them to simple additions by finding things equivalent to zero to add to them. It is an example of algebra in action because the method does not identify the actual numbers to be used to bring about the change to a simpler-form addition. Basically, the method says that any numbers will do as long as they add and subtract to 0. Thus, the example provides evidence for the power of big ideas, underlying mathematics structure, and teaching mathematics using algebra. Algebra teaching that is based on the structure of mathematics can enable students to learn big ideas that they can apply to particular examples right across mathematics.

Note: The other way to change without changing is to × 1. Combining + 0 and × 1 together gives a more generic way to show equals or equivalence (e.g. it includes equivalent fractions and proportion). This really big idea is called equivalence of expressions and is the basis of algebra.

In many schools, the teaching of arithmetic tends to focus on mathematics as disconnected parts, teaching the next activity as if it is a completely new thing, and relying on the weight of years for students to put together all the bits to make a whole. We call this **part-to-whole** teaching. The example above shows us that algebra enables us to teach the more powerful mathematics where we learn a big idea and use it in particular situations. We call this **whole-to-part** teaching or **holistic teaching**.

Thus, algebra based on the structure of mathematics gives us a chance to teach holistically from the big picture down to the special case.

1.1.3   Connections, big ideas and structures

As a consequence, YDM argues that knowledge of the structure of mathematics, particularly of connections and big ideas, can assist teachers to be effective and efficient in teaching content. This is because it enables teachers to:

- *determine what mathematics is important to teach* – mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present;

- *link new mathematics ideas to existing known mathematics* – mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem solving;

- *choose effective instructional materials, models and strategies* – mathematics that is connected to other mathematics or based around a big idea commonly can be taught with similar materials, models and strategies; and

- *teach mathematics in a manner that makes it easier for later teachers to teach more advanced mathematics* – by preparing the linkages to other ideas and the foundations for the big ideas later teachers will use.

Thus it is essential that teachers know the mathematics that precedes and follows what they are teaching, because they are then able to build on the past and prepare for the future. Algebra is particularly important here. The two major structural connections for algebra are as follows: (a) algebra generalises arithmetic and so follows on and extends number and operations; and (b) algebra, like arithmetic, is part of measurement, probability and statistics, so algebra also has connections here (particularly in measurement where it relates to formulae). Thus, with input from geometry, we see that algebra is connected to all mathematics strands (as in figures below).
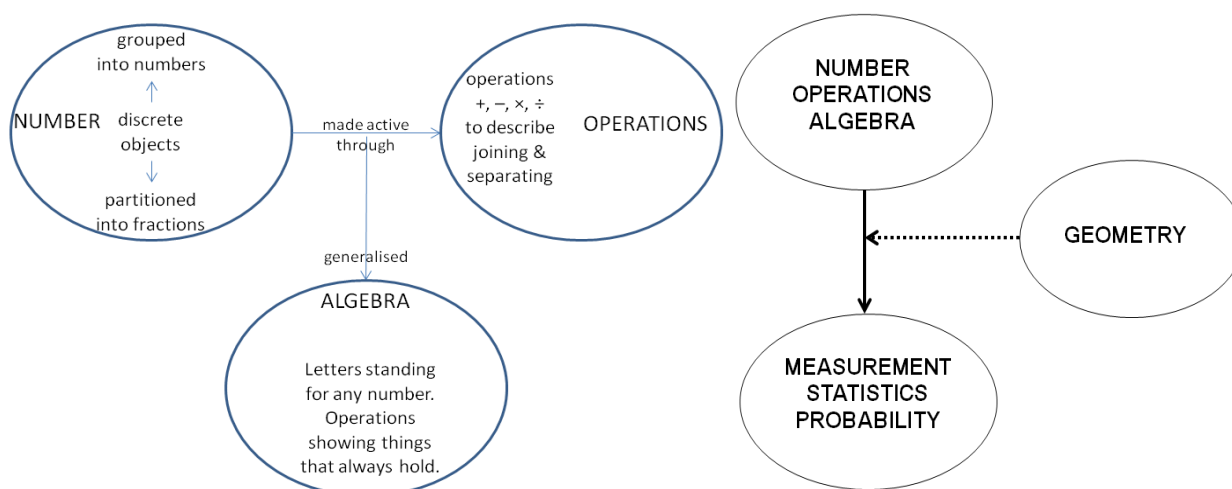
1.1.4 Algebraic big ideas

Big ideas are mathematical ideas which reoccur and are useful in many strands/topics of mathematics and across many year levels. There are five types of big ideas: global, concept, principle, strategy and teaching. The major big ideas for algebra are listed under these headings as follows.

1. **Global**. These are big ideas that apply very widely (some apply to all mathematics). An example is "symbols tell stories", that is, symbols are a concise shorthand language for describing the world. This concept applies to arithmetic, algebra, geometry and all of mathematics. The algebra global big ideas are those from operations: symbols tell stories, relationship vs change, interpretation vs construction, accuracy vs exactness, and part-part-total, plus extra emphasis on unnumbered to numbered.

2. **Concept**. These are meanings of central ideas, such as the concept of place value (also applies to mixed numbers and to measures) or the concept of subtraction (applies to all types of numbers and to algebra). Often there is more than one concept for each term (e.g. the part-of-a-whole concept of fraction and the division concept of fraction). Algebra concept big ideas are similar to those from operations but with some extra that are particular to algebra, as follows:

   - operation concepts – four operations, equals, order (> and <); and
   - new algebra concepts – generalisation, expression, equation, unknown, variable, linear, and function.

3. **Principle**. These are relationships whose meaning is encapsulated in the relationship between the components of the idea not in the actual content focus of the idea, such as the formulae for the volume of a cylinder or the distributive principle (e.g. $24 \times 3 = 20 \times 3 + 4 \times 3$; $6 \times 7 = 6 \times 2 + 6 \times 5$). Again, algebra principle big ideas are similar to those from operations but with some extra that are particular to algebra, as follows:

   - operation principles (field principles) – closure, identity, inverse, commutativity, associativity, distributivity, compensation, equivalence, inverse relation, and triadic relationships;
   - equals/order principles – reflexivity/nonreflexivity, symmetry/antisymmetry, and transitivity; and
   - new principles – balance rule, backtracking, expansion, factorisation, changing subject of a formula.

4. **Strategy**. These are general "rules of thumb" that point towards a solution of a problem or procedure. For example, the separation strategy for adding numbers of two or more digits (applies to subtracting mixed numbers, measures and variables as well as whole numbers), or the problem-solving strategy of make a drawing, diagram or graph (which is universally applicable to problems of any type). The strategies are similar to operations – separation, sequencing, and compensation.

5. **Teaching**. These are big ideas for teaching mathematics (not for mathematics itself) that apply to a variety of, if not all, teaching situations. An example is the RAMR reflection strategy of "reversing". For algebra, these are the same as for other areas and include the RAMR steps. This is elaborated further in section 1.3.

A complete list of big ideas can be found in the YDM *Big Ideas* supplementary resource.

1.2 Sequencing

This section looks at how YDM advocates that algebra ideas be sequenced. The sequence relates to how the ideas are presented in this book through the sections of the book and the sequencing within and across these sections. It is based on the underlying ideas discussed below.

1.2.1 Underlying ideas

The sections of the book have been constructed based on the following points.

1. **Process of generalisation**. Because algebra is the generalisation of arithmetic, students will have to be able to generalise – to be able to undertake the process of generalisation.

   This process of grasping a pattern from particular examples has the following attributes:

(a) there are **two parts** to generalisation – finding/determining the generalisation, and expressing the generalisation; and

(b) there are **four stages** in students' ability to express a generalisation from particular examples – using examples close to those that have been given (often using gestures to express themselves), using large numbers (called quasi-generalisation), using language, and using a variable (e.g. n).

2. **Change and relationship**. There are two perspectives from which algebra can be viewed and approached – transformation (e.g. change and functions) and relationship (e.g. equivalence and equations). These have to be integrated so that both support each other. However, it is easier at the start to develop them separately.

3. **Sequencing arithmetic to algebra**. This sequencing has to take into account the following.

(a) There is a big difference between arithmetic (e.g. 3 + 4) and algebra (e.g. x + 4) – arithmetic process (3 + 4) is separate from product or answer (7) while algebra process and product are the same (both x + 4). This means that the important ideas that sequence from arithmetic to algebra are concepts, principles and strategies not answers.

(b) There are three general ways to develop the concept of variable as an unknown number: (i) using letters to express a generalisation for a pattern (or a function machine); (ii) solving changes and relationships in terms of unknowns; and (iii) relating numbers through formulae (e.g. volume of a cylinder is $\pi r^2 \times h$).

(c) There are two sequences to be followed: (i) arithmetic → pre-algebra → full algebra; and (ii) one-operation arithmetic → two-operation arithmetic and one-operation algebra → two-operation algebra. **Pre-algebra** is where we have an unknown or variable but find it through calculations only with numbers (e.g. $3x$ + 1 = 16 is solved by 16 − 1 = 15 ÷ 3 = 5). Full algebra is examples like $3x$ + 1 = $2x$ + 7; solving this requires calculating $3x$ − $2x$ which is an algebraic calculation. **Two-operation arithmetic** is when we have two operations and students think about it as two operations rather than a series of single operations. For example, 3 × 2 + 7 = 6 + 7 = 13 is seeing 3 × 2 + 7 as a series of single operations. In algebra, $3x$ + 7 cannot be solved this way so students need to learn to think of it as two operations. This can be encouraged with examples like: *Solve (3 + 4) × 7 without adding 3 and 4*.

4. **Generalisations themselves.** Because algebra is the generalisation of arithmetic, it is important to know and understand common generalisations – to learn some of the important generalisations (i.e. concept, principle and strategy big ideas) that enable a smooth sequence from arithmetic to algebra, because they are in both topics. It is also important to develop new big ideas that cover the change from arithmetic to algebra such as the concept of unknown and concept of variable.

5. **Linearity and nonlinearity**. Early primary activities are generally additive: *I bought a shirt for some money and jeans for $50, how much did I pay altogether?* This example can be represented by x + 50. Upper primary becomes multiplicative: *I bought 3 shirts for some money and jeans for $50, how much did I pay altogether?* This can be represented by $3x$ + 50. Most upper primary and junior secondary algebra is of this form, known as **linear** because its graph is a straight line. However, primary and junior secondary mathematics has to prepare for, and begin doing, **nonlinear** algebra (where graphs are not straight lines) – notably squares (quadratics) and cubics. For example, the problem *I bought a square piece of carpet and a rectangular piece (2m by 4m), how much area did I buy?* is represented by a quadratic expression, that is, x^2 + 8 square metres. Thus, nonlinearity will be part of the activities in this book.

6. **Teaching big ideas**. In the early years, algebra is not about x's and y's; it is about doing and understanding arithmetic in a deeper way that builds arithmetic structure and prepares students for algebra. In the upper primary and early secondary years, it is about **understanding the world algebraically**, manipulating equations and expressions, solving equations, and expressing and representing functions. [*Note:* An expression is a number sentence without an equals sign, thus an equation is two expressions with an equals sign between them.] To achieve this, the book will focus on activities that are based on:

(a) teaching from unnumbered to numbered activities, and then to variables;

(b) finding and expressing generalisations (with the four stages of expression taken into account); and

(c) ensuring that abstract symbols such as $2y + 3$ are given meaning in reality (which means a heavy emphasis on symbols → reality and reality → symbols).

1.2.2 Major components of the book

Following from the underlying ideas above, the book is built around activities that teach: (a) the process of generalisations; (b) the transformation and relationship perspectives (and the big ideas within them); and (c) the major generalisations (concept, principle and strategy big ideas) themselves. This will mean four teaching sections as follows.

1. **Patterns**. Repeating and growing patterns will be used to teach generalising and to introduce the notion of variable. Patterns make students consider sequences of terms, both visual and numerical, and to determine the rule that determines, for example, the 10th term, the 100th term, the 256th term and the general term, namely, the nth term. Many curricula devalue the repeating pattern. However, as will be seen in this resource, repeating pattern activities enable powerful generalisations to be found. Nonlinear as well as linear examples will be included.

2. **Functions**. Transformations (change) and functions will be studied to build transformational algebra which is based on function machines, input–output tables, arrow diagrams, and inverse. It will also lead to solving equations through backtracking, and to functions. Once again, there will be linear and nonlinear examples.

3. **Equations**. Equivalence and equations will be studied to build relationship algebra which will be based on the balance principle and lead to solving for unknowns. Solutions will predominantly focus on linear equations as quadratics are introduced in Year 10, but some nonlinear outcomes will be considered.

4. **Arithmetic-algebra principles**. Activities will be shown to develop the major principles that are a result of generalisation, namely, the principles of equals and order (called the equivalence and order principles), the number-size principles, and the principles of operations or arithmetic (called the field principles). Attention will also be focused on formulae which contain quadratic and cubic forms (e.g. area, volume).

1.2.3 Sequencing

The overall sequence for algebra is given in the figure below. It relates to the four sections of the book (see 1.2.2 above): *Repeating and Growing Patterns*, *Change and Functions, Equivalence and Equations* (note the use of equivalence instead of equals), and *Arithmetic-Algebra Principles*.
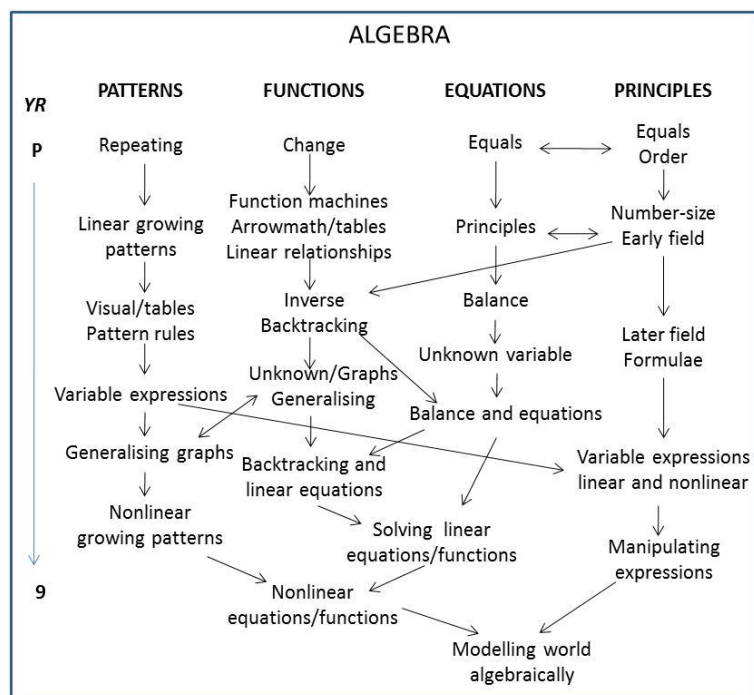
It begins with **patterns** as training in the act of generalisation by finding pattern rules and relating to graphs. It then moves onto **functions**, starting from change rules in transformations, using real situations, tables and arrowmath notation before equations and graphs, solving for unknowns by the use of the balance rule. After this it moves to relationships that in arithmetic and algebra are represented predominantly by **equations**, solving them by the use of the balance rule. The sequence is completed by focusing on arithmetic and algebraic **principles** and extending these to methods such as substitution, expansion and factorisation.

The sequence begins simply, in a separated manner, but by the time junior secondary is reached, the components are more integrated and connected to allow patterns, functions and equivalence all to be expressed in the same way (by equations), and for results to cover nonlinear as well as linear relationships and changes.

Within each of the sections the algebraic ideas will be sequenced as in the figure above. The sequencing will begin with unnumbered activities as these enable the big ideas to develop, move on to numbers and arithmetic situations and then move to generalised situations. More detailed sequences are given in Chapters 2 to 5.

The overall end point of algebra is modelling as well as manipulation of symbolics. Computers and special calculators can do the manipulations to simplify and solve for unknowns – what is important, like in arithmetic, is to apply the knowledge to the world and solve problems – to model the world algebraically. This is important because most students cannot see the relevance of, say, $x + y = 7$ to their everyday world. Yet, with understanding it is very relevant. It could mean that you bought two things at a shop for $7. Then the cost of the first thing (x) plus the cost of the second thing (y) is equal to $7. This gives parameters in which thinking can be used. Suppose we were working in whole dollars. Then the first thing could cost $1 and the second cost $6, or $2 and $5, or $3 and $4, and so on. Thus we need to teach students the role of symbols in telling stories (see **Appendix A**).

1.3 Teaching and cultural implications

This section covers implications for teaching and culture. More emphasis than normal falls on this section because algebra is holistic and best taught whole to part and this is similar to the cultural learning style of Aboriginal and Torres Strait Islander people (and low SES students). Thus, two outcomes coalesce: first, algebra understanding is best facilitated through big ideas, and second, this approach to teaching facilitates Aboriginal, Torres Strait Islander and low SES learning of algebra, and is a vehicle for mathematics understanding.

1.3.1 Teaching, algebra and big ideas

Two important points to note about big ideas are that, firstly, they are **generalisations** of particular ideas that applied initially to the particular numbers at which we were looking. Secondly, because they encompass many numbers and topics, they are **holistic** in their focus.

The principle big ideas are particularly powerful for both arithmetic and algebra; for example, the "turnaround" principle. This is an important principle for basic number facts; it says that a first number add a second number is the same as its reverse (the second number add the first number – e.g. 3 + 5 = 5 + 3). However, this principle also holds for large numbers (e.g. 3789 + 2094 = 2094 + 3789), fractions (e.g. $^2\!/_5 + {}^7\!/_8 = {}^7\!/_8 + {}^2\!/_5$), decimals (e.g. 4.39 + 6.68 = 6.68 + 4.39), algebra (e.g. $a + b = b + a$), and functions (e.g. $f + g = g + f$ where $f(y) = 2y - 1$ and $g(y) = 3y - 2$). Thus, as a mathematical idea, "turnaround" is a **big idea** because it applies to any content and across Years P to 9. As such, it has a special name within the structure of mathematics – the *commutative* principle.

Thus algebra and big ideas are almost synonymous. Both can be seen to reoccur across many topics and strands, be generalisations and be holistic in focus.

As a consequence, algebraic thinking and ideas, along with big ideas and generalities, are a very important part of mathematics because they have the following important attributes:

- they last learners many years;
- they connect many mathematical ideas;
- they cover many mathematical situations;
- they reduce the amount of mathematics that has to be learnt; and
- they are holistic.

Therefore, algebra is also very important and powerful in terms of teaching and learning mathematics in general. This can be seen in the examples in **Appendix B**.

1.3.2   Big ideas that affect algebra teaching and learning

A global big idea that is very important to how we learn and use algebra is relationship vs transformation. Let us consider three examples.

The first is **potatoes being cooked** into chips. We can consider this as a relationship, the potatoes and chips are the same food; we can consider this as a change, the potatoes have been changed to chips by cooking. This gives rise to two different ways of thinking about, and two different symbol organisations for, one mathematical idea – *same as* and *equals*, and *change* and *arrow*, as below.
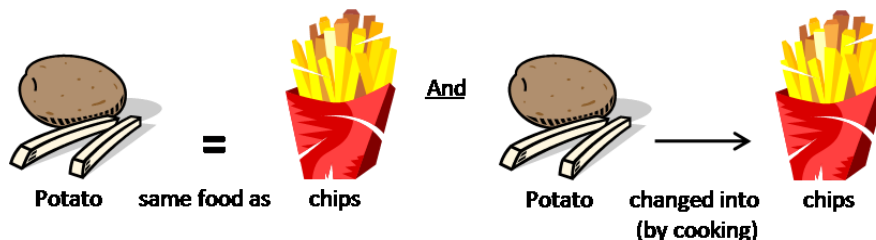
The second is **a balloon being blown up**. The half-filled balloon and the fully filled balloon can be considered to be related because they are the same shape although different in size. However, from another perspective, the little balloon could be considered to have been changed into the large by being blown into. Again we have two different symbols and two different ways of thinking (see below) for one mathematical idea.
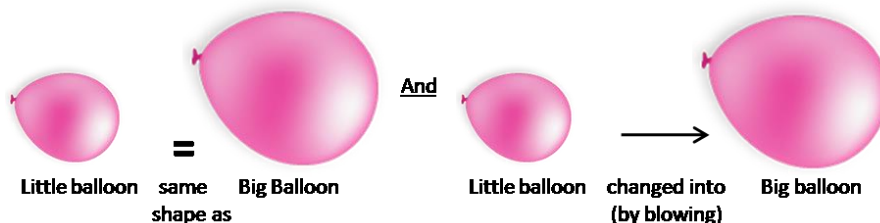
The third is **addition**. Consider 2 and 3. The joining of 3 to 2 could be considered as a relationship, *2 and 3 relate to 5 by addition*. However, it can also be considered as a change, *2 can be changed to 5 by the action of +3*. Again, one mathematical idea but two ways of thinking and two different forms of symbols, as seen below.

$$2 + 3 = 5 \qquad 2 \xrightarrow{\;+3\;} = 5$$

So, the everyday activities such as addition can be thought of in two ways and result in two symbol systems even though they only represent one mathematical idea. One way uses equations to express relationships such as that between 2, 3, 5 and addition; the other way uses arrows to show change such as how 2 can be changed to 5 by addition of 3. The first way leads to algebraic equations; the second to algebraic functions.

Thus, YDC advocates that an effective way to learn algebra is through both relationship and transformation perspectives. For example, when we are algebraically describing or modelling a shopping situation such as *the cost of a coat is $1 more than double the cost of the pants*, we can: (a) think of the situation as a relationship and express it as an **equation**, *the value of the coat (y) is equal to the value of the pants (x) multiplied by 2 plus 1,*

that is, $y = 2x + 1$ (in formal equation notation); and (b) think of the situation as a transformation and express it as a **function**, *the value of the pants (y) is found by doubling the price of the pants (x) and adding a dollar*, that is, $x \overset{\times 2}{\rightarrow} \overset{+1}{\rightarrow} y$ (in arrowmath notation).

As well, there are three important big ideas within the relationship and transformation perspectives:

(a) within relationship, the big idea is the **balance principle** – that to keep an equation equal, whatever is done to one side of the equation has to be done to the other side;

(b) within transformation, the big idea is the **backtracking principle** (based on inverse) – that change can be undone by reversing the operations in reverse order (this is covered in **Appendix B**); and

(c) within both perspectives, teaching and learning should focus on **unnumbered situations before numbered situations**.

Finally, because algebra is the generalisation of arithmetic, it will be necessary to focus on the development of the new concept of variable as standing for any number and on the big ideas from arithmetic that carry through into algebra (e.g. concepts of operations and equals, principles associated with operations and equals).

As a result of the above, YDC has structured the teaching ideas in this book to ensure that students understand:

- the relationship and transformation perspectives of algebra,
- the balance and backtracking big ideas,
- unnumbered before numbered pedagogy, and
- the equals and operations big ideas (because algebra is a generalisation of arithmetic).

There is one more aspect to take into account. Most of the work done in Years P to 9 with algebra is linear in that it relates/changes in terms of multiples or divisors of unknowns, not squares or cubes of unknowns. This means it is restricted to equations of type $3x - 2 = \frac{x}{2} + 1$. However, by Year 9, there are some nonlinear relationships, for example, area of a rectangle or volume of a cube. This means that we have to take into account some nonlinear activities to go with the predominantly linear activities. This will affect the end of each chapter in this book.

1.3.3 Indigenous culture, mathematics and holistic teaching

A danger in teaching Western mathematics (and science) to Aboriginal and Torres Strait Islander people is that teachers can make their teaching become a celebration of the growth and success of Western or European knowledge. It is particularly easy to represent Western knowledge as successful because it can be presented as continually advancing in terms of technology (e.g. cars, planes, rockets, computers) and as coming to dominate the planet. However, this same knowledge has been particularly unsuccessful in handling the intransigent and long-term problems of the planet such as destruction of the environment, poverty, war and violence, and climate change.

Teaching that presents mathematics as a celebration of this "linearly advancing technological process" can marginalise Aboriginal and Torres Strait Islander people, undermine the significance of their Indigenous identity and devalue Indigenous knowledges and cultures as simplistic societies (Matthews, 2003). Western mathematics places importance on number and arithmetic because this is where linear advancement in technological progress starts and what drives its progress. It can be argued that the invention of arithmetic was a consequence of a society in which material assets were considered more important than the individual. In Indigenous society, without the need to work out one's assets in fine detail, number was not developed to the same level as in Western culture. However, this does not mean that Aboriginal and Torres Strait Islander cultures do not have their own mathematical knowledge.

Considerations of Indigenous knowledge of mathematics require recognition and respect of such knowledge, which should, in turn, be reflected in the teaching and learning of mathematics to students. For example, as argued by Matthews (2003), Yolngu children, from a young age, have a good understanding of their kinship

system which governs the Yolngu way of life. This system is very complex and relies on cyclical and recursive patterns. Such patterns can be found within numbers themselves and other areas of mathematics (Jones, Kershaw & Sparrow, 1996; Divola & Wells, 1991) and forms a good basis for Yolngu children to start their journey into Western mathematics. As most Aboriginal and Torres Strait Islander knowledge systems are based on interactions within the environment and groups of people, they can form algebraic systems because they can relate numbers in flexible ways.

Traditionally, in Australian schools, mathematics and its teaching both reflect Western culture. Therefore, differences in mathematics performance can stem from a different cultural view of what it means to be good at mathematics. Commonly, in most school environments, this is determined by gauging students' performance levels from test items that reflect non-Indigenous learning styles, namely solving meaningless problems by pen-and-paper means. In those problems, there are often marked differences in errors between Indigenous and non-Indigenous students. One case study of Indigenous students' errors found that underperformance tended to reflect mistakes in procedures rather than understanding, reflecting the position of Grant (1998) that Indigenous students see the whole rather than the parts.

Therefore, it is important to teach mathematics on an equitable basis with Western mathematics reflecting "both ways" approaches (Ezeife, 2002). Western teaching is traditionally compartmentalised, resulting in an education system in schools (whether oral or written) focusing on the details of the individual parts rather than the whole and relationships within the whole. By contrast, Indigenous students tend to be holistic learners, appreciating overviews of subjects and conscious linking of ideas (Grant, 1998). In fact, Indigenous people have been characterised as belonging to "high-context culture groups" (Ezeife, 2002) which are characterised by a holistic (top-down) approach to information processing in which meaning is "extracted" from the environment and the situation. Low-context cultures use a linear, sequential building block (bottom-up) approach to information processing in which meaning is constructed (Ezeife, 2002).

What this means is that students who use holistic thought processing are more likely to be disadvantaged in mainstream mathematics classrooms. This is because Westernised mathematics is largely presented as hierarchical and broken into parts with minimal connections made between concepts and with the students' culture and community. It potentially conflicts with how they learn. If this is to change, curriculum and assessment need to be made more culturally sensitive and community oriented (QUT YuMi Deadly Centre, 2009).
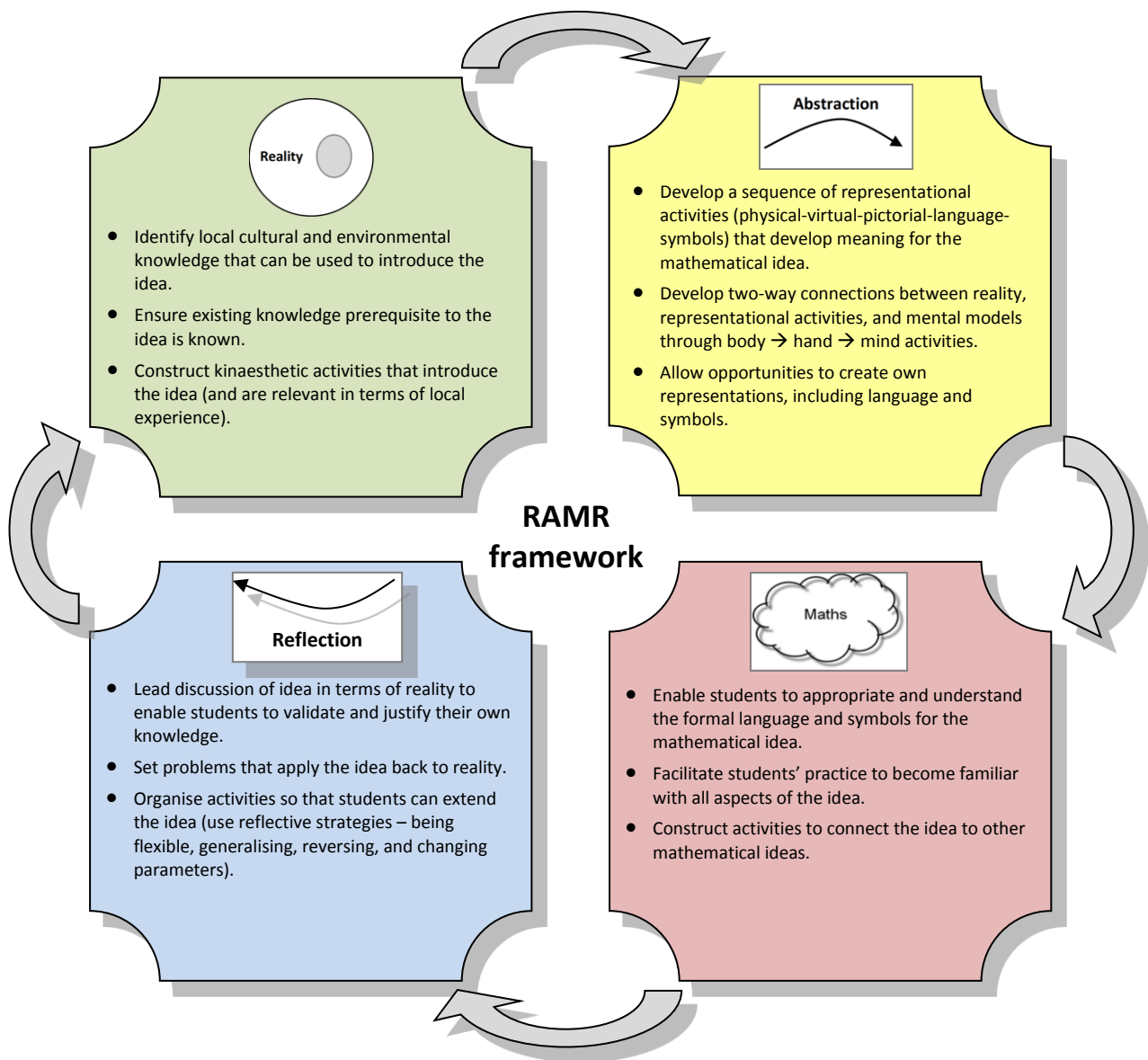
Thus, we have a confluence of results. Indigenous students are high context and learn best with holistic teaching (Grant, 1998; Ezeife, 2002). Mathematics in its most powerful form is based on structural understanding that is learnt best by holistic teaching. Algebra is the component of mathematics that is based on mathematical structure, and is capable of presenting mathematics holistically.

As a consequence, it seems that algebra is the form of mathematics that is most in harmony with Indigenous culture and learning style. Because of this, algebra understanding should be a strength of Indigenous students if it is taught through pattern and structure (rather than through sequential teaching of rules and algorithms). It seems likely that algebra is a subject in which Aboriginal and Torres Strait Islander students should excel. Finally, because of its relationship with arithmetic, this understanding of algebra should enable enhanced understanding of and proficiency with arithmetic.

1.3.4   Generic teaching strategies

YDM sees mathematics teaching as comprising three components – *technical* (handling materials), *domain* (the particular pedagogies need for individual topics) and *generic* (pedagogies that work for all mathematics). Interestingly, and fortunately, the domain section is not as complicated as it could be because mathematical ideas that are structurally similar can be taught by similar methods. For example, fractions and division are similar and both are taught by partitioning sets into equal parts – except that the set is seen as one whole for fractions and as a collection of objects for division. There are also some generic teaching methods that hold for any topic.

The **RAMR framework** (see the figure below) is very useful for algebra because of the generic teaching ideas contained in the framework. For a start, it grounds all mathematics in reality and provides many opportunities for connections, flexibility, reversing, generalisations and changing parameters, as well as body → hand → mind. The idea is to use the framework and all its components throughout the years of schooling and this will help prevent learning from collapsing back into symbol manipulation and the quest for answers by following procedures.

Reality

- Identify local cultural and environmental knowledge that can be used to introduce the idea.
- Ensure existing knowledge prerequisite to the idea is known.
- Construct kinaesthetic activities that introduce the idea (and are relevant in terms of local experience).

Abstraction

- Develop a sequence of representational activities (physical-virtual-pictorial-language-symbols) that develop meaning for the mathematical idea.
- Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.
- Allow opportunities to create own representations, including language and symbols.

Reflection

- Lead discussion of idea in terms of reality to enable students to validate and justify their own knowledge.
- Set problems that apply the idea back to reality.
- Organise activities so that students can extend the idea (use reflective strategies – being flexible, generalising, reversing, and changing parameters).

Maths

- Enable students to appropriate and understand the formal language and symbols for the mathematical idea.
- Facilitate students' practice to become familiar with all aspects of the idea.
- Construct activities to connect the idea to other mathematical ideas.

RAMR framework

Within the chapters of this book, many activities for teaching algebra are provided. Although they are based on the RAMR model of reality, abstraction, mathematics, and reflection, they are not presented in the formal RAMR framework format used in the other YDM books. Rather, the focus is on the sequencing of the activities across Years P to 9 and the sequence of steps that should be followed within each activity. It is expected that teachers will create their own RAMR lessons when teaching these activities. Section 1.4 provides more information on the components of the RAMR model that are particularly important in algebra teaching.

1.3.5 Cultural implications for teaching algebra

There are two implications for algebra from the discussion above: (a) what is the best way to teach it, and (b) what is the best way to teach it to Aboriginal and Torres Strait Islander and low SES students?

1. **Teaching algebra**. The power of mathematics lies in the structured way it relates to everyday life. Knowledge of these structures gives learners the ability to apply mathematics to a wide range of issues and problems. This is best achieved if the knowledge is in its most generalised form, which is algebraic form. Thus, the most

effective way to present mathematical knowledge is through algebra. However, any topic of mathematics can be presented instrumentally (as a set of rules). Although algebra is the direction for power in mathematics, it has to be algebra that is presented structurally, showing the generalisations that can be used in many examples. Powerful algebra teaching focuses on extending arithmetic to generalisations that can apply across all arithmetic. That is, teaching that builds holistic understandings of structure that can then be applied to particular instances (from the whole to the part). If students are fortunate enough to gain this structured understanding of mathematics, the subject becomes easy. This is because it is no longer seen as a never-ending collection of rules and procedures but rather the reapplication of a few big ideas.

2. **Teaching Indigenous students**. Aboriginal and Torres Strait Islander students tend to be high context. Their learning style is best met by teaching that presents mathematics structurally without the trappings of Western culture. Powerful Indigenous teaching is therefore holistic, from the whole to the part. As Ezeife (2002) and Grant (1998) argue, Indigenous students should flourish in situations where teaching is holistic (from the whole to the parts). Thus, holistic algebra teaching has two positive outcomes for Indigenous students: (a) it teaches a powerful form of mathematics, and (b) it teaches it in a way that is in harmony with Indigenous learning styles. Algebra taught structurally, then, is something in which Indigenous students should excel. However, this is just a general finding. What does this mean in practice for the teaching of algebra? It means that we will not be teaching rules for manipulating letters. Letters and algebraic expressions and equations will be understood in terms of everyday life and algebraic ideas will be generalised from arithmetic. This will mean a lesser focus on algorithms and rules, and a greater focus on generalisations and applications to everyday life.

3. **Teaching low SES students**. Interestingly, holistic teaching is also positive for low SES students. Three reasons are worth noting. First, low SES students tend to have strengths with intuitive-holistic and visual-spatial teaching approaches rather than verbal-logical approaches. Thus, an algebraic focus on teaching mathematics should also be positive for low SES students. Second, many low SES students in Australia are immigrants and refugees from cultures not dissimilar to Aboriginal and Torres Strait Islander cultures. They are also advantaged by holistic algebraic approaches to teaching mathematics. Third, many low SES students have themselves experienced failure in traditional mathematics teaching, and so have members of their families. This results in learned helplessness with regard to mathematics and what is called *mathaphobia*, where students believe that no effort on their part will enable them to learn mathematics. Holistic-based algebraically-oriented teaching of mathematics is sufficiently different that students may not apply their phobia to it – particularly if taught actively and from reality as in the RAMR model.

Thus, for the students that YDM was developed for, algebra is the key for mathematics success – not x's and y's but the generalised holistic thinking that is the basis of it.

1.4   Overview of book

This book consists of five chapters and two appendices:

Chapter 1: Purpose and overview – describing purpose, connections and big ideas, sequencing, and teaching and cultural implications;

Chapter 2: Repeating and growing patterns – describing activities for developing the ability to generalise for repeating and growing patterns and using this to introduce variable;

Chapter 3: Change and functions – showing how function machines develop the notion of inverse and can be used to build the concept of variable and backtracking to solve for unknowns;

Chapter 4: Equivalence and equations – showing how understandings of equivalence (equals) and equations can be developed and used to teach the balance principle and solve equations for unknowns;

Chapter 5: Arithmetic-algebra principles – showing how the major generalisations, the principles for equals and operations, can be taught;

Appendix A: Mathematics as Story Telling (MAST); and
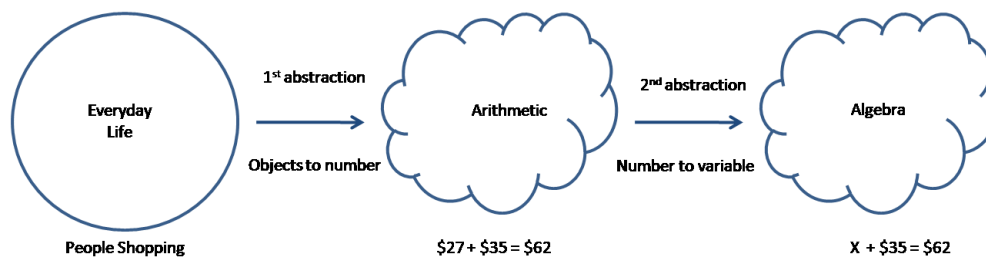
Appendix B: The power of algebraic big ideas.

The instruction for the ideas within the chapters is based on the RAMR model (reality, abstraction, mathematics, and reflection). However, like all strands, there are particular components of the RAMR model that should be highlighted because they have particular importance in algebra. Four of these are as follows.

1. **Initial abstraction as unnumbered**. Within most of the chapters, instruction begins with unnumbered activities. This is because students appear more easily able to look for patterns and generalisations in unnumbered activities than in numbered situations, where they tend to look for answers. Thus, the activities in this book, as far as possible, start with unnumbered activities, then move to numbered activities and then to variable activities. This involves generalisation in which students progress from working with small numbers, to large numbers (this is called quasi-generalisation), everyday language, and finally, formally with variables.

   The figure below illustrates this diagrammatically.

Unnumbered activities		
↓		Small to large numbers
Numbered activities	**GENERALISATION**	Everyday language
↓		↓ Variables
Variable activities		
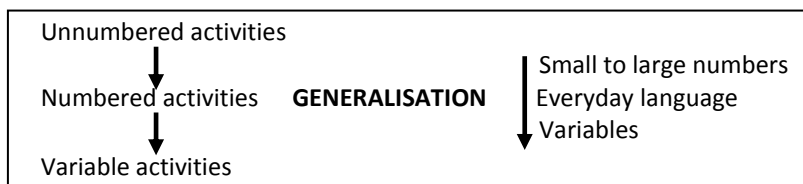
2. **Connecting symbols and reality**. As also discussed earlier, the relationship between everyday life and algebra is a two-step abstraction that goes through arithmetic (see below). This means, first, that the act of generalising is at the core of algebra and proficiency must be built in both the act of generalising (how to generalise) and the products of generalisation (the mathematical ideas/principles that result from generalising). Second, it means that the symbols of algebra, notably the letters, are far removed from everyday life and their meaning must be built with care through: (a) continuous connections being made between symbols and real-world stories; and (b) using sequences of materials and activities that become progressively more abstract.

   Everyday Life — People Shopping
   1st abstraction — Objects to number
   Arithmetic — $27 + $35 = $62
   2nd abstraction — Number to variable
   Algebra — X + $35 = $62

3. **Pre-empting and prerequisites**. The figure above also explains why algebra is difficult for many students. It shows that the step from everyday life (reality) to arithmetic must be well built because the step from arithmetic to algebra is built upon it. It is very difficult for students to invent stories for $2x - 1$, if they cannot invent stories for $2 \times 5 - 1$.

4. **Generalising and reversing**. As algebra is generalisation of arithmetic, the reflection step of generalising is crucial – all activities should be discussed in terms of what they mean in general – what would "any number" do? As well, to reach the generalisation often involves many steps – always **reverse** these steps in the next activity.

2   Repeating and Growing Patterns

There are two pattern types to explore in algebra to promote early algebraic thinking and introduce the notion of variable, namely repeating and growing patterns. Repeating patterns are simplest for introducing activities that engage students in noticing and identifying patterns, whereas growing patterns introduce more complex relationships between terms. Thus we begin with repeating and then show how these can be extended to growing patterns. However, as we show, repeating patterns can be used to introduce variable as well as equivalent fractions and ratio.

There are two types of growing patterns, linear and nonlinear. We will be focusing on linear growing patterns but there is a section on nonlinear at the end of this chapter. In linear growing patterns we focus on pattern rules (sequential and position) and on the techniques for identifying generalisation (using visuals and tables). After this, patterns are used to introduce variable, equations and graphs, with particular emphasis on the relationship between growing part and constant part and coefficient of variable/slope and constant term/y-intercept. As well, there is some time spent on patterns in other strands of mathematics that can lead to deeper understandings of those strands – this is particularly in terms of number and operations. This sequence is diagrammatically summarised on the right.

Repeating patterns
(identify repeat/determine position)

Linear growing patterns
(sequential/position rules)

Visuals and tables methods

Variable/Generalising
position and repeats

Variable/Generalising
pattern rules

Equations/Graphs
Linear relationships

Nonlinear
growing patterns
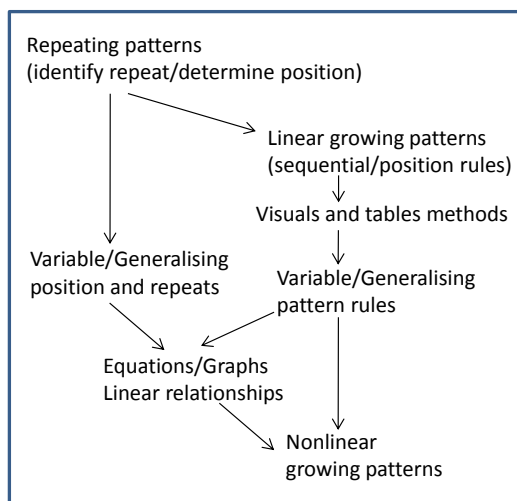
The sequence for patterns and this chapter is to look first at the mathematics content in terms of pattern types, sequences and materials (section 2.1), followed by very early repeating patterning (section 2.2), early to middle primary linear growing patterning (section 2.3), later linear growing and repeating patterning and applications to graphing (section 2.4) and later nonlinear patterning (section 2.5).

2.1   Pattern types, sequences and materials

In this section, we cover and summarise all the components of this chapter.

2.1.1   Repeating patterns

Repeating patterns are linear sequences of objects, pictures or numbers that form a pattern because a section of them repeats; for example:

0 x 0 x 0 x 0 x 0 x 0	repeating part: 0 x
o l l o l l o l l o l l o l	repeating part: o l l

The crucial skill is to be able to go from pattern to repeating part and repeating part to pattern. The normal sequence of activities is as follows:

(a) copying, continuing and completing repeating patterns and identifying repeating part;

(b) constructing repeating patterns (without direction and when given a repeating part);

(c) finding what object is at a position or finding positions for objects (e.g. what object is the 13th term?; what terms are square red counters?);

(d) breaking pattern into repeats and connecting to growing patterns, for example:

0 x 0 x 0 x 0 x … → Ox Ox Ox Ox … → Ox Oxx Oxxx Oxxxx … ;

(e) representing repeats on tables and generalising tabled numbers to variables; and

(f) using tables of repeats to introduce fractions, equivalent fractions, ratio and equivalent ratio (proportion).

The common materials to be used consist of objects of different colour, size, shape, and so on, with any one or more attributes determining the basis of the pattern, plus small numbered cards (1 to 5), tables, and calculators. It is useful to have magnetic copies of these objects/cards that can be placed on whiteboards.

The major ideas to be developed from repeating patters are: (a) generalisation, (b) representing generalisations with variables (introduction to algebra), (c) fractions and ratio, and (d) equivalent fraction and ratio.

2.1.2 Linear growing patterns

Growing patterns are series of terms where there is a fixed part and a growing part as on right. If the growing amount is always the same, then it is linear. In the pattern on the right, 0 is fixed and X is growing by one each time.

```
                              X
                       X      X
                 X     X      X
OX        OX     OX    OX     OX
```

When the growing part does not grow (grows by zero), you have a repeating pattern as on right.

```
OX        OX        OX        OX
```

It is possible for the fixed part to not exist (to be zero) as on right.

```
                         X
                  X      X
           X      X      X
    x      x      x      x
```

Pattern rules and graphing

The focus on growing patterns is to identify what is called the *pattern rule* which describes the growth. For patterns like that on the right, there are two types of rules.

```
                              X
                       X      X
                 X     X      X
OX        OX     OX    OX     OX
```

(a) *Sequential*: the nth term is the previous term + 1.

(b) *Position*: the nth term is $1 + n$, since:

1st term is	1 O and 1 X	(1 + 1)
2nd term is	1 O and 2 Xs	(1 + 2)
3rd term is	1 O and 3 Xs	(1 + 3)
and so on		

Position rules enable linear growing patterns to be used to introduce the notion of variable. They also can be used to plot graphs as straight lines. When this is done, a relation exists between the graph, the growing part and the fixed part − the growing part is the slope and the fixed part is the y intercept (for $y = x + 1$, slope is 1 and y intercept is 1) as shown on right.

Note: In previous times, sequential pattern rules (e.g. "1 more") were considered to be trivial. With the growth of computers this has changed. In the example above, the position pattern rule gives the function $y = x + 1$ but the sequential pattern rule gives the function $y(1) = 2, y(k + 1) = y(k) + 1$. This is now how functions are represented in programming.

$$y = x + 1$$

Slope = 1

y intercept = 1
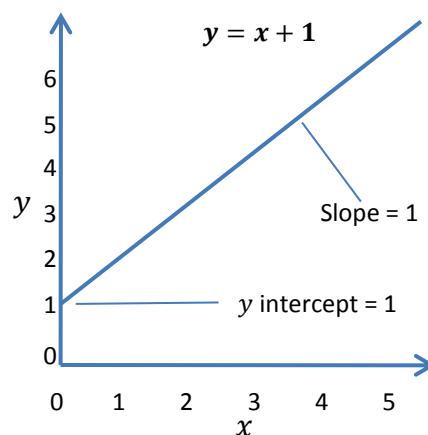
Reversing

For all activities, it is crucial to go from pattern to pattern rule **and** pattern rule to pattern. The normal sequence of activities is as follows:

(a) copying, continuing and completing growing patterns;

(b) constructing growing patterns (without direction and when given a pattern rule);

(c) finding what objects are at a position and what position has certain objects (e.g. what term has the 20th red circle?);

(d) identifying growing and fixed parts of visual patterns;

(e) identifying pattern rules (sequential and position) from visual patterns with and without use of number tables;

(f) identifying growing and fixed parts and pattern rule from number tables;

(g) identifying different versions of pattern rules (leads to equivalence of expressions – number sentences with no equals), and justifying why it works for all terms;

(h) using pattern rules to introduce variable and algebraic expression; and

(i) representing patterns with graphs and relating growing and fixed parts to slope and y intercept of graph respectively.

Activities, materials and major ideas

Students gain a better understanding of patterning if given experience finding number pattern rules without using a table. An example of this is below. (*Note*: We are starting the patterns from 0 so as to accommodate the y-intercept in later years.)

Example

Consider the following:

	X	XX	XXX	XXXX	XXXXX
X	XX	XXX	XXXX	XXXXX	XXXXXX
0	1	2	3	4	5

The fixed part is the one X from term 0, the growing part is two extra Xs each new term. In a table it is easy to see that term 0 is 1, term 1 is 3, term 2 is 5, term 3 is 7 and so on, leading to a pattern of $2n + 1$ for the nth term. However, if we stay with visuals, then more is possible. The visuals can be interpreted as

- *two rows, the top is n and the bottom is $n + 1$, making the pattern $n + n + 1$;*
- *a double row of length n and an extra X, making the pattern $2n + 1$; and*
- *a double row of length $n + 1$ with a missing X, making the pattern $2(n + 1) - 1$.*

The focus on visuals gives the students an understanding that there are different equivalent algebraic expressions for a pattern rule. The different interpretations of the visuals also provide arguments to support that the pattern rule holds for all items; they provide justification.

The common materials to be used consist of numbers or objects of different colour, size, shape, and so on, with the number of objects (or the numbers themselves) determining the basis of the pattern, plus small numbered cards (1 to 5), tables, and calculators. Again it is useful to have magnetic copies of objects, cards and numbers.

The major ideas to be developed are

- generalisation and justification of generalisation,
- visual manipulation and numerical tabulation,
- representation of generalisations with variables (introduction to algebra), and

- introduction of line graphs and relation of slope and x-intercept to growing and fixed parts of growing patterns.

As well, the difference between use and non-use of number tables can be identified; that is, number tables make identification of position pattern rule easier but determining the rule from visuals alone enhances students' ability to justify their rule and to find more than one version of the rule (which helps develop equivalence of expressions).

2.1.3 Nonlinear growing patterns

Growing patterns can be constructed in such a way that they do not grow in a constant manner (i.e. by the same amount each time). For example, the pattern on the left below (open square) grows by 4 each time while the pattern on the right (filled square) grows by increasing amounts:
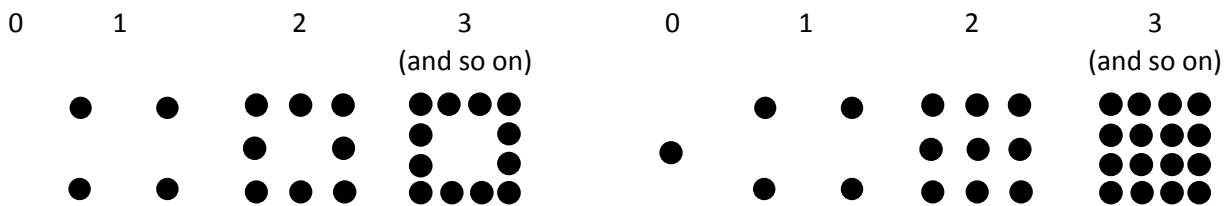
The left-hand pattern is 0, 4, 8, 12, 16, and so on (the multiples of 4) which is the rule $4n$ (a linear equation) and the right-hand pattern is 1, 4, 9, 16, 25, and so on (the squares starting at 1^2) which is the rule $(n+1)^2$ (a quadratic, thus a nonlinear equation).

There is a relationship that gives whether the pattern is a quadratic or a cubic and so on. Consider the patterns below:

Pattern A   1, 3, 5, 7, 9, 11, and so on
Subtracting consecutive terms gives 2, 2, 2, 2, and so on (all the same)
so the rule has n in it (it is linear).

Pattern B   2, 3, 5, 8, 12, 17, and so on
Subtracting consecutive terms gives 1, 2, 3, 4, 5, and so on
subtracting consecutive terms again gives 1, 1, 1, 1, and so on (all the same)
so the rule has n^2 in it (it is a quadratic and nonlinear).

Pattern C   2, 5, 10, 18, 30, and so on
Subtracting consecutive terms gives 3, 5, 8, 12, and so on
Subtracting consecutive terms again gives 2, 3, 4, 5, and so on
subtracting consecutive terms again gives 1, 1, 1, 1, and so on (all the same)
so the rule has n^3 in it (it is a cubic and nonlinear).

Note: If we had to go through the subtraction four times, we would have n^4 in the rule, and so on.

2.1.4 Patterns in other mathematics strands

Patterns in other strands can be used to obtain understanding of ideas and recall of facts in these strands. For example, the order of place-value positions, the relationship between adjacent place-value positions, counting patterns and the odometer principle, multiplication basic facts, higher decade facts (3+4=7 → 30+40=70) and repeated addition are all examples of understandings and facts that can be obtained by seeing patterns.

Much of this is covered in the YDM Number and YDM Operations resource books.

2.2   Very early patterning activities

Patterns are an excellent way to teach students the act of generalisation and to introduce variables, algebraic expressions, and graphs. There are two types of patterns, repeating and growing; and three types of pattern rules to identify – the repeating part for repeating patterns, and the sequential and position rules for growing patterns. Materials used are objects with different attributes (e.g. size, shape, colour) and cards with numbers 1, 2, 3, 4 and 5 on them; and a teacher set of these materials magnetised for placing on magnetic whiteboard. This section looks at repeating patterns – how to continue them and relate object to position and reverse.

2.2.1   Copying, continuing and creating repeating patterns

The steps here are as follows:

1. Use movements and sound (e.g. body movements, clapping, music and dance steps) to follow patterns of activity.

2. Extend these movement and sound activities to where students have to copy and continue a pattern that repeats.

3. Encourage students to create their own patterns of movements and sound and lead others in copying their patterns.

2.2.2   Copying, continuing, completing and constructing a recorded repeating pattern

The sequence of activities here is as follows.

1. The first activity is for the students to **copy** a pattern (e.g. O X X X O X X X …) where O is a red block and X is a blue block. Teacher places out the pattern with materials and students build a copy. It seems to be a better teaching sequence and easier for students if they copy patterns before continuing them.

2. The second activity is for students to **continue** a pattern. Teacher starts the pattern (e.g. O X X X O X X X O) and students continue it (in this example, X X X O X X X O X …).

3. The third activity is for students to **identify the repeating part**. Students often identify only part of the repeating part (e.g. X X X not O X X X), so language must direct students to find **all** of the repeating part.

4. The fourth activity is to **complete** a pattern – to fill in the empty spaces (e.g. what shapes will fill the gap in X X O O O X X O O O X X O _____ X O O O …)

5. The fifth activity is to have the students **construct** their own patterns and identify the repeating part. Here students sometimes construct a symmetric design which does not go on linearly forever (e.g. X X X O O X X O X X O O X X X). This type of pattern could be a repeating pattern if the sequence was repeated; however students must show that they understand there is a repeating part and must show the repeats, so that it is a continuing repeating pattern like the following: X X X O O X X X O O ….

6. Finally, the sixth activity has the student reversing the direction of the previous activities and **constructing a repeating pattern when given the repeating part** such as O O X X X (e.g. O O X X X O O X X X O O X X X …). This activity can be made much more difficult if the repeat given is O X X X O, because the repeating part starts and finishes with the same object. So start with repeats that are simpler.

The following points should also be noted:

- All examples above have only two objects (X and O). Three or more can be used to make more difficult repeating patterns.
- Younger students find it easier to work with repeating patterns if the objects are very different. This often means two attributes different – both colour and shape (e.g. red O, blue X).

2.2.3 Determining what object is in a position in a repeating pattern

In this activity, the teacher provides the start of a repeating pattern (e.g. X O O X O O …) and numbers each object as below.

X	O	O	X	O	O	__	__	__	__
1	2	3	4	5	6				10

The teacher then asks the students to identify the object (X or O) that is in a particular position, e.g. the 10th position as in the example above. The sequence involved in this is as follows:

(a) initially allow the students to put out extra objects until the position is reached before asking them to work it out in their head;

(b) initially give the students positions to find that are five or less ahead of the last item placed before going to positions further out (e.g. finding the 10th position is easier than the 13th position); and

(c) initially allow students to copy the pattern when they are asked to determine the object in positions before asking them to find a position in a pattern that the teacher has put out.

It seems that determining what object is in a position requires students to coordinate two things in their minds, the pattern and the position number; or, more difficult, synchronise these two things as moving, in their mind, along the pattern of objects and along the number for the position of the objects. It also requires students to identify the whole repeat and recognise its components (e.g. one X and two O's). This is easier to do if the students can put out the objects as they go, the number for the position past the last object placed is within the students' subitisation range (normally less than or equal to 5), or the students have familiarised themselves with the pattern by placing it out themselves.

As students get older and gain skip counting proficiency or improved understanding of multiplication, they can use the pattern to find the object and so larger numbers can be set as below, e.g. find the 27th term:

X	O	O	X	O	O	__	__	…	__	O	__	__	__
1	2	3	4	5	6					27			

The students can determine that, for example, the 27th term is O because they see that X O O … is a pattern of three and 27 as a multiple of three must be the last object in the pattern of three (i.e. the students can jump in threes). As a beginning to this, students start to manipulate the objects emphasising features as below – they start to think of the pattern in terms of its last object.

X	O	O	X	O	O	X	O	O	__	__	…	__
		3			6			9				27

Sometimes, the students use a wrapping technique as below – here the students appear to be using multiples of six not three and seeing 27 as three more than 24 and counting halfway along the six objects.

X	O	O	X	O	O
X	O	O	X	O	O
__	__	__	__	__	__

Because skip counting by five and the five times tables are more familiar to students, five-object patterns are easier than three- or four-object patterns; for example, students find that a pattern like O O X X X … is easier than a pattern like O X X X … . Finally, students seem to find patterns like O O X … easier with respect to finding objects in positions than patterns like X O O … because the students can tag the third or repeat ending object.

In summary, determining what object is in what position is difficult and should also be part of patterning when growing patterns are being developed. It requires students to: (a) **coordinate** (synchronise) counting and pattern; (b) identify the **whole** repeat and the number of objects in it and relate this to the position of the object to be found; and (c) **tag** the last term and use this for skip counting towards the required term.

2.3 Early to middle linear patterning activities

This section looks at moving from repeating to growing patterns and then undertaking all activities with these growing patterns. It introduces the technique of identifying the fixed and the growing part of a growing pattern and using this to find the pattern rule. It compares visual to tabular techniques and concludes by relooking at generalisation from repeating patterns and how patterning can be connected to some real-life experiences.

2.3.1 Moving from repeating to linear growing patterns

The following sequence of steps should be used.

1. Set up a repeating pattern; say X O O X O O X O O … .

2. Ask the students to identify the repeating part and then to break the pattern into repeats and to **separate** the repeats as below. Note that when separating the repeats it can be useful to discuss and trial other ways of representing the repeat, as shown below.

<pre>
                                              O   O   O
         XOO   XOO   XOO   or   XO   XO   XO
</pre>

3. Have the students construct a set of number cards and place these under the repeats as below. (*Note: Repeating patterns have the same term structure as growing patterns but they do not grow. Here having the zero makes sense as it is the original and the 1 is the first repeat.*)

<pre>
         XOO   XOO   XOO ……… and so on

          0   1   2
</pre>

4. Ask students to pick one of the objects and grow it, as shown below.

<pre>
                          X   O   O
                  X   X   O   O   O
         XOO   XOO   XOO   or   XO   XO   XO

          0   1   2   0   1   2
</pre>

The teacher can provide a variety of activities – grow one object, grow both objects, grow both at different rates, change the way the objects are presented, and so on. This presents repeating patterns as a precursor to growing patterns and links the two together.

2.3.2 Copying, continuing, completing and constructing linear growing patterns

Once growing patterns are introduced, students can be asked to: (a) copy, continue, and complete growing patterns set up by the teacher; and (b) create their own growing pattern and explain how it is growing. It is important to use a variety of objects and to build the patterns using a variety of attributes (e.g. colour, size, shape).

This activity is the major one in the middle years. The steps are similar to those for the repeating pattern activities (section 2.2.2) and are as follows.

1. Set up a growing pattern up to the 3rd term, as on the right. [This example has X increasing by two and O increasing by one in each new term.]

2. Have the students copy this – make their own copy.

<pre>
                                              X
                                      XO   XO
                              XO   XO   XO
                      O   XO   XO   XO
                     XO   XO   XO   XO

                      0   1   2
</pre>

---

3. Ask the students to continue this pattern for the next few terms; for example, this could involve making the 4th and 5th terms as in the diagram on right. It is useful for students to have number cards and place these under the terms.
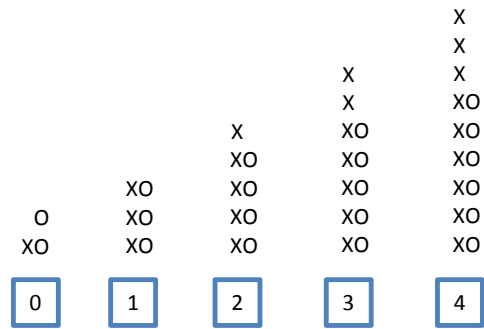
4. Ask the students to make some further terms (or tell what is involved in these terms) such as the 7th, 10th or 20th term. This requires some understanding of the pattern rule but can be completed by considering what happens term by term.

```
                                    X
                                    X
                            X       X
                            X       XO
                    X       XO      XO
                    XO      XO      XO
            XO      XO      XO      XO
    O       XO      XO      XO      XO
    XO      XO      XO      XO      XO
  [ 0 ]   [ 1 ]   [ 2 ]   [ 3 ]   [ 4 ]
```

5. Ask the students to make the first five terms of their own growing patterns and to explain what is involved in the patterns and how the terms are growing.

6. Finally, have students complete a pattern – this is where there are gaps in the example given, as on the right. It should be noted that it is more difficult to complete than continue a pattern. It is also harder if the terms given are not regular, e.g. when you are given the 1st, 2nd and 5th terms. [The example on the right has X increasing by one and O increasing by two in each new term.]

```
                                    O
                                    O
                                    O
                                    OX
                            O       OX
                            OX      OX
                            OX      OX
    X                       OX      OX
  [ 0 ]   [ 1 ]   [ 2 ]   [ 3 ]   [ 4 ]
```
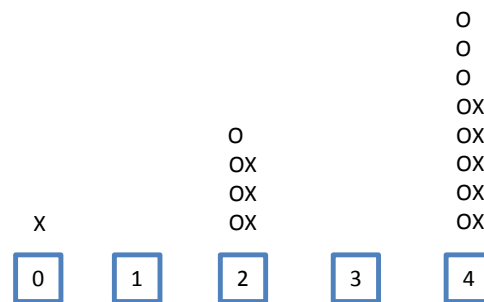
2.3.3 Determining growing and constant/fixed parts and finding and using pattern rules

The important part of growing patterns is to identify the general rule for the pattern that enables any term to be determined – this is called the **pattern rule**. Determining pattern rules is assisted by identifying in the pattern what grows and what stays fixed – the growing part and the fixed part rule. Again, there are six steps that should be followed.

1. Look at patterns and discuss what **grows** and what is **fixed**. It is important to start this process with simple growing patterns such as that below.
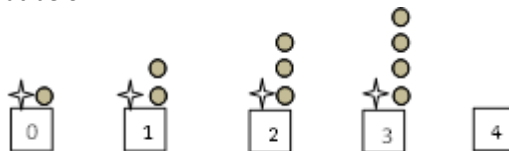
2. Use the above information to work out what **each term** will look like. This is usually done for some numbers and then considered for any term. For the example above, the 10th term is one star and 10 circles, the 20th term is one star and 20 circles, and so on.

3. Next, **identify the pattern rule**. There are two types of pattern rule – the **sequential** pattern rule which gives the difference between sequential terms, and the **position** rule which relates number of objects to the term position. The sequential rule for the pattern above is "add one circle", and the position rule (number of objects) is 1 + position number. We should accept messy language at this stage; later we can give the rule in terms of n.

4. The fourth step is **application to real-world problems**. For example, a table in a restaurant sits 4, two tables pushed together sits 6, and so on. How many will 15 tables pushed together into a row sit? [32]
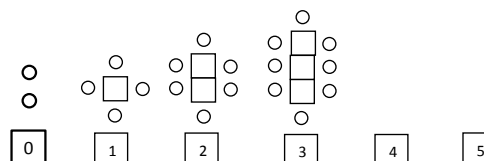
5. Next, **reverse the process** and find the position of the term with a given number of objects. For example, in the first pattern above, what position has 62 objects? [the 61st position]. In the table pattern above, how many tables have to be pushed together in a row to sit 18 people? [8 tables].

6. Finally, teach students to **solve patterns both visually and by use of tables**. To assist with generation of rules from visual cues, it is important to teach students to visualise objects in **different ways**. Consider the pattern on the right.
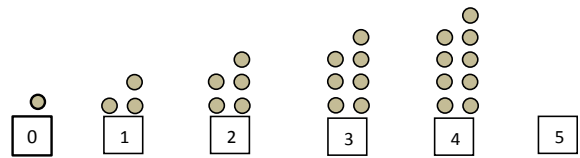
The pattern consists of two towers and these can be viewed in at least three ways, as shown below. Person A sees one double tower and an extra, Person B sees two single towers with one tower having one more than the other tower, and Person C sees a double tower with one missing.

Person A   Person B   Person C
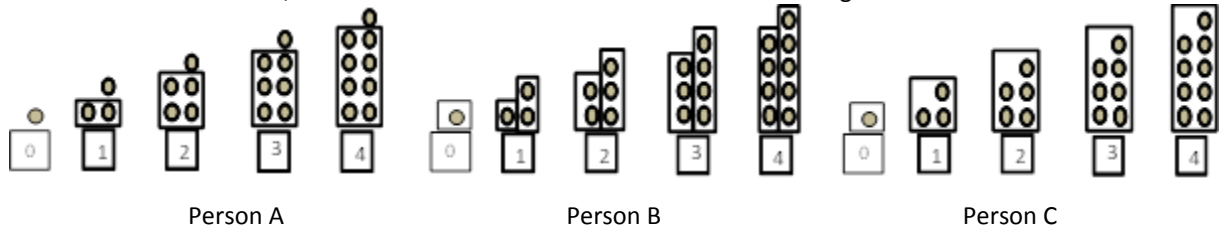
The three cases have the same sequential rule which is "add 2". However, each case has a different but equivalent position pattern rule. For the third term, A is a 2 × 3 tower plus an extra 1, B is a 3 tower plus a 4 tower, and C is a 2 × 4 tower minus 1. Thus, for any term or position, A gives the position pattern rule of "2 times the position plus 1", B gives "position plus position plus 1", and C gives "2 times (position plus 1) minus 1". In algebra these are $2n + 1$, $n + n + 1$, and $2(n + 1) - 1$ for any n, which are the same thing.

2.3.4 Tables versus visuals for growing patterns

There has been debate over when tables of numbers should be used as a way of finding position pattern rules, as against determining the pattern from the visuals. It is important that students be taught all strategies; however, for growing patterns in numbers, tables are the main strategy (as below).

Number	1	3	5	7	9	and so on
Position	0	1	2	3	4	

This book recommends that both the visual and table strategies be taught. Therefore, in the second half of the middle years, introduce tables to scaffold thinking as below.

1. Teacher provides a growing **pattern** as on right.

2. Students construct and complete a **table** as below right.

3. Students find the sequential and position **pattern rules**. For the sequential rule, look down the table; for the position rule, look across the table. Looking down the right-hand column, the sequential rule can be seen as "add 2". Looking across the table, the position rule is found by checking if there is some multiplication and/or addition/subtraction that works for all numbers. For example, 1 × 2 = 2 which is 1 less than 3, 2 × 2 = 4 which is 1 less than 5, so try this on next (yes, 3 × 2 + 1 = 8, 4 × 2 + 1 = 9 and so on). Thus, the position rule is that, for a number like 256, the number of objects is 2 × 256 + 1, and in everyday language, as a generalisation, it is "2 × term number + 1".
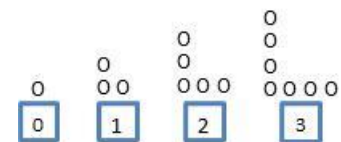
No. of term	No. of objects
0	1
1	3
2	5
3	7
4	9

It is a good idea to ask the students what would it be for any position number n but not expect all to get the answer as an algebraic expression.

4. **Reverse the direction** by giving students a rule such as "3 times the position plus 2" and ask them to construct a pattern for this rule. Do the same for a sequential pattern rule.

Note: The table is simpler to use to find the position rule. However, the visual method gives the reason for the rule and gives more than one rule.

2.3.5 Tables in repeating patterns

It is also possible to use repeating patterns to develop generalisation. To do this, a repeating pattern is displayed, the students are asked to break it into repeats, and then complete a table for these repeats from which generalisations can be found. An example of this is below.

Pattern: XXOXXOXXO …

Original + repeats: XXO XXO XXO

Table: as on right

Generalisations: Look for "patterns" or "rules" **down and across** the table; the down generalisations include: originals + repeats go up by 1, X's go up by 2, O's go up by 1, and total goes up by 3; and the across

No. of original + repeats	No. of X's	No. of O's	Total
1	2	1	3
2	4	2	6
3	6	3	9

generalisations include number of X's is twice the number of original + repeats, number of O's is equal to the number of original + repeats, and total is three times the number of original + repeats.

2.3.6 Patterning applications

YDM is based on relating mathematics to reality – in particular, the RAMR cycle argues that mathematics should come from reality (through abstraction) and return to reality (through reflection). In this book, we have focused on the sequence in building algebraic ideas. This is particularly so with patterns where we have given emphasis to how we build from repeating to growing patterns to finding the pattern rule with little focus on real-life examples. This small subsection is to provide balance and to discuss how we can ensure that we start and end with reality.

Repeating pattern examples

The major sources of reality for repeating patterns are as follows.

1. **The built environment**. Many components of the built environment will repeat; this could be as simple as window, window, balcony repeated, or window, window, door, window, garage, repeated. Paving is often a repeating pattern as is tiling – colours and shapes. Garden beds, trees and lawn can follow repeating patterns – particularly with respect to edges.

2. **Art and design**. Many forms of art repeat – for example, weaving can produce a colour pattern for a scarf, knitting and crocheting can follow complex repeating patterns of different stitches. All fabric designs are repeats as are many of the artistic designs on buildings (e.g. frieze patterns). This area is a real opportunity to involve different cultures and their art and design.

3. **Dance and music**. Drumming and clapping rhythms are repeated patterns as are rhythms using double basses and guitars. Playing music is a great way to introduce repeating patterns. YDC has developed with a music group called JAM (Join Australian Music) a set of mathematics lessons that use repeating patterns to enable students to develop mathematics knowledge based on drumming and clapping rhythms, pitch, loudness, and notes/beat (for further information, contact ydc@qut.edu.au).

4. **Poetry**. Some poetry and song lyrics follow repeating patterns (e.g. limericks). The mathematics of the repeating patterns can be introduced as a way to analyse the writing.

Growing pattern examples

It is harder to find good reality examples for growing patterns. However, some examples are as follows.

1. **Growing shapes**. There are some fun ones in this area, like using counters to grow shapes (e.g. squares, triangles, pentagons, L's, X's, Y's, N's, W's, and so on). Be careful to ensure these are linear if focusing on linear relationships.
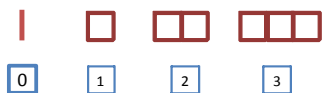
2. **Real-world situations**. The real world can also provide examples like the relationship between chairs and tables in a restaurant. However, be careful, because many of these such as the number of ways one could go up a number of stairs are nonlinear examples that lead to square or triangular numbers (1, 3, 6, 10, 15, ...) and Fibonacci sequences (1, 1, 2, 3, 5, 8, 13, ...). Nonlinear patterns are covered in section 2.5.

3. **Mathematics itself**. Many of the relationships in mathematics start off as patterns. For example, interior angle sums for n-sided polygons, the number of diagonals in an n-sided polygon, the rule for place-value positions when multiplying/dividing decimal numbers, and so on. This is a good opportunity to integrate aspects of mathematics – use discovering of a formula as a chance to solve growing patterns.

4. **Scientific experiments**. One way to have something with reality for any growing pattern is to argue that you are testing a chemical on tree growth. The tree starts with the fixed part, say two leaves, then after each day it has grown three extra leaves, and so on (this is pattern $3n + 2$; pattern $4n + 3$ would start with three leaves and grow four each day). Talking about days and leaves helps students work out the pattern. For this experimental approach, it is appropriate to start from 0 because the extra leaves grow after 1 day, 2 days and so on. The tree with its two starting leaves is day 0.

2.4 Later linear patterning activities and applications to graphing

This section continues the patterning activities as a way of introducing variable and graphing for both growing and repeating patterns, ensuring that all developments are reversed. It shows how repeating patterns introduce equivalent fractions and ratio (proportion). It concludes by looking at graphing skills such as slopes and midpoints of lines, and by looking across patterns and their graphs to investigate relationships between characteristics of patterns and characteristics of graphs.

2.4.1 Building algebraic generalisations from growing patterns

In the later years, it is important to ensure that students can find and express generalisations from patterns in quasi-generalisation form (for large numbers, e.g. 3 × 674 + 2), in language (informally, e.g. "3 times the position plus one"), and in algebraic form (e.g. $3n + 2$). This is developed in the following sequence.

1. Present a **pattern** as on right, e.g. starting with one stick, how many sticks to make, 1, 2, 3, ..., squares in a row.

2. Identify the **fixed and growing parts** of the pattern (the first stick is fixed and then it grows by three sticks, as on right).

3. Determine the **sequential rule** ("add 3") and the **position rule** ("number of sticks is 3 times position plus 1"). To assist finding the positional rule, do the following:

    (a) record the number of sticks for each term – term 1 is 1 + 3 = 4 sticks, term 2 is 1 + 3 + 3 = 7 sticks, term 3 is 1 + 3 + 3 + 3 = 10 sticks, and so on;

    (b) develop quasi-generalisation – e.g. for term 7, number of sticks is 1 + 3 × 7 = 22, for term 100, number of sticks is 1 + 3 × 100 = 301, for term 345, the number of sticks is 1 + 3 × 345 = 1036;

    (c) state the generalisation in language – "1 stick at start plus 3 for every term".

4. Put the generalisation in **algebraic terms** (any number represented by the letter n) as $1 + 3n$ (or $3n + 1$).

5. Use the generalisation to **reverse the position-to-number activity** to number-to-position, e.g. how many squares use 28 sticks? [Position 9 has 1 + 3 × 9 sticks, so answer is 9 squares.]

6. **Reverse everything** and construct a pattern from a pattern rule, e.g. $2n + 3$ (as on right).

This activity can be extended to graphing. Once the position pattern rule has been determined as an algebraic expression, a graph can be constructed (see section 2.4.2).

Note: If term 1 of the pattern was one stick (instead of one square) and then went to the squares as on right, the pattern is the same in terms of fixed and growing part but the pattern rule is different because the pattern is 1, 4, 7, 10, ... instead of 4, 7, 10, ... In this case, the pattern is 1 for term 1, 1 + 3 × 1 for term 2, 1 + 3 × 2 for term 3, 1 + 3 × 3 for term 4 and so on. This means that for position 10 it is 1 + 3 × 9 and for position n it is $1 + 3 \times (n-1) = 1 + 3n - 3 = 3n - 2$. That is 3 less than the original pattern, $3n + 1$. This is $(3n + 1) - 3 = 3n - 2$.
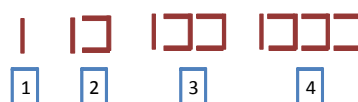
2.4.2 Extension to graphs for linear growing patterns

Begin with same pattern as in subsection 2.4.1 above and complete steps 1 to 6. The remaining steps are as follows.

1. Construct a **table** and place early values in it:

Number	1	4	7	10	13	16	and so on
Position	0	1	2	3	4	5	
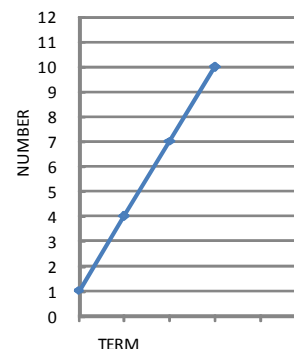
2. Use the table to draw the **graph** (as on right).

3. **Reverse everything** – provide a graph as below and make up a pattern to match this graph (use the fact that term 0 = 2, term 1 = 4, term 2 = 6, etc. to construct a pattern – see below right).
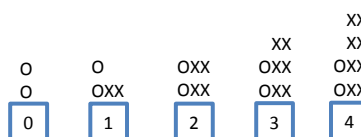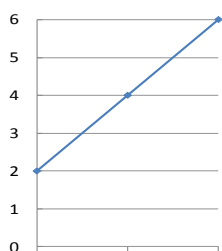
There is a relationship between graphs and patterns that holds generally for growing parts and, in a way, for fixed parts. The slope of the graph is the growing part, and the y intercept is the fixed part of a growing pattern. It works on the two examples above. The pattern at the start of subsection 2.4.1 grows by 3 and has a fixed part of 1, and gives a graph with slope of 3 and a y intercept of 1. The graph above has a slope of 2 and a y intercept of 2 and the pattern it comes from grows by 2 and has a fixed part of 2.

Thus, we can build the generalisation of graphs of straight lines, $y = mx + c$, where m is slope and c is y intercept, by graphing patterns that grow by m and have a fixed part of c.

2.4.3 Moving from physical to mathematical patterns

Up to now, we have focused on patterns that emerge from using counters or sticks, for example:
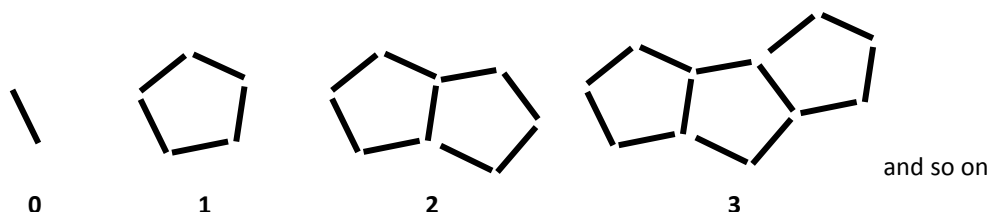
and so on

However, physical objects cannot show negatives and these are perfectly acceptable in patterns. Thus, we now move from physical to mathematical patterns, for example:

Number	7	5	3	1	−1	−3	?	
Position	0	1	2	3	4	5	6	and so on

We can even give the pattern in the middle, for example:

Number	?	?	?	1	4	7	?	?
Position	0	1	2	3	4	5	6	7

The steps are the same, as follows.

1. Construct a table:

Number	−8	−5	−2	1	4	7	10	13
Position	0	1	2	3	4	5	6	7

and so on

2. Look for a pattern: growing part is +3, starting is −8; pattern is ×3 −8

3. Give the pattern rule: n^{th} term is $3n - 8$.

4. Reverse the pattern rule: position for amount k is $(k + 8) \div 3$.

5. Draw graph: slope = 3, y-intercept = −8.

6. Reverse everything: go from graph to pattern.
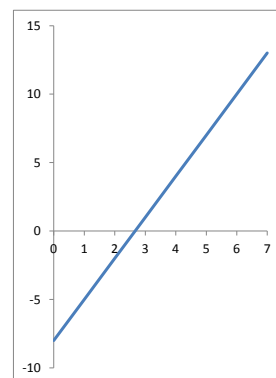
It is also possible for the graph and growing part to be negative, as in the following example.

1. Table:

Number	11	7	3	−1	−5
Position	0	1	2	3	4

and so on

2. Pattern: growing part is −4, start is 11; pattern is ×−4 +11.

3. Pattern rule: n^{th} term is $11 - 4n$.

4. Reverse: position for amount k is $(11 - k) \div 4$.

5. Draw graph: slope = −4, y-intercept = 11.
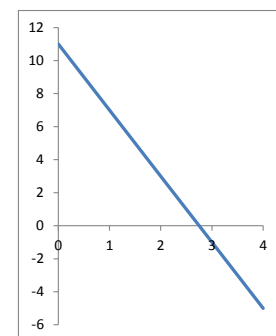
2.4.4 Building algebraic generalisations from repeating patterns

The process in subsection 2.4.1 can be applied to repeating patterns to also teach generalisation to algebraic expressions as follows. (It is based on considering numbers of objects after 1, 2, 3, and so on repeats.)

1. Teacher presents a **repeating pattern** – we'll use the following pattern: X X O X X O X X O … XXO XXO XXO … in repeats.

2. Complete the **table** below and find the down and across patterns.

No. of original + repeats	No. of X's	No. of O's	Total
1	2	1	3
2	4	2	6
3	6	3	9

3. **Generalise** how to find the numbers for n repeats. Do this in language first – O's are the same as number of original + repeats, X's are double number of original + repeats, and so on.

4. State generalisations as **algebraic expressions** – if n original + repeats, number of X's is $2n$, number of O's is n, and total number is $3n$.

5. **Reverse** the **original+repeats-to-number** activity (to number-to-original+repeats) – how many original + repeats are needed for 26 X's? [$26 = 2n$ so it is 13 original + repeats.]

6. **Reverse** the activity **algebraically** (e.g. if n X's, then $n/2$ original + repeats).

7. **Graph** the relationship between number of X's and number of original + repeats.

8. **Reverse overall** – construct a repeating pattern of X's and O's for a given graph, e.g. the graph on right. This graph has 3 X's for original, 6 X's for original + 1 repeat, so it is OXXX, OXXX ... or OOXXXOOXXX, etc. *Note:* The y-intercept is 0. This is because for no original or repeats this is always zero.
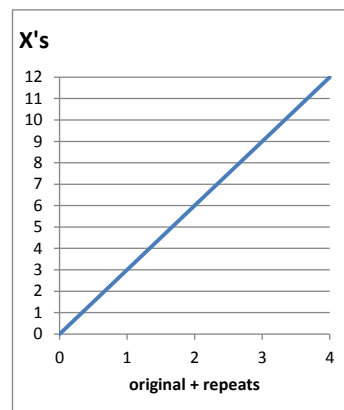
2.4.5 Extending repeating patterns to fractions and ratio/proportion

The use of repeating patterns is only the start – repeating patterns are excellent for introducing fractions and ratios and equivalent fractions and ratios, as follows.

1. **Pattern.** Start with a repeating pattern, e.g. X X O O O X X O O O X X O O O

2. **Repeats.** Change it to original + repeats, e.g. XXOOO XXOOO XXOOO

3. **Table.** Complete a table that contains fractions and ratios.

No. of original + repeats	No. of X's	No. of O's	Total	Fraction X's	Ratio X:O
1	2	3	5	$2/5$	2:3
2	4	6	10	$4/10$	4:6
3	6	9	15	$6/15$	6:9

4. **Generalisation.** Look at what n original + repeats will give (repeats to O's and X's).

5. **Reversing.** Go from numbers of O's and X's to number of original + repeats (e.g. 72 O's means $72/3$ original + repeats = 24 original + repeats).

6. **Equivalence.** Discuss whether fractions and ratios are equivalent (or in ratio terms, in proportion). For example, is 2:3 = 4:6?

   Note: A student explained it to class as shown on right. She stated that 2:3 is XXOOO and 4:6 is XXXXOOOOOO. However, she rearranged the objects as on right and argued that this showed that XXXXOOOOOO is the same as XXOOO.
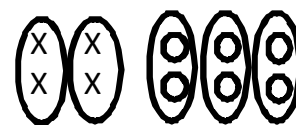
Same as XX OOO

7. **Reversing overall.** Make up a repeating pattern where the ratio is 2:5 (e.g. O O X X X X X O O X X X X X O O X X X X and so on).

Notes: (a) The teaching approach is to have class discussion. Let students propose generalisations. Don't say if generalisations are right or wrong. Focus on down and across generalisations. Encourage students to justify their point of view. (b) Repeating patterns can be done with number, e.g. 1 2 2 1 2 2 1 2 2, ..., but numbers here act like objects.

2.4.6 Relationships between linear growing patterns and graphs for linear equations

Relationships between patterns and graphs of linear equations can be found by relating solutions to different patterns looking for similarities and differences.

1. Provide examples of patterns that have fixed growing parts (pattern rule is a linear equation) and are related in a way that enables relationships with regard to graphs and functions to be seen. For example:
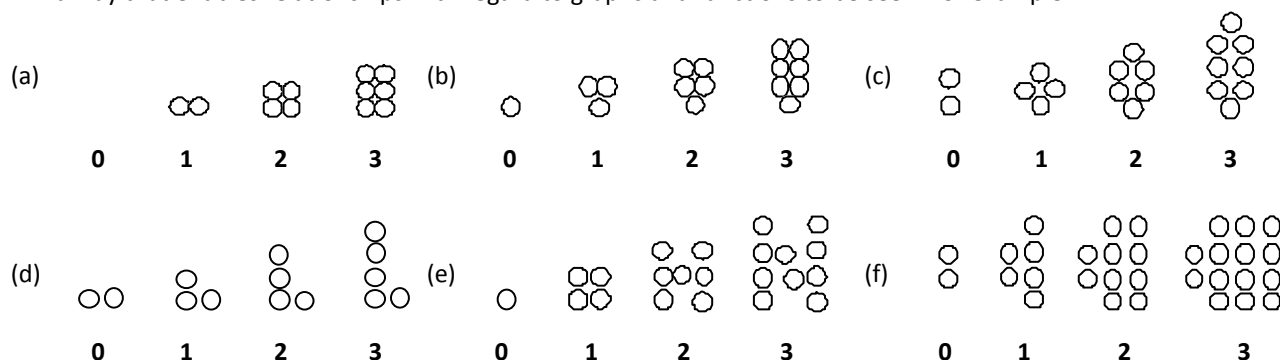
Note: The composition of, and position rules for, the patterns and their graphs are as follows. Examples have to be chosen so that the linear relationships that are wanted are available from the patterns.

PATTERN	COMPOSITION	POSITION RULE	GRAPH CHARACTERISTICS
(a)	Grows by 2, no fixed part, starts with 0	$2n$	Slope 2, y-intercept 0
(b)	Grows by 2, fixed part of 1, starts with 1	$2n + 1$	Slope 2, y-intercept 1
(c)	Grows by 2, fixed part of 2, starts with 2	$2n + 2$	Slope 2, y-intercept 2
(d)	Grows by 1, fixed part of 2, starts with 2	$n + 2$	Slope 1, y-intercept 2
(e)	Grows by 3, fixed part of 1, starts with 1	$3n + 1$	Slope 3, y-intercept 1
(f)	Grows by 4, fixed part of 2, starts with 2	$4n + 2$	Slope 4, y-intercept 2

2. For each pattern, analyse the values visually and identify the fixed and growing parts.

3. For each pattern, find the value of some positions (e.g. 10, 100, 39, 64), find the position of certain values (e.g. 17, 71, 149 for pattern b), write the pattern rule in English, find the value for position n, and find the position for value k.

4. Draw the graph of each pattern and identify the slope and y-intercept.

5. Look for patterns between characteristics of pattern, position rules, and characteristics of graph (can be useful to complete a table like that above). First compare (a), (b) and (c) noting similarities and differences – what changes (fixed part) and what does not change (growing part). Then compare (d) with (c) – now growing part changes and fixed part stays the same. After this compare (e) and (f) – both growing part and fixed part are different (but how does this difference relate to graphs, its slope and y-intercept?).

What follows is the **general rule for slope**: if the growing part is p, the pattern rule starts with pn and the graph has slope p. This relationship is even more obvious when we restrict ourselves to numbers. For example:

Number	7	5	3	1	−1	−3
Position	0	1	2	3	4	5

and so on

The growing part is −2, the fixed part is 7, so the pattern is 7−2n. This makes the slope −2 and the y-intercept 7. This means that the function for a linear equation is $f(x) = mx + c$ where m is the slope (and the growing part if from a linear growing pattern) and c is the y-intercept and the constant part for the zero term ($n = 0$).

Note: It is useful to compare the patterns above to triangular numbers; a pattern which does not have fixed growth.

2.5 Nonlinear patterning activities

Linear patterns are the important patterns for up to Year 9 for two reasons. First, they are the simplest form of algebra and so allow learning of forms such as $2n$, $n + 3$ and $2n + 3$. Second, they lead to understanding of graphs of lines and linear equations which are the major focus up to Year 9. However, there are also opportunities for nonlinear pattern work to pre-empt quadratics and other nonlinear forms in Year 10 onwards.

2.5.1 Nonlinear patterns and graphs

Here are some steps for investigating linearity and nonlinearity.

1. **Comparing linear and nonlinear**

   A. Consider the two patterns below:
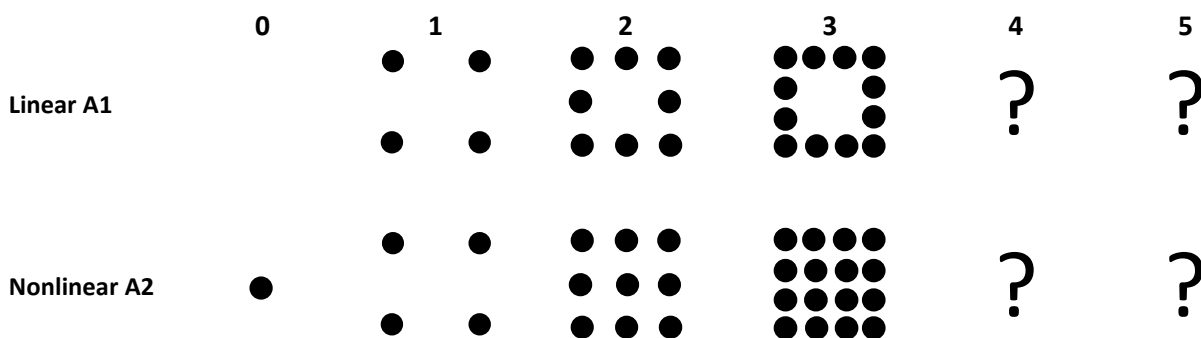
   (a) Complete terms 4 and 5 by drawing or with counters.

   (b) Look at term 5 visually and use this pattern to determine the 10th term, 100th term and 156th term for A1 and A2.

   (c) Describe the positional rule for A1 and A2 (in language and as an algebraic expression).

   B. Consider the two patterns below:
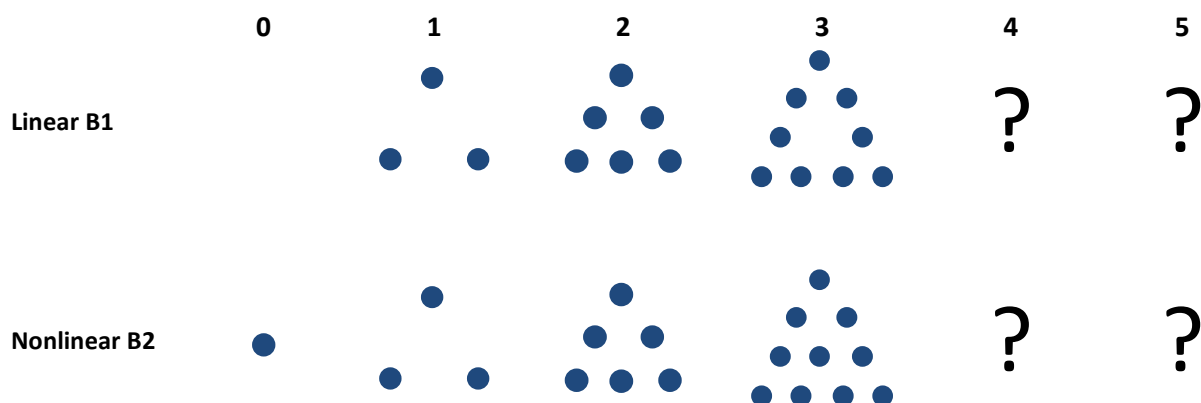
   Repeat questions (a) to (c) above.

   Hint: To find the rule for B2, put two triangles together, see that they form a parallelogram as on right – and so the triangle is half of this.

2. **Comparing graphs of linear and nonlinear**

Plot points on a table and then transfer them to graphs.

	0	1	2	3	4	5	6	7
A1	0	4	8	12	16	20	24	28
A2	1	4	9	16	25	36	49	64

A1

$4n$
(line)

A2

$(n+1)^2 = n^2 + 2n + 1$
(parabola)
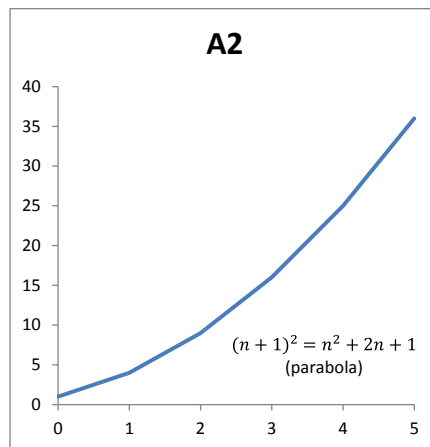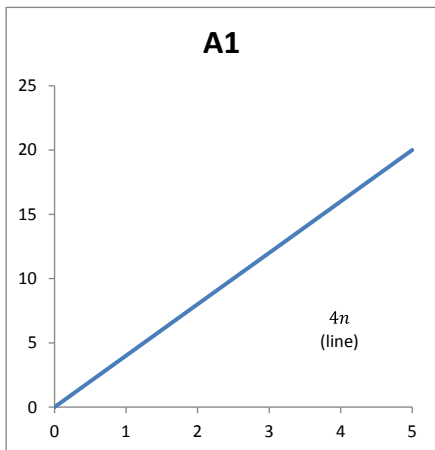
	0	1	2	3	4	5	6	7
B1	0	3	6	9	12	15	18	21
B2	1	3	6	10	15	21	28	36

B1

$3n$
(line)

B2

$\frac{(n+1)(n+2)}{2} = \frac{n^2+3n+2}{2}$
(parabola)
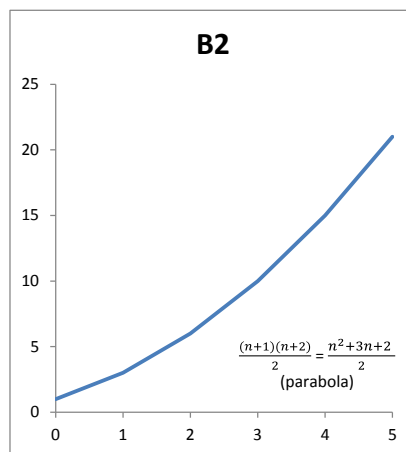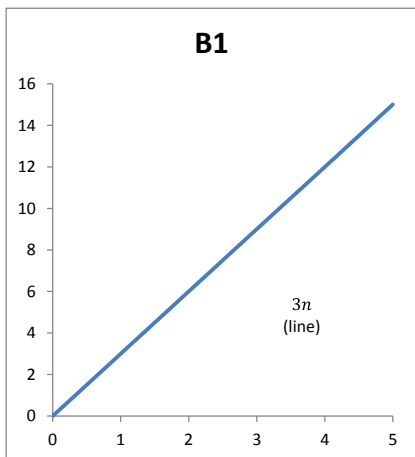
3. **No comparison**

Find the position rule for the following nonlinear patterns.

	0	1	2	3	4	5
C					?	?
D				?	?	?
E					?	?
F					?	?

Note: Drawing F is showing a pattern which doubles every time.

| G | | | | | | |

Note: Drawing G is showing a pattern which adds the previous number to the one before that.

| H | | | |

three steps

one step   two steps

one way   two ways   four ways

Note: Drawing H is showing the pattern for going up stairs if we can go up 1, 2, 3 or any number of steps at a time; how many different ways can we go up 1 step, 2 steps, 3 steps, ... any number of steps:

STEPS	WAYS	STEPS	WAYS
1	1	2	2
3	4 or 2×2	4	8 or 2×2×2
5	16 or 2×2×2×2	6	32 or 2×2×2×2×2

The result of this investigation is that n steps means 2×2 ... ×2 $n-1$ times; we can use this to introduce the notation 2^n, 2^{n-1}, and so on.

4. **Reversing**

Start with a nonlinear pattern, for example $n^2 + 3n$, and construct a pattern.

(a) Numbers:

	0	1	2	3	4	5
I	0	4	10	18	28	40

(b) Construct a pattern to fit this.

Repeat this for:

(c) J with the pattern rule $2n^2 + n$.

(d) K with the pattern rule $3n^2 - n + 2$.

(e) L with the pattern rule $n^3 - n$.

2.5.2 Growing and fixed parts (challenge)

In linear patterns, we found that the growing and fixed parts of a pattern related directly to the graph. For example:

```
O O     O O     O O     O O     O O
        X       X X     X X X   X X X X
        X       X X     X X X   X X X X
        X       X X     X X X   X X X X

0       1       2       3       4
```

The above pattern has a fixed part of 2 and a growing part of 3. The growing part of 3 can be seen in the table by looking at differences – these are all 3 and are the same, thus the growing part is 3. Therefore the equation for the position rule for this pattern has a $3n$ in it and the graph has a slope of 3. The y-intercept is related to the fixed part which is 2, but it depends on what happens when n is zero. Here it is 2.

We can now say that the pattern rule is $3n + 2$, that the coefficient or power of n is 1, and that the relationship and graph is linear, with slope of 3 and y-intercept of 2.

Position	0	1	2	3	4	5	6	7
Number of objects	2	5	8	11	14	17	20	23
Difference		3	3	3	3	3	3	3

The question is whether something similar to this holds for nonlinear. To explore this we look at an example of a quadratic (e.g. example A2 which is $n^2 + 2n + 1$).

1. Take quadratic A2 and put the information into a table.

Position	0	1	2	3	4	5	6	7	8
Number	1	4	9	16	25	36	49	64	81

2. Now subtract differences and continue until differences are the same or constant.

Position	0	1	2	3	4	5	6	7	8
Number	1	4	9	16	25	36	49	64	81
		3	5	7	9	11	13	15	17
			2	2	2	2	2	2	2

3. It took two differences to get the constant. Is there a pattern here – one difference to constant for linear like $3n + 2$; two differences to constant for quadratic like $n^2 + 2n + 1$. Is it possible that all quadratics take two differences to get a constant?

4. The constant is 2, this is the coefficient of n in $n^2 + 2n + 1$. Is this always the case for quadratics?

5. Repeat the above for the quadratic patterns in examples B2, C, J and K. Does it hold?

6. Try this difference technique for cubics F and L. Is there a similar rule for cubics?

7. Is there a rule that relates difference to coefficient of n?

3   Change and Functions

This chapter on change and functions explores how to represent everyday life in terms of change or transformation. Thus, it studies the symbols, notation and rules for change and functions (including input–output tables and arrowmath symbols and their relation to equations and graphs). In the long run, it returns to representing functions using equation notation. It is important that arrowmath and equations be related at the end so that both relationship and change ideas can be applied to algebraic situations using the same notational forms (the expression and the equation).

To get to this point requires studying (a) change in unnumbered situations; (b) linear change in arithmetic situations (numbers and operations but with no squares, cubes, and so on); (c) linear change in algebraic situations (variables and operations); and (d) nonlinear change. Similarly to patterns, most of the change and function work for Years P to 9 is in linear form. However, the groundwork for nonlinear changes should be laid to pre-empt the needs of Year 10. The teaching sequence is shown in the figure below.

The meaning of change in arithmetic

↓

Function machines/Linear change/
Arrowmath notation

↓

Inverse of linear change/Backtracking

↓

Unknowns and linear change

↓

Solving linear equations/
Relation to graphs

Materials
Function machines/
Input-output tables

↓

Number lines/Arrows

↓

Arrowmath/Equations

Generalising change rules (variables)/
Translating arrowmath ←→ equations

Non-linear change/Inverse/Graphs

The sections in this chapter are constructed around the activities suitable for different year levels. The chapter begins by looking at major ideas and models (section 3.1), then linear activities across Years P to 9 under the headings *Very early change and function activities* (section 3.2), *Early to middle linear change and function activities* (section 3.3) and *Later linear change and function activities* (section 3.4). The chapter concludes with a section on function applications and nonlinear change (section 3.5).

3.1   Major ideas and models

This section looks at the major mathematical ideas that underlie change and functions. It describes the two approaches to mathematics, discusses how change leads to identity and inverse/backtracking and the notation called arrowmath, and lists the major ideas to be developed. It describes the two main models used (number line and function machine). It shows how the function machine model leads to variable and graphing.

3.1.1   Major ideas

Real-world situations can be translated into relationships or into changes. For example, "2 joining 3 to make 5" is a **relationship** if considered as 2+3 = 5 and is a **change** if considered as 3 changing to 5 by +2 (as on right).

$$3 \xrightarrow{+2} 5$$

These real-world situations do not have to involve operations. For example, two triangles being similar can be considered as a **relationship** of angles being equal and sides in ratio, or can be considered as a **change** due to a projection that enlarges the first shape to the second shape, as on right.
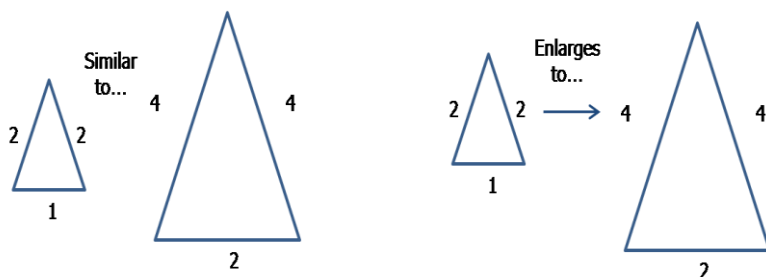
Identity and inverse

The major ideas to be covered in change and functions deal with mathematical forms that describe change (such as functions) and with the major ideas that emerge from changes, namely,

(a) changes that do not change anything (e.g. +0, ×1, a 360° turn) – the **identity principle**; and

(b) changes that reverse other changes (e.g. −6 reversing +6, ÷8 reversing ×8) – the **inverse principle** and **backtracking**.

Relationships are most often represented as equations and this form of notation is good for seeing equals as balance and for applying the balance principle (that there is a left-hand side and a right-hand side and they have to stay in balance).

Arrowmath

Changes can also be represented as equations but it is easier to understand them if they are represented by arrowmath notation. For example, the situation *I bought some $3 pies and a $5 chocolate, how much did I spend?* cannot be calculated because there is not enough information given. However, if we knew the number of pies, we could calculate the answer by multiplying this number by 3 and adding 5. Thus, the notation can be thought of as the equation $n \times 3 + 5$ (or $3n + 5$), or in arrows:

$$n \overset{\times 3}{\to} \overset{+5}{\to}$$

The arrowmath notation makes studying the change easy. First, changing forward, it is easy to work out what money will be paid for differing numbers of pies, for example, if the number of pies is 7, then the answer is $26.

$$7 \overset{\times 3}{\to} \overset{+5}{\to} 26$$

Second, by reversing the change, it is possible to find the number of pies if I paid $38 for the pies and chocolate (see below). We use the inverses of the operations and backtrack (as shown in the bottom arrows) to the answer of 11 pies.

$$n \overset{\times 3}{\to} \overset{+5}{\to} 38$$

$$11 \overset{\div 3}{\leftarrow} 33 \overset{-5}{\leftarrow} 38$$

Ideas developed by change

Thus, the major ideas that can be developed in this section relate to inverse but include the following:

(a) developing the notion of change and inverse (backtracking) in unnumbered situations;

(b) extending the notions of change and inverse to numbers and operations (first with addition and subtraction, second with multiplication and division, and third with more than one operation);

(c) introducing drawings (number lines and function machines), tables and arrowmath notation to describe changes and inverses;

(d) relating change and inverse (backtracking) to real-world situations and vice versa;

(e) generalising change and inverse and using this to introduce variables and algebraic expressions and equations (including conversions between arrowmath and equation notations);

(f) interpreting real-world problems in terms of change and using backtracking to solve for unknowns;

(g) representing generalised change with graphs and relating real-world situations, arrowmath and equation notation, and graphs and change to graphs in all directions.

3.1.2 Main models

There are two major models that can help with change, namely number lines and function machines.

Number lines

Operations can be represented on number lines by arrows, for example, the problem with the pies and chocolate ($3n + 5 = n + n + n + 5 = 38$) can be represented on a double number line as in the diagram on the right.
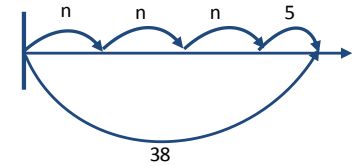
The number of pies (n) can be worked out by first crossing out the 5 which gives $3n = n + n + n = 33$ and then sharing the 33 evenly between the three n's as on the right. This gives the answer of 11 pies.
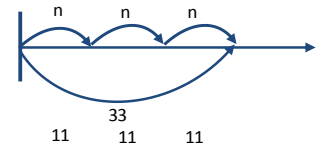
However, this reflects the balance approach from equivalence and equations (see Chapter 4) and does not use backtracking (inverse).

There is another way to represent the problem on a number line as change. This way shows operations as changes on the line as shown below.
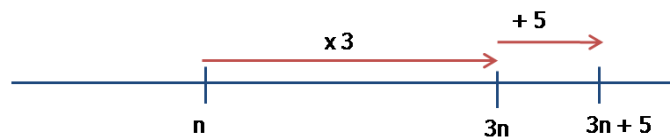
Backtracking is going back along the line and undoing what has been changed as seen below.
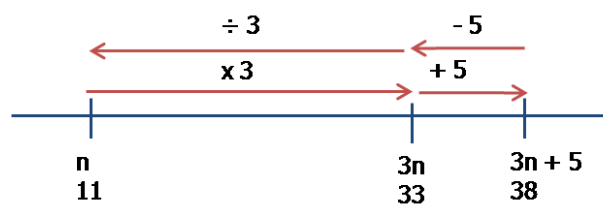
Function machines

This is the major model. A "machine" is constructed. It can be a whiteboard, or blackboard or a box with holes in it and takeout cards, for example:
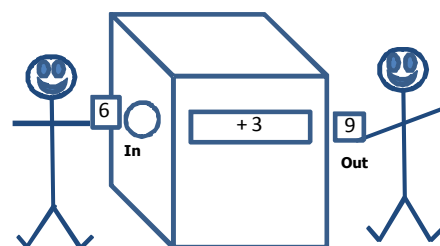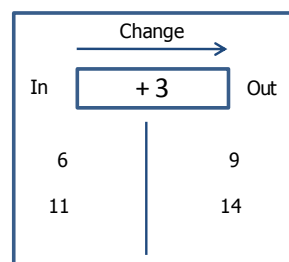
Function machines operate as follows.

1. Situations are described: *I sold small prints for $20 each, I paid $200 to rent the site, how much money did I make?*

2. Situations are translated to activities on an unknown or variable, e.g. P is the number of prints so:

$$P \xrightarrow{\times 20} \xrightarrow{-200}$$

3. These are translated to two function machines and the changes are acted out by students with numbers on cards.
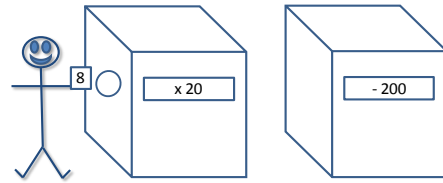
4. Examples are put on an input–output table (shown on right) starting with input numbers, and students act out the change.

Input	Middle	Output
8	160	−40
10	200	0
20	400	200

5. Numbers are then put into the middle or output and students act out how to find the other sections of the table.

Input	Middle	Output
?	240	?
?	?	100

6. A variable amount, say n, is considered and discussed at the function machines and responses included.

Input	Middle	Output
n	$20 \times n$	$20 \times n - 200$
n	$20n$	$20n - 200$

7. Then different examples (including variables) are given at output and students **backtrack** to find input using the function machines and recording on input–output table.
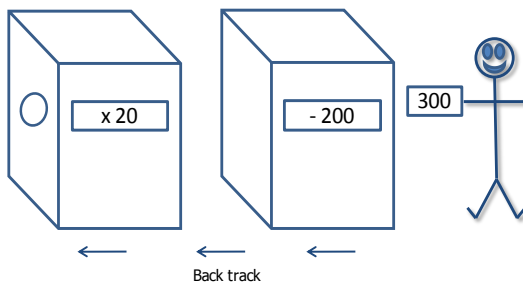
Input	Middle	Output
22	440	240
25	500	300
$\dfrac{m + 200}{20}$	$m + 200$	m

Back track

8. The changes are reversed, and the inverse operations and inverse sequence is found and written in arrowmath notation, as shown on right.

$$P \xrightarrow{\times 20} \xrightarrow{-200} m$$
$$P \xleftarrow{\div 20} \xleftarrow{+200} m$$

9. Then the change and inverse change are written as equations, as shown on right (it takes time to get used to how these two are connected).

$$P \xrightarrow{\times 20} \xrightarrow{-200} m \text{ same as } 20P - 200 = m$$
$$P \xleftarrow{\div 20} \xleftarrow{+200} m \text{ same as } \frac{m+200}{20} = P$$

10. The change is then graphed as on right below.

11. Everything is reversed: a graph is drawn and students have to find equations (forward and inverse), change equations, use arrowmath notation, put examples on the input–output table, and construct the function machines.
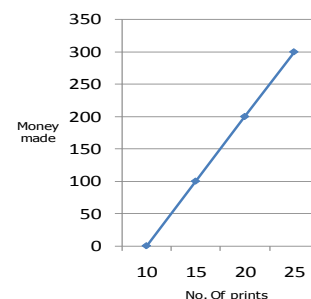
Notes: (a) the 11 steps above are the final endpoint of teaching across P–9; only a few of these steps would be undertaken for young students; (b) if this approach is learnt, it makes such things as the inverse of a function easy. For example, a function $f(x) = 2x + 1$ can be considered in arrowmath notation as shown below left. Thus, the inverse function, $g(x)$, is this arrowmath notation in reverse (backtracked) as shown below right. Changing this backtracking to an equation, this means that the inverse function is $g(x) = (x - 1) \div 2$.

$$x \xrightarrow{\times 2} \xrightarrow{+1} f(x) \qquad\qquad g(x) \xleftarrow{\div 2} \xleftarrow{-1} x$$

3.1.3 Applications and nonlinear changes

The changes considered in this section are predominantly linear, consisting of one or two operations, such as $3 \times p + 2$ or $\frac{p}{5} - 6$. This is because, up to Year 9, linear functions are the major form of function used and the development of backtracking is used for solving linear equations with unknowns in algebra.

There are two applications of the change approach and associated backtracking: percent, rate and ratio problems and nonlinear changes.

Percent, rate and ratio

Backtracking can be used for percent, rate and ratio problems in upper primary and junior secondary school, as we have seen in the YDM Number book.

Percent

What is 36% of $85?

$$\$85 \xrightarrow{\times 0.36} = \$30.60$$

If 36% is $85, how much is 100%?

$$\$? \xrightarrow{\times 0.36} = \$85$$

$$\$236.11 \xleftarrow{\div 0.36} = \$85$$

Rate

If petrol is $1.65 per litre, how much petrol can you buy for $90?

$$?L \xrightarrow{\times \$1.65} = \$90$$

$$?L \xleftarrow{\div 1.65} = \$90$$

$$? = 54.54 \text{ litres}$$

Ratio

The ratio of sand to cement is 7:2. How much sand for 24 tonnes of cement?

$$\text{sand} \xrightarrow{\times \frac{2}{7}} \text{cement} \qquad ? \text{ tonnes sand} \xrightarrow{\times \frac{2}{7}} = 24 \text{ tonnes cement}$$

$$? \text{ tonnes sand} \xleftarrow{\div \frac{2}{7}} = 24 \text{ tonnes cement}$$

$$? = \frac{24}{1} \times \frac{7}{2} = 84 \text{ tonnes of sand}$$

Nonlinear changes

Changes can be **nonlinear** involving quadratics and cubics. For example, *The carpet was sold by m², the salesman measured the length of a square room and added 1 m² for error. How many square metres were sold?* Here the change is: length → squared → 1 is added.

This gives a function machine with the following changes.

If we only deal with positive numbers, this change can be reversed as follows.

We can even solve this simple problem by backtracking: *If 26 m² was ordered, what was the length of the square room?*
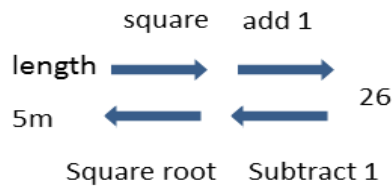
3.2 Very early change and function activities

Change and functions covers operations as change and is designed as a precursor to functions. The sequence for teaching is to move from unnumbered to numbered activities, addition and subtraction to multiplication and division, and one operation to multiple operations. It introduces input–output tables and arrowmath notation. It develops inverse of one operation, inverse of sequence of operations, and backtracking to solve for unknowns. It relates arrowmath notation to equations, and graphs the results, relating all aspects in all directions. It continually relates symbols and models back to everyday situations so that it can model reality. Finally, it allows change to be generalised into a rule using algebraic notation, thus introducing variable and leading to function.

The early years are when unnumbered activities are used to develop the language of change (the words change, input and output) and the notion of what change and reversing change are.

3.2.1 Introduce notion of change

Discuss with students how things change (e.g. we clean things, we grow things, weather gets hotter, things are moved around, we change clothing, change hair colour, and so on). Discuss before and after – before washing hair, after washing hair, before putting on a dress, after putting on a dress. Take photographs – use these as before and after discussion points (e.g. what happened here?). Look at relationship patterns – what goes with what? For example, shoes with feet, shirts on bodies, hats on heads, and so on. Focus on how before and after have to be related in some way.

Play card games like "switch" (or its commercial form, UNO). If a spade is put down, you have to follow it. To change suit you can put the same value on top, and so on. Play Snakes and Ladders – things change if you land on a snake or a ladder.

3.2.2 Unnumbered change activities

Set up a function machine that will have an input and output and a rule for change, see example on right for whiteboard.

Give students a small copy of the board to record their changes (doing this with a small chalk board was very successful in one school).

Choose an unnumbered change and put this in the RULE box. Discuss what input and output are – act out some changes. Have students walk in on the left with a picture of a thing to be changed and stick this on input. Discuss what it could change to. Give students a picture of this change to put on RHS. Students should also record on their recording sheets or chalk boards, the input and output (the In and the Out).
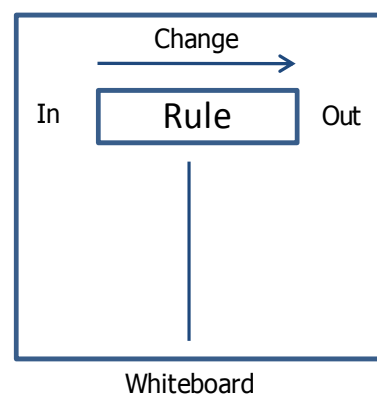
Examples of unnumbered change include:

- (a) lower case to capital letter (e.g. input h and output H);
- (b) "cook it" (e.g. input a picture of a potato and output a picture of chips);
- (c) "wash it" (e.g. dirty car to clean car);

(d) "wear it" (e.g. hand to glove, foot to shoe);

(e) add "at" (e.g. b to bat, fl to flat);

(f) add "ing" (e.g. r to ring , s to sing); and

(g) move first letter of word to end of word.

Involve students in bringing out picture cards (or potatoes or letters or whatever is relevant for the RULE) and working out what the output will be. Get students to discuss what is happening, and encourage students to think of things to change and even to think of changes.

Note: Attribute logic blocks or pattern blocks can also be used – changes can be blue to red or large to small or triangle to square. The problem here is that attributes that are not mentioned should not change.

3.2.3 Inverse of unnumbered activities

Use the function machine from 3.2.2 above to look at the notion of inverse. Make up a matching set of pictures before and after cooking (e.g. pasta in a packet to spaghetti bolognaise, and so on). Organise students to come out front and pick an input card. Get students to show the picture to the class and stick it on the input side of the table. Discuss what would be on the output side. Select the likely picture from output cards.

After doing this for some time, ask students to select an output card and stand on the RHS. Discuss what the input card could be. Repeat this as often as required. Initially, get students to think what input could give this output, e.g. *We have mashed potato, what could we cook to get this?* But after a while, get them to "think backwards", e.g. *What do we get when we "uncook" the mashed potato?*

This is easier done with, for example, the "add *at*" RULE. Here, input of *s* goes to output of *sat* – we have "added" the "at". When we look at an output of *rat*, it is fairly straightforward to consider removing, or "taking away", the "at" to find an input of *r*. Similarly, input and output makes *h* a capital *H* if "capitalise" is our change rule, and we can think of output to input as "uncapitalising", e.g. *R* to *r*. Practice with many activities.

3.2.4 Extending change

There are two ways we can begin extending what we have done in the above.

1. **Consider two changes** – one after the other (e.g. two function machines together), for example:
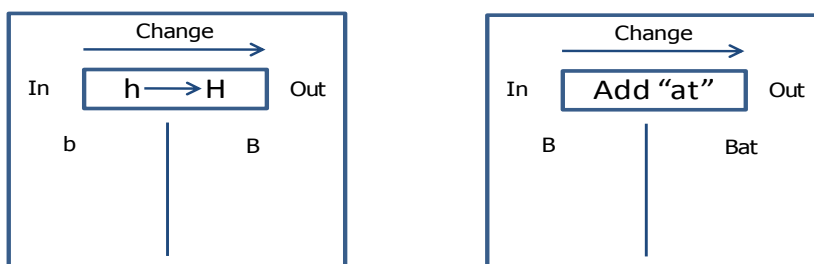
Now students can move through two changes – first change from *b* to *B* and second change from *B* to *Bat*. Can also try to reverse both changes, e.g. if we ended with *Fat*, then this goes back to *F* and then *f*. (It should be noted that changes like *e* to *Eat* could be challenging as well as *fl* to *Flat*.)

2. **Consider bringing in number**. This could be done initially by adding two extra counters or removing three counters from plastic bags with counters, or using input and output cards showing sets of counters, as for the function machine on right.
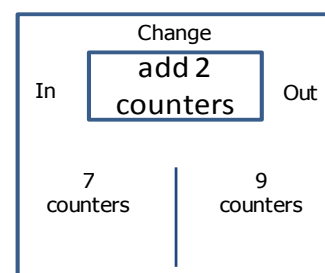
For this function machine, an input of 4 counters would give an output of 6 counters, while an output of 9 counters comes from an input of 7 counters.

3.3 Early to middle linear change and function activities

In the early to middle years of primary, change and functions work progresses to include numbered activities – first with the operations of addition and subtraction and then with multiplication and division. The numbered activities begin with one operation and involve presentation of change with formal symbols for the first time. Then the activities move on to two-operation function machines. At all times, it is important that this work begins from reality (from real-world problems), involves tables, inverse and arrowmath, and reverses everything (problem → rule for change, and rule for change → problem).

3.3.1 One operation – addition and subtraction

Discuss with students how addition and subtraction could be thought of as change. For example: *What happens if you have two more toys or dollars – what if you start with 3 toys or 3 dollars? What does 7 toys become if you give 3 toys to a friend?*

Set up a function machine that adds or subtracts a small number. The whiteboard function machine is still excellent but for this we will move on to the "robot" function machine. Basically it is a large box (in which students can stand) with a small "head", two openings each side, a rule hung around the neck or from the top (if there is no "head") as in example on right.
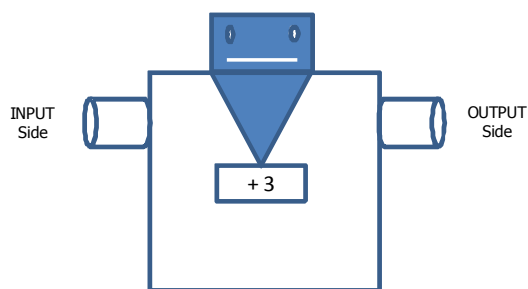
Two card sets are made – numbers 1 to 20 for input cards and 1 to 30 for output cards. Students, in twos, bring an input card to LHS of robot and place it in the opening. Students inside add 3 and push output card out RHS opening. Remaining students have a calculator to check that correct change has been made (e.g. 6 to 9) and a worksheet on which to record input and output numbers.

The following is a sequence of activities found useful.

1. Give students a **real-world problem** that adds/subtracts a small number. For example, *It costs $5 to have a present wrapped. What is the total cost of present and wrapping?* Discuss what we can do with this. [We cannot get answer as is but there are two things that can be done: (a) if given the present's cost, we can work out the total cost, e.g. present is $36, total cost is $41; and (b) if given the total cost, we can work out the present's cost, e.g. total cost is $24, present is $19.]

2. Students consider problem as a **change** – ask "what is the operation" and then draw a function machine as on right.
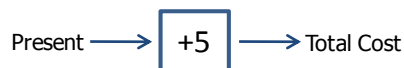
3. **Act out change with the function machine**. Organise a student to go into robot with output cards. Give other students input card numbers and ask them to bring them out front, in turn, to Input and then collect a changed card at Output.

4. **Fill in input–output table**. Students should follow the function machine activity with a calculator, checking calculations and filling in input–output tables. Ask students to complete tables without watching a student at the front use the function machine.

5. **Reverse the change**. Teacher directs a student to collect an output card without showing input. Ask class what was the input card. Walk the student backwards from Output to Input as you are doing this. Discuss options and how to find this inverse number. Teacher provides a series of input and output numbers for students to fill in on their input–output tables. Have large numbers as part of this.

6. **Develop inverse**. Teacher leads discussion on quick ways to find the inverses and encourages students to see that –5 gives inverse of +5.

7. **Use arrowmath notation**. Students are directed to write both changes as arrowmath diagrams using examples (as on right).

$$6 \xrightarrow{+5} 11 \qquad 8 \xleftarrow{-5} 13$$

Change   Inverse change

8. **Generalise the change and its reverse**. Choose a student and ask to go to Input on function machine. Tell other students that this student has a number to input but does not know what it is. Get class to discuss what the output would be. Do the same for any output number – what would be the input? Give students a variety of large numbers to say what the change would be. Get students to write their rules in language.

9. **Move on to symbols** (but do not push for accuracy or for everyone getting the answer). Ask what the output would be if input was n? Ask what the input would be if output was k? See if students can write $n + 5$ and $k - 5$.

10. **Reverse everything**. Give students a generalisation of a change (can give it as language or as examples of numbers). Ask the students to represent the change and its inverse, with examples, using arrowmath notation, fill in an input–output table for some values, draw the change as a function machine, and create a problem for it. For example:

Input $n \to n - 3$ Output

$$6 \xrightarrow{-3} 3 \qquad 11 \xleftarrow{+3} 8$$

Change   Inverse change
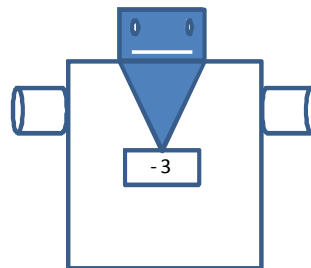
Input	Output
5	2
10	7
18	15
	84
112	

3.3.2 Arrowmath activities

It is useful for students to think of operations as change. So build the idea of "arithmetic excursions", travelling from number to number by actions or operators, as below.

$$2 \xrightarrow{+7} 9 \xrightarrow{+26} 35 \xrightarrow{-2} 33 \xrightarrow{+61} 94$$

Some ideas for arrowmath excursions include using calculators to:

(a) go from 2 to 62 by 4 changes;

(b) go from 68 to 1001 by 7 changes;

(c) go from 687 to 23 going through 2099 on the way; and

(d) make a long journey from 3679 to 9763 passing through 2 000 001 on the way.

Encourage students to use calculators without first working out in their head where they are going. The aim is for them to understand that making a number larger is achieved through adding, multiplying, or dividing by a fraction; similarly making a number smaller is achieved by subtracting, dividing, or multiplying by a fraction. If they get a decimal, just subtract it on the next move.

Use arrowmath notation to study principles for a variety of numbers. Some examples of activities are below.

1. The following examples use ? to represent "any number".

(a) Does the change on the left give the same output for given input as the change on the right?

$$? \xrightarrow{\times 4} \xrightarrow{\times 3} ? \qquad\qquad ? \xrightarrow{\times 3} \xrightarrow{\times 4} ?$$

(b) Does the change on the left give the same output for given input as the change on the right? Relate this answer to (a) – what does this mean?

$$? \xrightarrow{\times 4} \xrightarrow{+3} ? \qquad\qquad ? \xrightarrow{+3} \xrightarrow{\times 4} ?$$

2. Reverse arrowmath excursions and use arrowmath to study inverse, as below.

$$6 \xrightarrow{\times 8} 48 \xrightarrow{+64} 112 \xrightarrow{\div 4} 28$$

$$6 \xleftarrow{\div 8} 48 \xleftarrow{-64} 112 \xleftarrow{\times 4} 28$$

3. Importantly, there is a need at the end of middle school to begin relating arrowmath notation to equations, as below.

Change $\qquad 6 \xrightarrow{\times 8} 48 \xrightarrow{+64} 112 \xrightarrow{\div 4} 28 \qquad$ is $\qquad \dfrac{6 \times 8 + 64}{4} = 28$

Inverse $\qquad 6 \xleftarrow{\div 8} 48 \xleftarrow{-64} 112 \xleftarrow{\times 4} 28 \qquad$ is $\qquad \dfrac{28 \times 4 - 64}{8} = 6$

3.4 Later linear change and function activities

In the final years of P–9, change and function activities are used to generalise, introduce variable and algebraic expressions, draw graphs, and introduce functions.

3.4.1 One operation – multiplication and division

This repeats the activity from "One operation – addition and subtraction" (section 3.3.1). The function machine is set up for changes like ×5 and ÷4. The input and output cards have to be specifically selected. For change ÷4, input cards are 4, 8, 12, 16 and so on to 80, while output cards are 1 to 20. As for addition and subtraction, a student is put in the robot function machine while other students bring up input cards and receive output cards, and later bring up output cards in order to work out the inverse that gives the input. The idea is to cover the steps described below (using example of ÷4):

(a) start with a real-world problem;
(b) draw and set up a function machine for this problem;
(c) fill in an input–output table;
(d) draw arrowmath diagrams (forward and reverse);
(e) conceptualise inverse as ×4;
(f) generalise change in language and using variables, e.g. $n \div 4$ for input n, and $k \times 4$ for output k; and
(g) reverse the whole process – go from, say, $n \times 3$ right through to real-world problem.

Use calculators for arrowmath excursions, similar to the activities in section 3.3.2; for example, *Go from 654 to 268 by multiplications only*.

Two function machines – all operations

This repeats the activities from one operation (sections 3.3.1, 3.3.2 and 3.4.1) for two operations using two function machines, for example:
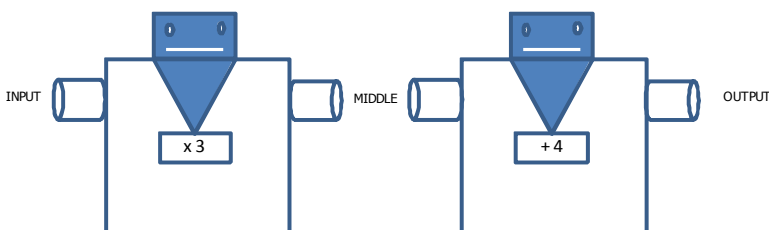
Input	Middle	Output
⑦	21	25
64	192	⑲⑥
ⓝ	$n \times 3$	$n \times 3 + 4$
p	③ⓟ	$3p + 4$
$\dfrac{k-4}{3}$	$k - 4$	ⓚ

This means that there are three columns in the input–output table and three generalisations, see above right for example (circled item is starting point).

Inverse is important here, as is the arrowmath, as it shows that, as well as inversing all operations, the order of the operations is also reversed, e.g.

$$6 \xrightarrow{\times 3} 18 \xrightarrow{+4} 22 \qquad\qquad 11 \xleftarrow{\div 3} 33 \xleftarrow{-4} 37$$
$$\quad\;\; 1^{st} \qquad 2^{nd} \qquad\qquad\qquad 2^{nd} \qquad 1^{st}$$

Once again it is important to go both ways: (a) from real-world problems to drawing to table to arrowmath diagrams to generalisation; and (b) then reverse from a generalisation to arrowmath to table to drawing to real-world problem. Do not be tempted to miss the real-world problem; it is crucial to relate the function machine to everyday life.

Note: For middle years, do not expect or require students to successfully generalise for a variable n. Students go through the following stages in generalising:

> Not being able to generalise
>
> Quasi-generalisation – doing it for any number given
>
> Saying it in language
>
> Writing it with letters
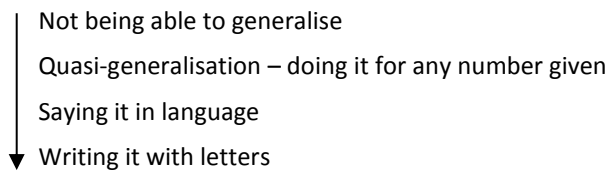
3.4.2 Two operations and backtracking

This activity repeats and extends the earlier activities by formalising the process of backtracking. The steps are as follows.

1. Begin with a two-operation problem: *I bought every student a Child's Meal for $3 and myself a Burger Meal for $7. How much did I spend?*

2. Discuss what can be done with this problem; encourage students to realise that if they know how many students, they can work out how much was spent; and if they know how much was spent, they can work out how many students. Discuss that these can be expressed as changes: forward – students to spending; and backward – spending to students.

3. Work out operations used and construct/draw function machines, e.g. robots as below.
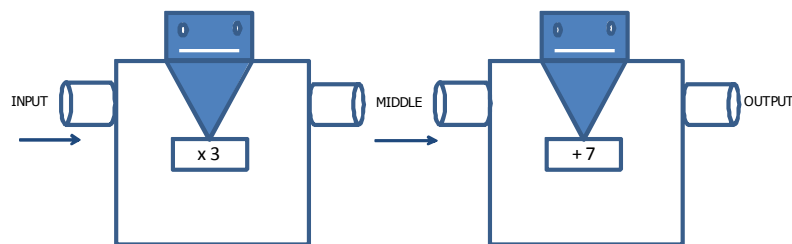
4. Do examples and record on input–middle–output tables. Students in robots with cards act out what happens. Other students check with calculators as well as record on tables. Note that in the table on right, the circled numbers are given and the rest have to be worked out and filled in.

Input	Middle	Output
5	15	(22)
9	(27)	34
18	54	(61)

5. Look at reversing the process. Again act this out, check with calculators and record on input–middle–output tables. Again, the circled numbers are given and the rest are worked out and filled in by students.

Input	Middle	Output
(3)	9	16
(11)	33	40
(156)	468	475

Discuss with students until they see that reversing involves the inverse of the operations, e.g. ×3 goes to ÷3 and +7 goes to −7. Also ensure students see that order of operations is reversed, e.g. ×3+7 goes to −7÷3. Get students to stand on Output side with say 22 on a card and walk them backwards to the Middle and the Input side showing the inverses as the students walk backwards. Introduce the term "backtracking".

6. Record forward and backward (reverse) as arrowmath, e.g. $9 \xrightarrow{\times 3} 27 \xrightarrow{+7} 34$ and $18 \xleftarrow{\div 3} 54 \xleftarrow{-7} 61$.

7. Generalise the forward and backward changes by using letters and requiring students to complete input-middle-output tables as on right. Again, the circled letters are starting points – the rest are worked out.

Input	Middle	Output
(n)	3n	3n + 7
p	(3p)	3p + 7
k-7/3	k - 7	(k)

8. Reverse everything – start with a generalisation, e.g. $2n + 3$, and work through arrowmath, chart and function machine to a real-world problem.

9. Use backtracking to solve real-world problems as follows.

   (a) Start with problem: *Each team member is to carry 3 litres of water and a truck is to carry 25 litres. How many in the team if there are 58 litres of water to be carried?*

   (b) Put this into arrowmath notation: $? \xrightarrow{\times 3} \xrightarrow{+25} 58$

   (c) Reverse this (backtrack) to show there are 11 team members: $11 \xleftarrow{\div 3} 33 \xleftarrow{-25} 58$

   (d) Use a worksheet where one of the four columns below is filled in and the rest are completed by students.

Real-world Problem	Forwards Arrowmath	Backwards Arrowmath	Solution

Motivating activity

Use the inverse of change and relation to equations to make up **"talking calculator"** activities. Discover which numbers upside down form letters (e.g. 0 is O, 1 or 7 is L, 3 is E, 4 is h, 5 is S, 8 is B). Make up a number which upside down is a word (like "shells"). Take this number, make changes to it with an arrowmath excursion and follow

What do you find on the beach? — Reverse Calculation — Hint

these with calculator. Reverse the sequence of changes and write as an equation – it should equal the original number but leave this place blank (i.e. do not show the original number as the answer to the reverse calculation). Google "talking calculator" for more examples.
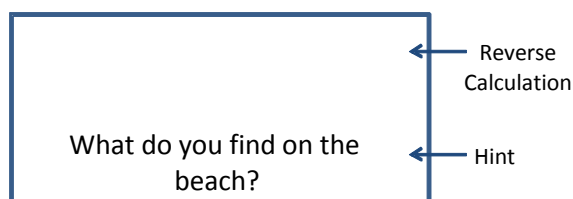
3.4.3 Guess my rule

As well as starting with a real-world situation, it is useful to start with the changes, determine the change rule and then construct a real-world situation. This is an example of reversing. To do this, follow the procedure below.

1. Teacher states that he/she has a change in his/her mind (a change rule). To reduce the number of probabilities, it can be useful to restrict numbers to 1, 2, 3 or 4, i.e. $4 \times p + 1, \frac{p}{3}$ and so on.

2. Teacher asks students to give him/her a number one at a time. The teacher then begins to fill in an input–output table and gives the output back to the student. For example, see table on the right.

Input	Output
2	4
7	19
11	31

3. Students have to identify the rule. (*Note:* May have to restrict the number of guesses from each student.)

4. Students have to construct a function machine for the change rule and a problem that leads to the function machine operations.

Note: For the example above, the rule is ×3 and −2, so the problem for this could be: *I bought bottles for $3 each and received a $2 discount*. This can require students to work a problem in a given context (e.g. school, sport, shopping, fishing, and so on).

5. Worksheets can be set up for students to practise this on their own. For example:

    (a) Look at the input–output table on the right.

    (b) Determine the change rule.

    (c) Construct a problem that leads to this change rule (driving cars context).

Input	Output
4	4
7	$5\frac{1}{2}$
13	$8\frac{1}{2}$
22	13

6. This can also be practised in the form of a group game for four players:

    (a) One player (called the rule setter) chooses a rule (e.g. ×2 +3).

    (b) Other players in turn give numbers to be changed (e.g. give 4, rule setter gives 11).

    (c) After each number is given, the player gets a chance to guess the rule.

    (d) The rule setter scores the number of guesses it takes for the group to guess the rule correctly.

    (e) The role of rule setter moves to the next person clockwise.

    (f) After a few rounds, the highest score wins.

Note: "Guess my rule" can be played with logic blocks (see Chapter 2 of the YDM Number book).

3.4.4   Linear equations and backtracking

The **first stage** here is to relate real-world problems to equations.

1. Begin with arrowmath excursions, e.g. $6 \xrightarrow{\times 5} \xrightarrow{-16} 14$

2. Write these as equations and discuss relationship between arrowmath and equations: $6 \times 5 - 16 = 14$

3. Introduce letters into arrowmath, e.g. $p \xrightarrow{+6} \xrightarrow{\times 4} 88$

4. Write these as equations: $(p + 6) \times 4 = 88$

5. Solve these by backtracking (gives $p = 88 \div 4 - 6 = 22 - 6 = 16$): $p \xleftarrow{-6} \xleftarrow{\div 4} 88$

The **second stage** is to translate real-world problems to equations and solve by thinking backtracking.

1. Give problem: *4 teams of players plus 9 adults got on buses. There were 57 people on the buses. How many in each team?*

2. Translate to arrowmath and equation (have to identify unknown and give it a letter, e.g. t):

$$? \xrightarrow{\times 4} \xrightarrow{+9} 57 \qquad t \times 4 + 9 = 57 \quad \text{or} \quad 4t + 9 = 57$$

3. Backtrack to get t: $t \xleftarrow{\div 4} \xleftarrow{-9} 57$

The **third stage** is to go from equations to answers by thinking backtracking, as shown below.

Equation: $2x + 5 = 17 \longrightarrow x = 17 - 5 \div 2 = 6$

Thinking: $x \xrightarrow{\times 2} \xrightarrow{+5} 17$ $\qquad 6 \xleftarrow{\div 2} 12 \xleftarrow{-5} 17$
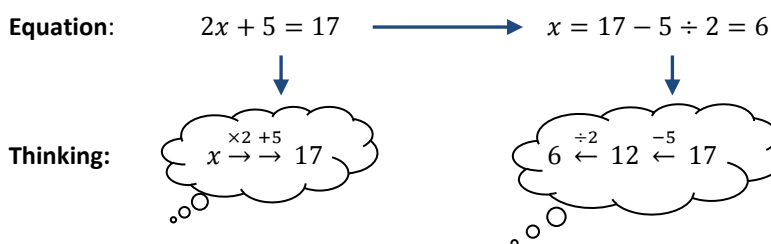
However, it is always important to relate to real-world situations.

Function machines and graphing

This extends the above activities to graphing and functions. The steps are as follows.

1. Start with a **problem:** Five fishermen each caught the limit of fish. They gave 7 fish away. This left them with 33 fish. What was the limit?

2. Translate to **function machine** (i.e. translate problem to changes) and identify change operations.

$$\longrightarrow \boxed{\text{x 5}} \longrightarrow \boxed{\text{- 7}}$$
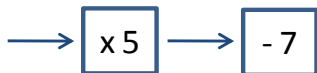
3. Complete input-middle-output **table**, identify reverse (inverses), and write change as **arrowmath** and **equations** and use backtracking (or reverse equation) to solve the problem for an unknown (in this example we use ? for unknown, but could use a letter).

$$? \overset{\times 5}{\to} \overset{-7}{\to} 33 \qquad 8 \overset{\div 5}{\leftarrow} \overset{+7}{\leftarrow} 33$$

$$5? - 7 = 33 \qquad ? = \frac{(33+7)}{5} = 8$$

4. **Generalise** forward and backward change, e.g. input n gives output $5n - 7$. Output k gives input $\frac{k+7}{5}$. Practise this also.

5. **Reverse from and to this point**. Start from a generalisation, e.g. start with ? × 3 – 5 (or $3n - 5$), fill in an input–output table, write arrowmath equations, draw the function machine and then develop a real-world problem that leads to this generalisation via a function machine. Go both ways.

6. **Extend to graphing**. Start from a problem, construct function machine, table, arrow equations and generalisation.

    (a) Fill in an input-middle-output **table**. Use table points to plot a **graph** (as below and on right for $5n - 7$)

Input	Middle	Output
1	5	-2
2	10	3
3	15	8
4	20	13
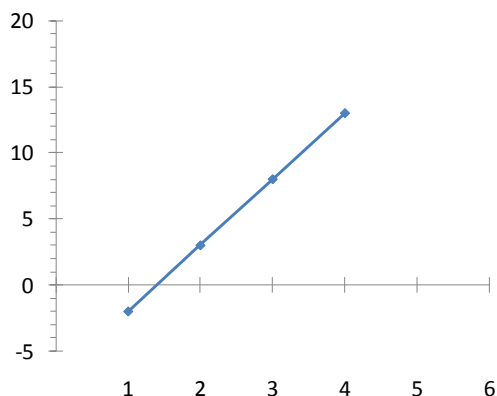
    (b) Do this a few times and relate the equations to the slopes and y-intercepts of the graph. For the example $3n - 5$, the slope is 3 and the graph cuts the y axis at –5.

    (c) Notice the generalisation: equation $y = mx + c$ gives graph with slope m and y-intercept c.

7. **Reverse** by going from graph to problem and then problem to graph again.

3.4.5 Linear functions and reverse linear functions

Functions can be considered in this "change" way by the following steps.

1. Look at **change** from x to y, as follows:

   (a) A function $y = 4x - 7$ can be considered as a change from x to y; for this example, the change is $\times 4 - 7$.

   (b) This can be written in arrowmath as on right: $x \overset{\times 4}{\rightarrow} \overset{-7}{\rightarrow} y$

2. **New notation** can then be introduced to represent function as f with $f(x)$ to denote variable being used, that is:

   $f(x) = 4x - 7$ is the same as $x \overset{\times 4}{\rightarrow} \overset{-7}{\rightarrow} f(x)$ or y

3. This leads to **tables and graphs** as below.

x	f(x) or y
0	-7
1	-3
2	1
3	5
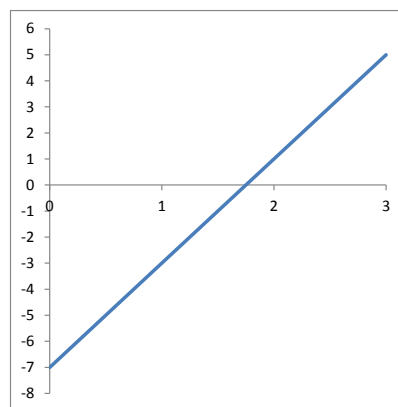
4. **Use to find inverse function**. A function, like an equation, can be used to construct input–output tables and to draw graphs. Backtracking can be used to find the inverse function as below.
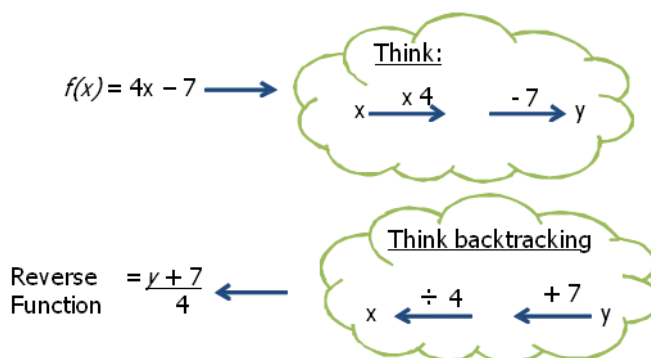
   Thus, if we stay with x as variable for all functions and label the inverse function as g, we have inverse function for $f(x) = 4x - 7$ being the backtracking:

   $$g(x) = \frac{x + 7}{4}$$

5. **Reverse everything** by starting from function or graph and using these to construct arrowmath notation for change and reverse change, input–output table, drawing of function machine, and story (real-world situation). Then go back the other way.

6. **Developing relationships between change rules and characteristics of graphs of linear functions** can be done using the same approach as in the later linear patterning activities (section 2.4). For example, we can show that, for change rule $y = 4x + 7$, the slope of the graph is 4 and the y-intercept is 7. (*Note:* In many ways function machines do what patterns do – focus attention on the relationship between input/position and output/term – however, function machines do this randomly while patterns do 0, 1, 2, 3, and so on, in order.)

3.5   Function applications and nonlinear change

This section looks at applications of the function machine approach to percent, rate and ratio and nonlinear changes.

3.5.1   Applications

One of the major big ideas of mathematics advocated by YDM is that all maths ideas can be conducted as relationships or changes. Thus, we can represent most problems in a change format:

$$\text{start} \xrightarrow{\text{change}} \text{end}$$
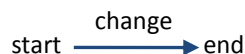
Problem types

If we can do this, then problems are easy to solve because they are based on three types:

- **Type 1** problems (end unknown) are solved by multiplying the start by the change.
- **Type 2** problems (start unknown) are solved by dividing the end by the change (reversing the change).
- **Type 3** problems (change unknown) are solved by dividing the end by the start (relationship between start and end).

One of the most prevalent and difficult problems with operations is multiplicative comparison which is:

$$\text{start} \xrightarrow{\text{multiplier}} \text{end}$$
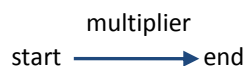
Here, the problem types are even simpler:

- **Type 1**: start × multiplier
- **Type 2**: end ÷ multiplier
- **Type 3**: end ÷ start.

Interpreting

So now we simply have to interpret the multiplicative value problems in terms of multiplication. There are three types of multiplicative comparison problems prevalent in upper primary, junior and secondary schools which are percent, rate and ratio. Interpret the problem in terms of change as follows.

Percent

$$\text{amount} \xrightarrow{\times \text{ percent}} \text{percentage}$$
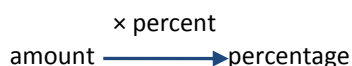
e.g. *85% of \$640 is ?   \$640 × 0.85 → ?*

Rate

$$\text{"per" attribute} \xrightarrow{\times \text{ rate}} \text{original attribute}$$
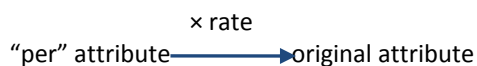
e.g. *carrots are \$3.65 per kg, how many kgs can be bought for \$20?*
? kg × \$3.65 → \$20

Ratio

$$\text{1}^{\text{st}} \text{ attribute} \xrightarrow{\times \text{ 2}^{\text{nd}} \text{ number / 1}^{\text{st}} \text{ number}} \text{2}^{\text{nd}} \text{ attribute}$$
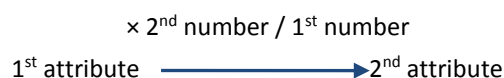
e.g. *flour:butter = 3:2, how much flour is needed for 1 kg of butter?*
? flour $\times \frac{2}{3}$ → 1 kg of butter

This approach means that all problems can be quickly solved if put into a change perspective.

(a) John sold his house for $480 000 which was a 27% profit. How much did it cost?

$$\$? \text{ cost} \xrightarrow{\times 1.27} \$480\,000 \quad \text{so} \quad \$? \text{ cost} = \$480\,000 \div 1.27$$

(b) Petrol was $1.65 per litre. How many litres can be purchased for $100?

$$? \text{ litres} \xrightarrow{\times 1.65} \$100 \quad \text{so} \quad ? \text{ litres} = \$100 \div 1.65$$

(c) Sand and cement is mixed 7:2. How much cement for 20 tonnes of sand?

$$20 \text{ tonnes sand} \xrightarrow{\frac{2}{7}} ? \text{ cement} \quad \text{so} \quad ? \text{ cement} = 20 \times \frac{2}{7} \text{ tonnes}$$

3.5.2 Nonlinear changes (Challenge)

Function machine activities can be repeated for nonlinear changes (e.g. quadratics and cubics).

Look at the problem from section 3.1.3: *The carpet was sold by m², the salesman measured the length of a square room and added 1 m² for error. How many square metres were sold?* This is the quadratic $n^2 + 1$. Go through the following steps: (1) construct function machine, (2) complete table, (3) construct arrowmath notation, (4) see if will backtrack, (5) construct a graph, and (6) reverse the procedure.

This problem will meet the requirements of the function machine – it even backtracks.

Consider another problem: *John bought USB memory sticks for his school. He paid the same price per stick as the number he bought. Then he also bought an extra 4 sticks for himself at the same price. How much did each stick cost?*

1. Construct the function machine:

$$\longrightarrow \boxed{\text{square}} \longrightarrow \boxed{\text{plus} \times 4}$$

2. Complete the input-middle-output table:

Input	Middle	Output
$7 sticks	7×7=49	7×7 + 4×7=77
4	4×4	4×4 + 4×4=32
11		
9		
		96
	25	
p		
		k

3. Construct arrowmath symbols and equations:

$$p \xrightarrow{square} p^2 \xrightarrow{+4p} p^2 + 4p$$
$$y = x^2 + 4x$$

4. See if there is backtracking. In this example, there is no reversing of the arrowmath change because we do not know what p is. However, $p^2 + 4p = p(p + 4)$, so we are able to use this for backtracking.

   If the output is 45, then the input is 5 because 45 = 5×9 and 9 is 5+4, that is, 45 = 5(5+4).

   What is the input for the following outputs?

   (a) 21   (b) 192   (c) 320   (d) 515

---

5. Construct a graph of change by plotting points from the input–output table. The graph should be part of a parabola, as below.
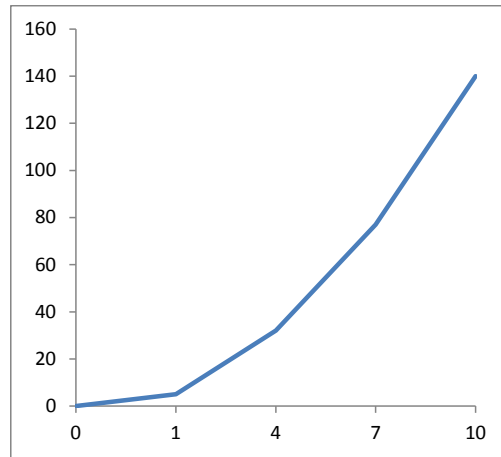
6. Reverse the procedure. Provide students with the graph for a quadratic, say $\frac{x^2}{5} + 2$ (see graph on right). Then the students have to do the following:
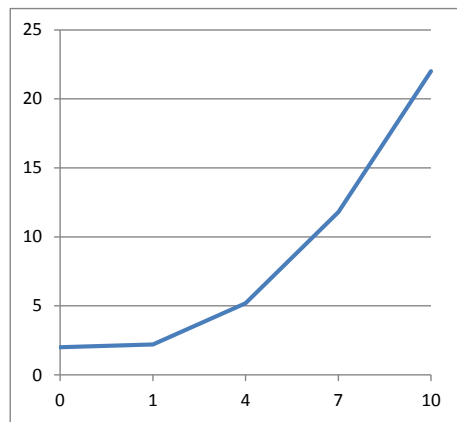
(a) write the equation;

(b) put equation into arrowmath notation;

(c) fill in an input–output table;

(d) draw a function machine; and

(e) write a problem that gives the function machine and leads to the equation.

Note: $y = \frac{x^2}{5} + 2$ is $\xrightarrow{\text{square}} \xrightarrow{\div 5} \xrightarrow{+2}$, that is $x \xrightarrow{\text{square}/5} \xrightarrow{+2}$. If reversed, it is $\xleftarrow{\sqrt{}} \xleftarrow{\times 5} \xleftarrow{-2} y$. This could also be seen as three function machines:
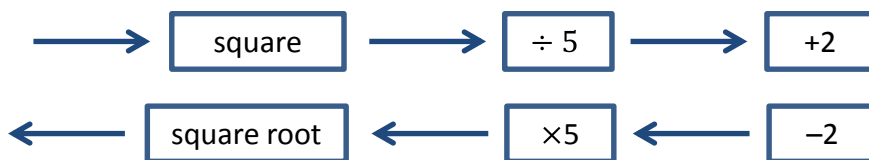
4   Equivalence and Equations

Equivalence and equations explores how to represent everyday life in terms of **relationships**. Thus, it studies the symbols, notation and rules for equations (number sentences with equal signs). In the long run, this is equations with numbers, operations and letters. To get to this point requires the study of the following:

(a) equations in arithmetic (no variables or letters);

(b) equations with unknowns for which calculation is in arithmetic form (called pre-algebra by some curricula); and

(c) equations with variables where calculation is in algebraic form (considered to be full algebra).

It is imperative that students learn symbols as ways of telling stories about everyday life. Initially, the symbols will be as in arithmetic (e.g. numbers and operation symbols). However, as problems move to where not all numbers are provided (e.g. *I bought a hat for $88 and a coat and spent $227 altogether*), the symbols will include unknowns (pre-algebra) and variables (algebra).

Thus in this chapter, we look first at the meaning of equals, greater than and less than, at number sentences with these symbols (equations and inequations), and finally at their principles (properties). Then the balance rule is introduced (first for unnumbered situations and then for numbered), and used to find unknowns. This leads to the method of solving equations which has the widest application because it can be used with variables on both sides of the equations. The chapter concludes with nonlinear equations (those with variables as squares or cubics), and sets the scene for modelling algebraic situations. This sequence is shown below.

Equals, equation and inequation (relation to reality)
↓
Principles (equivalence and order)
↓
Balance rule (unnumbered → numbered)
↓
Unknown/Solving linear equations
↓
Nonlinear equations/Formulae
↓
Modelling algebraically

Materials
Physical balance/Lengths
↓
Pictures and counters/
Number lines
↓
Abstract 'Math' balance/
Drawings/Equations
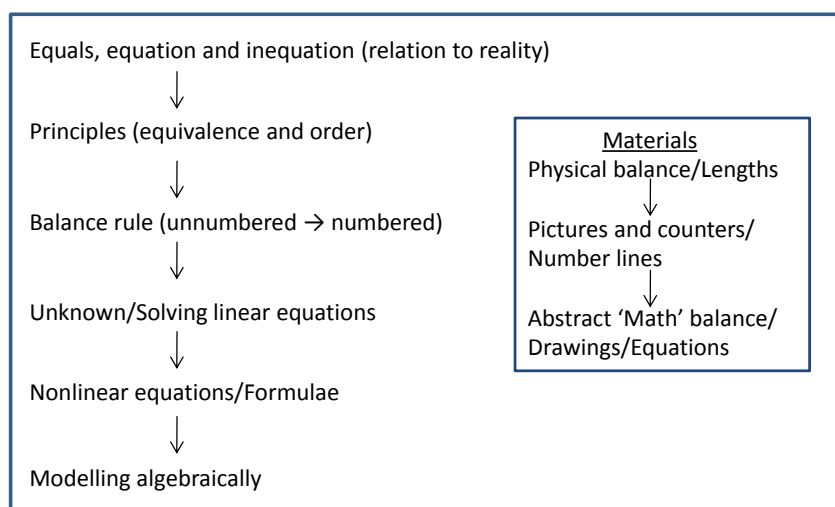
The physical balance and length materials restrict the operations that can be used for addition, subtraction and simple multiplication. They are also kinaesthetic and time consuming – excellent for beginning the teaching. The diagrams of balances and number lines enable more activities to be completed but still have restrictions on operations. Thus, it is important that students understand that the balance and the lines must become **abstract mathematical balances and lines** and able to represent any operation (including division). This means combining the abstract balances/drawings with equations. This leads to the final step which is to dispense with materials and just use equations.

The chapter begins with the major ideas and models (section 4.1) and then, similar to other chapters, looks at very early equivalence and equation activities (section 4.2), early to middle equivalence and equation activities (section 4.3), and later equivalence and equation activities (section 4.4), before finishing with nonlinear equations, formula activities and modelling (section 4.5).

4.1 Major ideas and models

This section covers major ideas, lists sequences of activities, discusses the main models, and looks briefly at nonlinear equations.

4.1.1 Major ideas

Equivalence and equations covers developing understanding of number sentences involving numbers, operations, variables, and equals, greater than and less than signs. These are called equations, inequations and expressions, and are defined as follows.

(a) An *equation* is a sentence, usually involving numbers, operations and variables, that has an equals (=) to show a relationship between two things (e.g. $2 + 3 = 5$, $2x + y = 16 + y2$).

(b) An *inequation* is an equation with greater than or less than symbols (< or >) showing an order relationship (e.g. $2 + 3 > 4$, $2x + y < 16 + y2$).

(c) An *expression* is one side of an equation; it has numbers, operators and variables but no equals or greater or less than symbols (e.g. $2 + 3$, $16 + y2$). Thus an equation is the equivalence of two expressions and an inequation is an order relationship between two expressions. To study equations and inequations is also to study expressions.

The major ideas to be covered in equivalence and equations deal with using equations to model real life and manipulating and solving the equations to solve these real-life problems. The sequence of activities designed to build this understanding and proficiency is as follows:

(a) introducing the notion of same and different and relating this to introduce equal, unequal, greater than and less than in length and mass (balance) situations;

(b) using mass and length in unnumbered situations to build understanding of equals and equations and to develop the equivalence and order principles;

(c) using mass and length in numbered situations to build understanding of arithmetic equations and reinforce the equivalence and order principles in numbered situations;

(d) relating arithmetic equations to real-life situations and vice versa (e.g. telling stories about the world);

(e) using mass and length models to introduce the balance principle that equations stay equal if the same thing is done to both sides of the equation;

(f) using mass and length models to introduce unknowns, and relate equations with unknowns to real-world situations and vice versa;

(g) extending mass and length models to mathematical versions (in picture form) where all operations are possible;

(h) using the balance principle to find solutions to equations with unknowns;

(i) developing rules/principles that enable expressions to be manipulated (including simplification and substitution); and

(j) introducing graphical representations of equations and showing how graphs, equations and everyday life relate.

4.1.2 Main models

The main models to be used are mass (the balance beam) and length (strips of paper and the double number line). Each of these will now be described. It should be noted that each of these models has to move from physical model to virtual or pictorial model and then to abstract maths model.

Mass

The materials/pictures begin with real balances (which can only really cover adding and some subtracting) and move on to pictures of "mathematical balances" (which can cover all operations) added with symbolic equations.

Equals is shown by the balance being "in balance", and not equals by the balance being "out of balance" (see examples below).

Drawing of mathematical Balance

4 + 8

3 x 4 + 2

$3 \times 4 + 2 \neq 4 + 8$
$3 \times 4 + 2 > 4 + 8$

Equation: $3 \times 4 + 2 = 16 - 2$
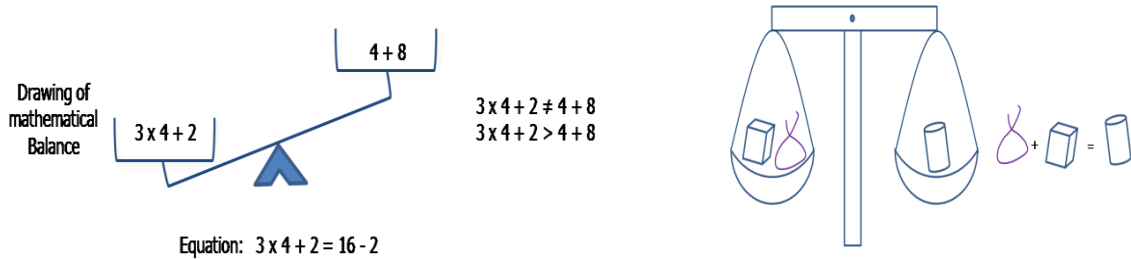
The balance is good for developing the balance rule and, in picture form, handling all operations.

2 + 4   6
Remove 1 from the left hand side
1 + 4   6
Remove 1 from the right hand side
1 + 4   5

balanced   unbalanced   balanced
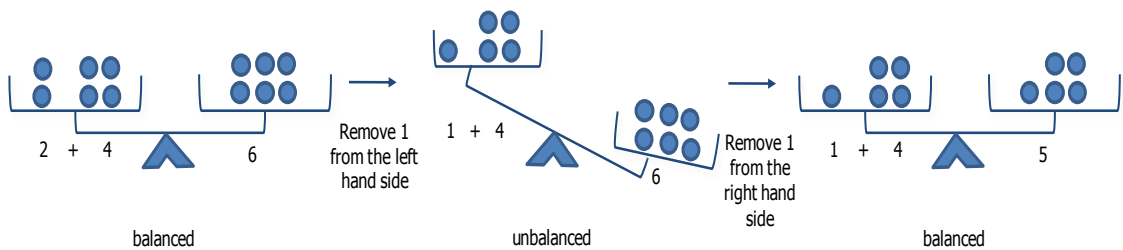
Virtual balances are also available online which provide a combination of the movement of the real balance with the range of operations available when using the picture balance.

Length

The materials for this model are strips of paper and single and double number lines (see below for diagrams). All these length models can handle addition and subtraction and the line and double number line can handle simple multiplication as well. However, more complex operations (particularly if they go into negative) are not possible. The number lines can be horizontal as shown here or vertical as shown later. The vertical line has the advantage of enabling an = sign to be placed under it and a left-hand and right-hand side to be identified (as in an equation).

Strips of paper

Green
Red   Blue

Green = red + blue

Single number line

–11
+5
n

$n + 5 - 11$

Double number line

2   4   –1
7

$2 + 4 = 7 - 1$

x   x   7
x   11

$2x + 7 \neq x + 11$
$2x + 7 > x + 11$
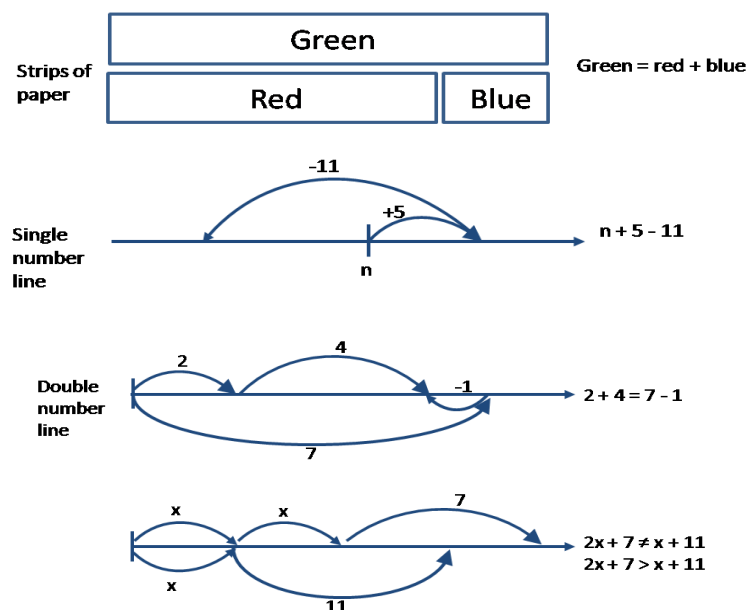
Length is also a useful material for teaching the balance principles, for example:

A = B + C | remove part of C to make D | A > B + D | remove the same part from A to make E | E = B + D
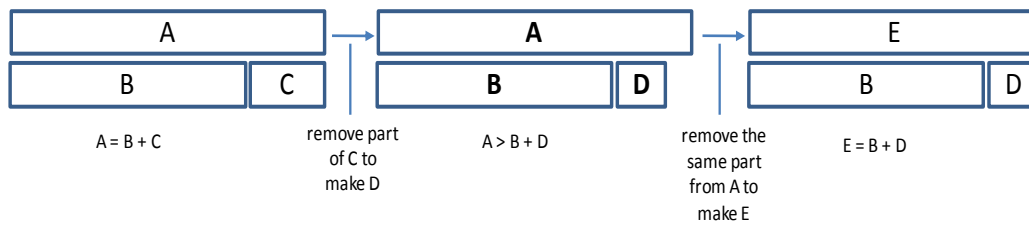
4.1.3 Advanced models

As we go up the years, the equations become more complicated. Physical balances and number lines can be used to show and solve simple equations, usually linear. But it is difficult to use them to show subtractions, divisions, complicated multiplications, and nonlinear examples such as quadratics. At this point, we need to move on to imaginary extensions of balances and lines, where anything is possible. One can subtract, divide, multiply, and go into negatives and so on. Anything mathematical is allowed.

Maths balance

Maths ruler
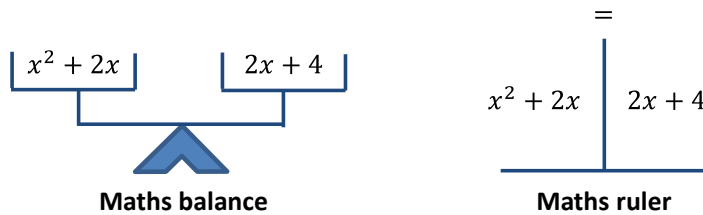
One simply thinks of the expressions being in balance or of the same length. So we can subtract $2x$ from both sides and then square root both sides. For example:

$$x^2 + 2x = 2x + 4$$
$$x^2 = 4$$
$$x = \sqrt{4}$$
$$x = 2$$

Subtract $2x$ from each side
Square root both sides
(remember negatives)
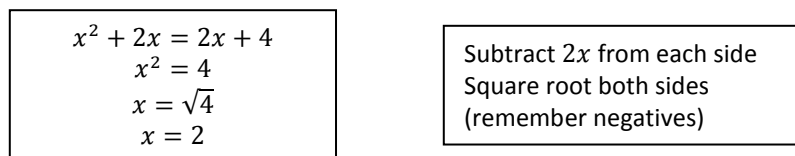
Once we have unknowns or variables, we can introduce models for unknowns. The following are useful:

(a) a bag covered in question marks into which weights can be placed to act out ? + 3 = 11;

(b) boxes of various shapes into which counters could be placed to act out Δ + O + 4 = Δ + 7; and

(c) cups ☐ and counters O to act out equations with variables (cup acts as variable and counters as ones) to act out ☐ + ☐ + OOO = ☐ + OOOOO.

4.1.4 Nonlinear models

For nonlinear examples such as quadratics, we can use advanced models but there is also an opportunity to think of algebra in terms of area, for example:

$x(x + 1)$ is ... is $x^2 + x$
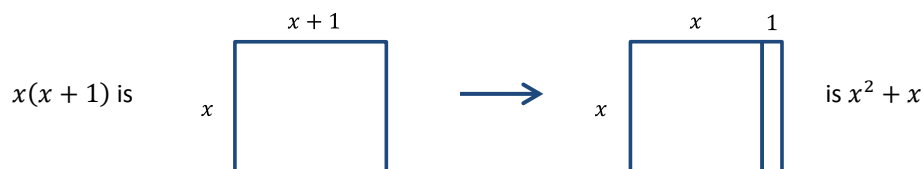
This leads to two models:

(a) area model for multiplication so we can multiply and factorise (see above); and

(b) tiles of three types: $x \times x$, $x \times 1$ and 1×1 so we can represent quadratics (see diagram below).

| | 1 | Small square | | x | Long rectangle which is the width of the small square | | x^2 | Large square whose length is the height of the x tile (i.e. $x \times x$) |
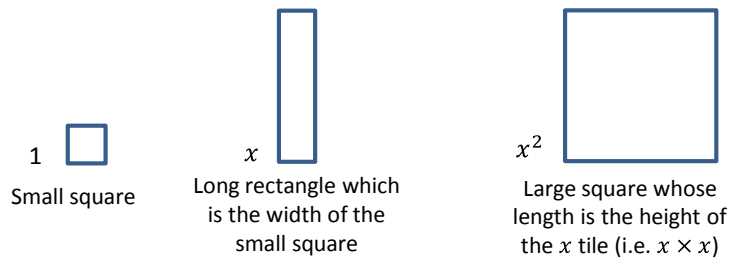
The tiles can be used as follows for example $2x^2 + 3x = x^2 + 4x + 2$.

1. Replace the x^2, x and numbers with tiles:

$2x^2 + 3x$ is 2 large squares and 3 rectangles

$x^2 + 4x + 2$ is 1 large square, 4 rectangles and 2 small squares
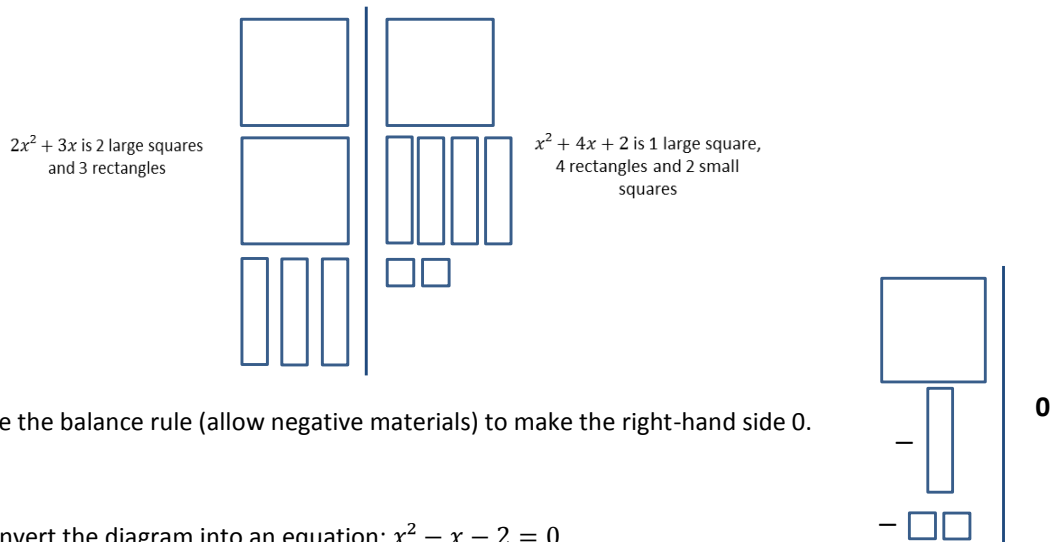
2. Use the balance rule (allow negative materials) to make the right-hand side 0.

3. Convert the diagram into an equation: $x^2 - x - 2 = 0$

4. Factorise the quadratic to get the answer (see section 4.5).

4.2 Very early equivalence and equation activities

Equivalence and equations studies arithmetic (and algebra) as relationships. It builds understandings of equations and inequations, unknown and variable, algebraic equations, and the balance principle. The major models on which teaching is based are balance (mass) and length models, as discussed in section 4.1. Thus, the techniques used in equivalence and equations are similar to the arithmetic principles. Equivalence and equation activities can begin in Prep as the following sequence shows.

4.2.1 Same and different

1. Students identify objects which are the same and which are different. They learn to describe what is the same and what is different about two objects. They sort objects into those that are the same and notice that different groups are different.

2. This understanding of same and different is then considered in terms of, particularly, mass and length. For mass, the teacher takes two plastic bags and places on students' arms, puts things in the bags and allows children to feel when things are the same and when they are different (as on right).

    For length, look at objects (e.g. paper strips) and see if the same or different lengths.

SAME   DIFFERENT

3. Once same and different are understood, the material can be used to introduce the formal language for same and different, namely, equals, not equals, greater than, less than. After the formal language the symbols are introduced, namely =, ≠, <, and >. For example:

| SAME | Slide on the word Equal | SAME | Slide on the symbol for Equals | SAME |
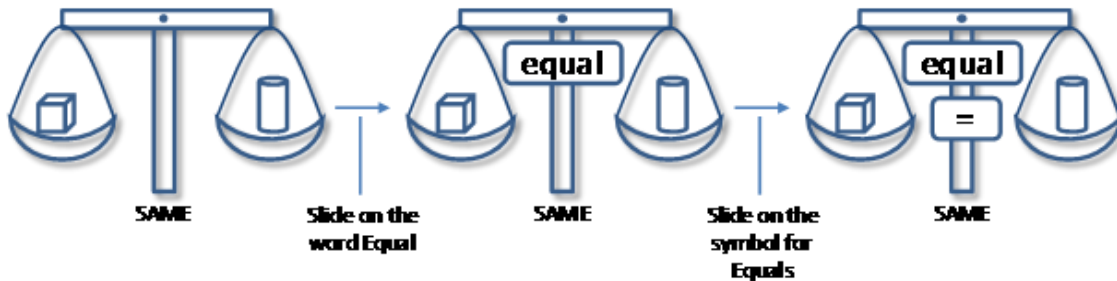
The technique is to discuss what is happening with respect to the balance [it is balanced] and introduce words and symbols by sticking the language and symbols on the balance. Relate the notion of balance to "equals" and imbalance to "not equals". Early on in the primary years, similar techniques could be used to introduce "greater than" (>) and "less than" (<).

Length can also be used in this way, but maybe not as strongly. For example:

A		A	
B	SAME Equal =	B	DIFFERENT Not equal ≠

4. Move hands along the balance to introduce equations for objects:

LHS	Balance Point	RHS
Box	In Balance so, Box =	So: Box = Ball

4.2.2   Unnumbered activities

1. The first formal activities should not use number; just different objects and different lengths. These are explored for equals and not equals and, later, greater than and less than. Students find different things to balance and not balance and record these as "equations" (not in the strictest sense), see example on right.

LHS

Balance so that Box + Ball = Jar + Scissors

2. With direction, exploration with materials and objects can be used to find the equivalence principles (see below).

Reflexivity:

Ball = Ball

Symmetry:

Box = Ball → Turn the balance around. → Ball = Box

Transitivity:

Ball = Box **And** Box = Jar → Find something (a box) that balances the ball — then something (a jar) that balances the box — check the ball balances the jar. → Ball = Jar
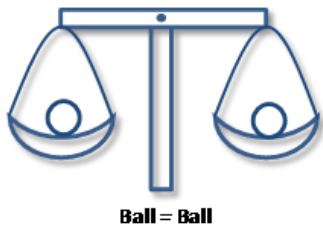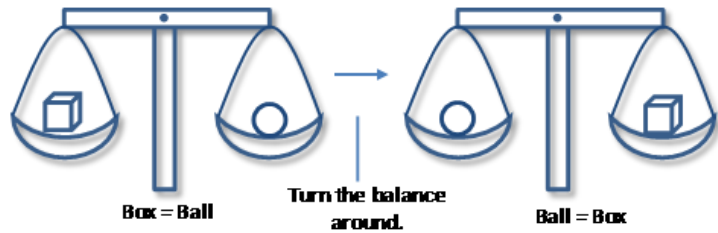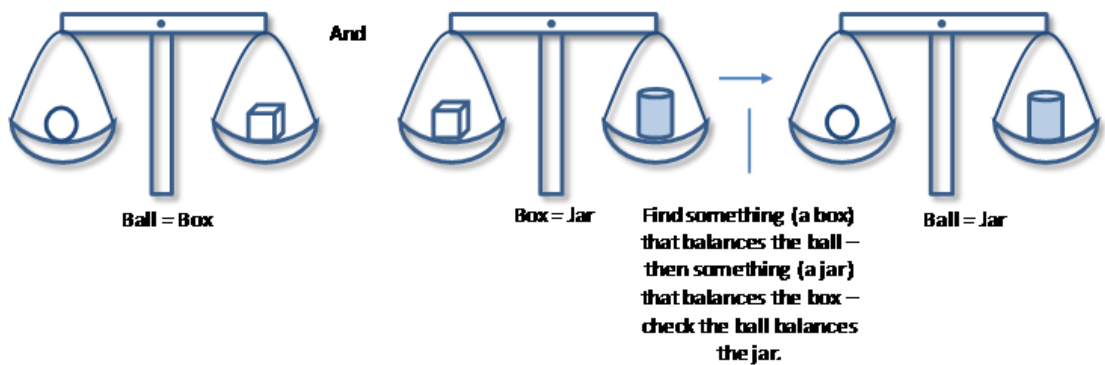
As pioneered by Davidov in Russia, unnumbered activities are an excellent way to begin work in a mathematics area. The lack of numbers appears to allow the students the freedom to explore structures and principles.

3. Similar work can be done with length and, because of ability to put things side by side, the length model can represent some things strongly. For example:

Reflexivity
A
A
A = A

Symmetry
A
B
Turn 180°
B
A
A = B → B = A

Transitivity
A
B
B
C
A = B and B = C
A
C
→ A = C
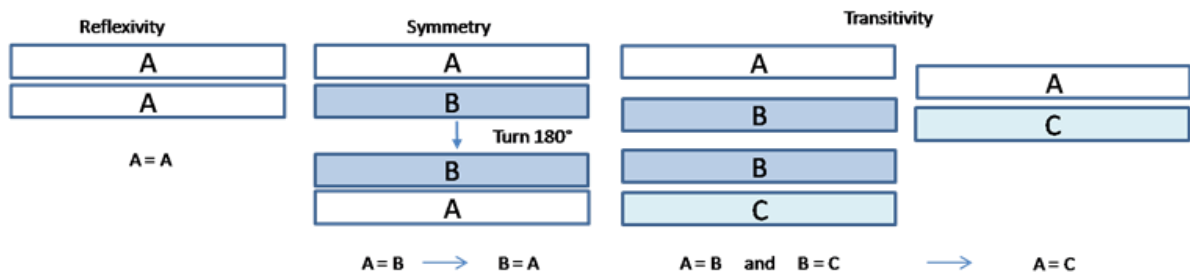
4.2.3 Numbered activities

1. Once unnumbered situations have been explored, numbers can be introduced by using same-size weights (we recommend small cans of baked beans) and same-size lengths (e.g. Unifix cubes). Then use models to represent equations with numbers, for example:

2 + 4 = 6

2 + 4 = 6
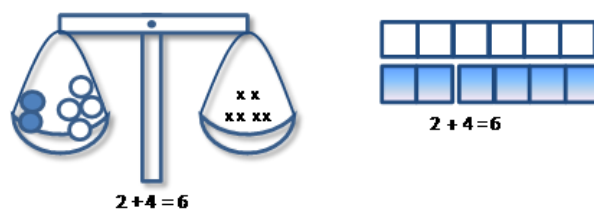
It is important to read the equations from the materials, e.g. 2 cans plus 4 cans balances 6 cans so 2 + 4 = 6; and to reteach the principles, particularly symmetry:

$3+5=8$ $8=3+5$
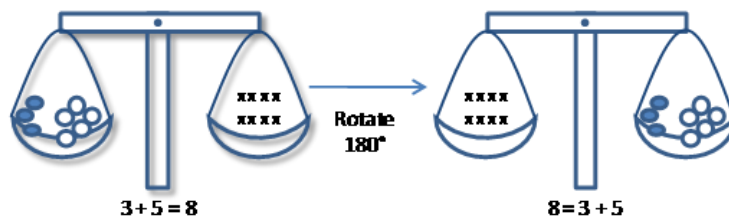
2. Extend the models to inequations (e.g. greater than and less than):

$2+3<7$ $7>2+3$
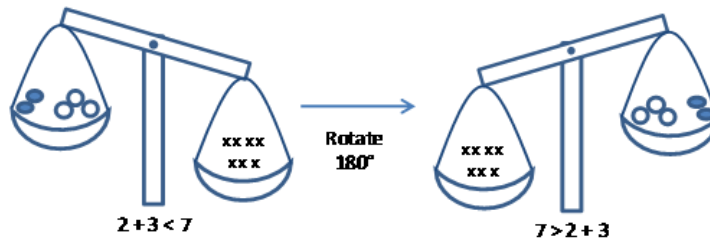
3. As students' experience grows, extend models to introduce mathematical-balance pictures and double number lines which can handle more operations:

$3+5=8$ $2+4=6$

$2+4=6$ $2+4=6$
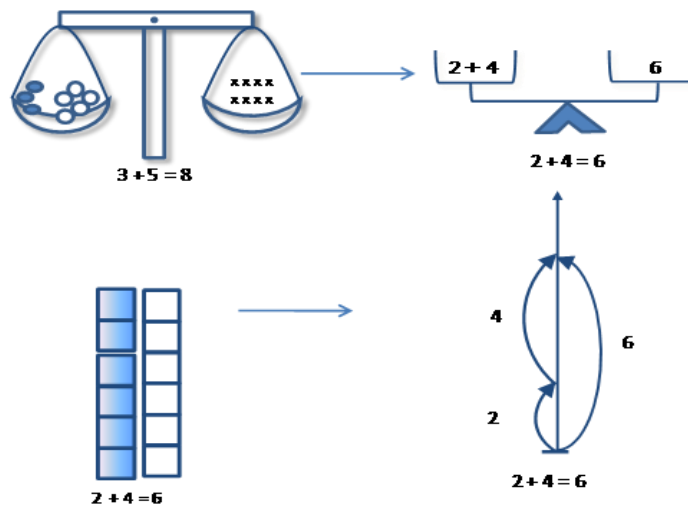
> *Note:* We have turned the cubes and the number line vertical because it shows LHS and RHS. This does not have to be done but it makes the equation easier to relate to the picture.

4.2.4   Relating equations to real-world situations

It is important to relate real-world situations to equations. To do this, relate stories to actual components of the equation, for example, see table below.

STORY	SYMBOLS
Two boys join three others	2 + 3
Two boys join three others, how many in all?	2 + 3 =
Two boys join three others to make five boys	2 + 3 = 5
Two boys join three others and this is the same number as six boys and one leaves	2 + 3 = 6 − 1

A good way to teach the relation of symbols and stories is to use a material context. Two examples are *The rice and the soap are the same as the pasta and the sugar,* and *The two weights and the four weights are the same as six weights*.
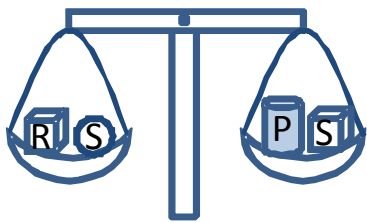
Rice + Soap = Pasta + sugar

2 + 4 = 6
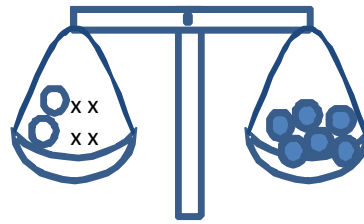
It is important to reverse this process, for example, *What is a shopping story for 3 + 5 = 8?*
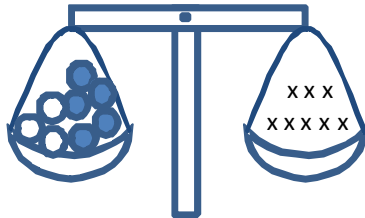
3 + 5 = 8

I bought a chocolate for $3 and a hamburger for $5, and spent $8.

One can also use length processes, for example:

There were 7 boys and 1 left, which came to the same number as the girls, where 4 joined 2.
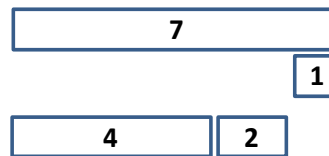
4.3 Early to middle equivalence and equation activities

From the models developed in the early years, activities can now build the important principles for solving equations.

4.3.1 Balance principle activities

Balances can be used to explore what happens when extra weights (e.g. one weight) are added or removed from a balanced equation.

2 + 3 = 5

Remove one

1 + 3 < 5
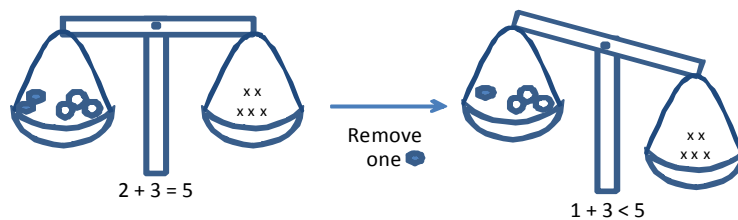
Students can be asked how to balance the equation again. There are three possibilities for the example above: (a) put the weight back again (this returns the equation to 2 + 3 = 5); (b) add another weight to the 3 on the LHS (this makes the equation 1 + 4 = 5); and (c) remove a weight from RHS (this makes the equation 1 + 3 = 4). The third possibility is the beginning of the balance principle and should be the focus of questioning (see diagram).

1 + 3 <5

Remove one x

1+ 3 = 4
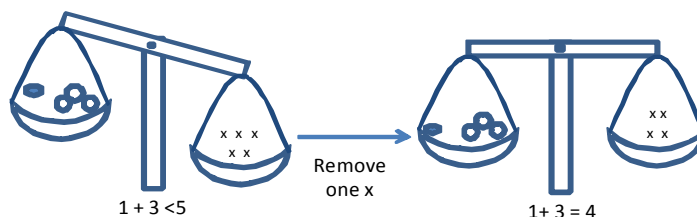
Direct the students to add and remove different weights and to rebalance. With questioning, try to get students to generalise this process to the full balance principle (e.g. "whatever you do to one side you do to the other"). The balance principle can also be introduced and demonstrated with length models (as shown below).
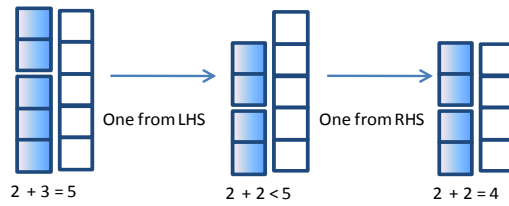
The balance principle can be reinforced with mathematical-balance pictures and double number lines as follows.
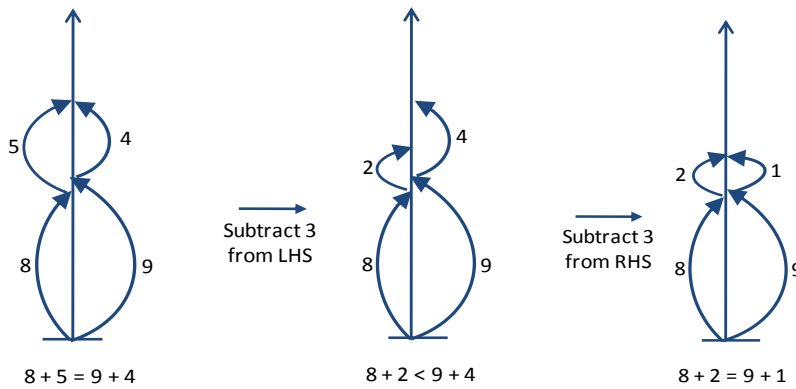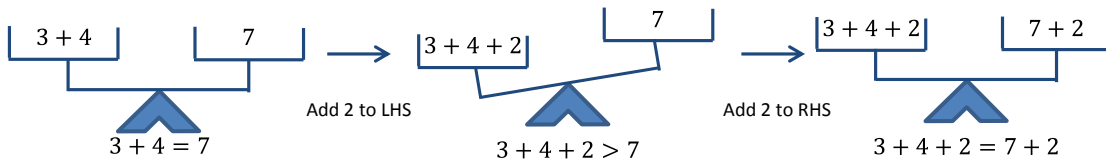
4.3.2 Relating real-world situations to equations

It is important to continue to reinforce the relationship between real-world situations and equations. The relationship is best taught by experience in which equations are deconstructed into parts and related to stories and vice versa (i.e. stories are deconstructed and related to equations). Some examples are as follows.

Story to equation: *I bought 3 chocolates for $4 each and a pie for $6. I spent the same as June, who bought a meal for $14 and drinks for $4.*

$3 \times 4 + 6 = 14 + 4$
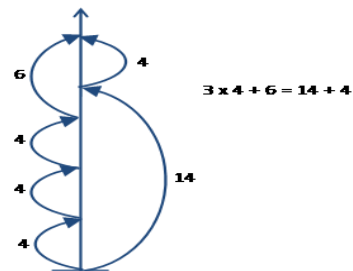
Reversing – equation to story: *The equation is 2 × ? + 6 = 3 × 8; what story can this tell?*
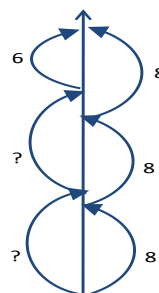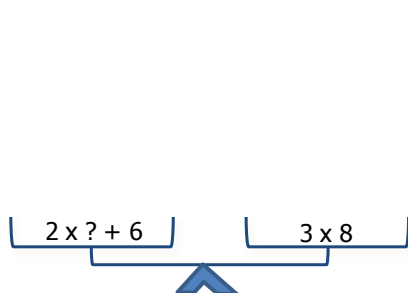
"My dad and my mum gave me the same amount of money. I already had $6, so I was able to exactly pay for 3 meals at $8 each."

One way to reinforce these relationships is with worksheets with headings as below in which teachers fill in one space in each row and the students fill in the other spaces. (*Note:* Students tend to have the most difficulty with creating their own stories.)

Story	Number line	Balance	Equation
1. I was given…….			
2. ……..			
3. ……..			
4. ……..			____ = ____

4.3.3 Introduce unknown and solve it

Discuss with students what they see an unknown as (students tend to choose "?"). Make a bag with the chosen symbol on it (e.g. ?), put in 3 weights and then balance it and 2 weights on one side with 5 weights on the other side.

? + 2 = 5
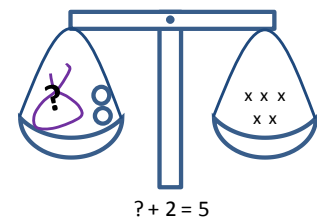
Discuss that the bag is unknown. Ask how we could find what is in the bag without opening it. Most students will know it is 3 because 3 + 2 = 5, but ask students to find a way to work it out without this knowledge – state that the numbers could be large.

Talk about how we can get the unknown on its own. Students can usually see that we can get unknown on its own by removing 2 weights and that this means removing 2 weights from RHS as below.
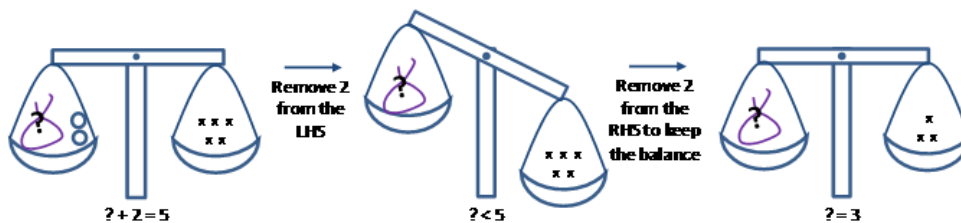
This can be translated to mathematical-balance diagrams and many examples done, for example:

The number line can also do this, first with blocks and then, more abstractly, with double number lines:

$? + 4 = 8 − 1$ Remove 4 from LHS → $? < 8 − 1$ Remove 4 from RHS → $? = 8 − 5 = 3$
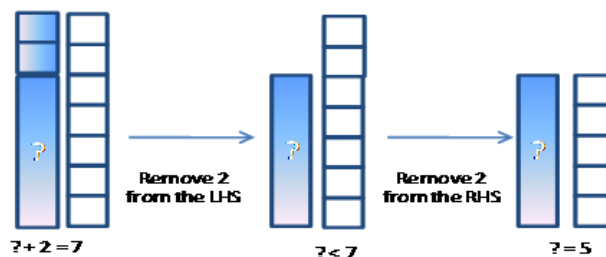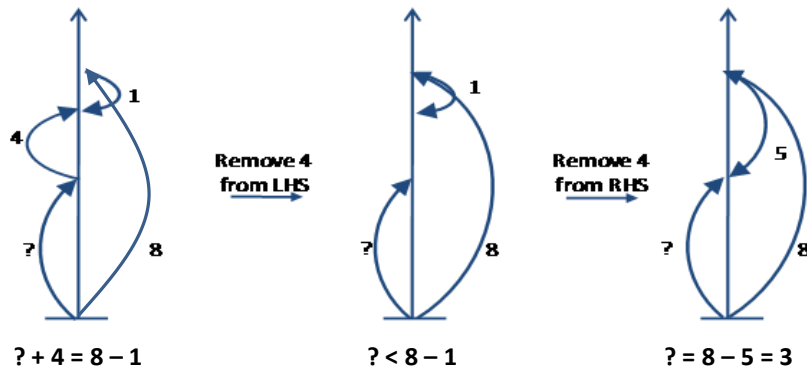
It is quite easy to extend this to more than one unknown and more than one type of unknown (see examples below).

One type of unknown: $(? + 7 = 2? + 3)$

$?+7 = 2?+3$ Remove 1 and 3 from both sides to maintain balance $4 = ?$
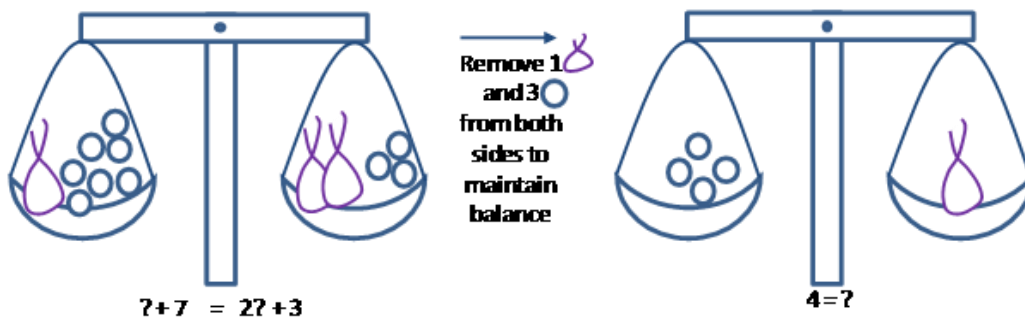
Two different types of unknowns: $(A + B + 3 = A + 7)$

$A + B + 3 = A + 7$ Remove A and 3 from both sides to maintain balance $B = 4$
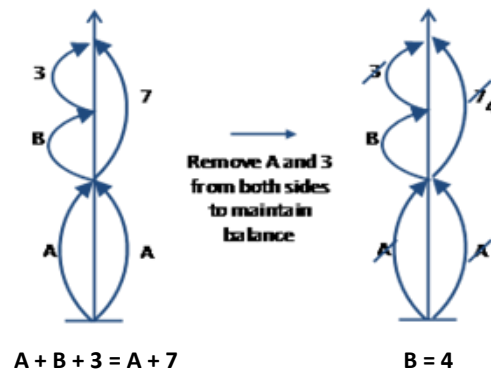
At this point, the process can be extended to symbolic equations. The balance drawings are the best way to make the transition because they can do all operations. Continue to stress that the drawings show a "mathematical balance" that can do all operations.

Picture		? notation		Variable notation
$2 ×? +3$　　15		$2 ×? +3 = 15$		$2x + 3 = 15$
remove 3 and ↓ divide by 2	$−3$ $÷ 2$	$2 ×? = 12$	$−3$	$2x = 12$
$?$　　6		$? = 6$	$÷ 2$	$x = 6$

4.3.4 Number-line activities

At this point, it is useful to introduce single number-line activities. In these activities, the line is used to act out real-world situations. For example, *I had $20, I spent $8, got an extra $10 and then spent $15, how much do I have now?* Moving along the line will give the answer.
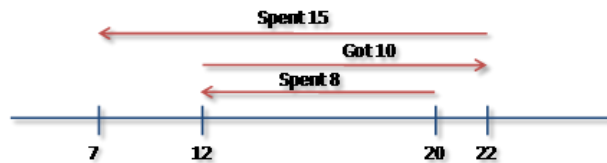
It is possible to use this number-line model when there is a variable. For example, *My Dad gave me some money, I spent $12, I received $8, then my Mum gave me the same money as my Dad, how much do I have now?* Need to come up with a symbol for the unknown – could be n or ?. Then the line helps.
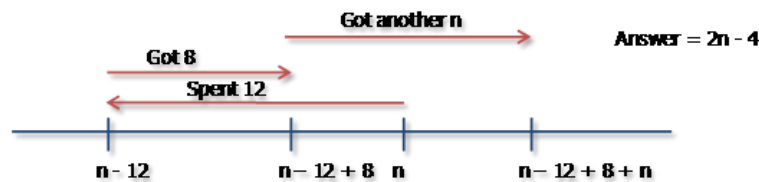
The line is useful for teaching inverse. This is particularly so for the inverse that makes an unknown on its own when solving an equation such as ? − 3 = 11. For example, *How do we change ? − 3 so that we get back to ? on its own?* The following line work can help students understand, and the line is a good model to explain or act out the process:
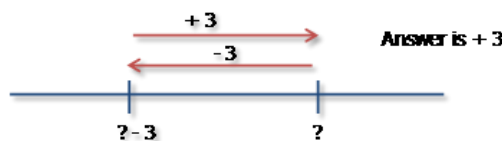
Thus, the number line can be used to find the answer to unknowns. For example, *I went out and bought a CD for $23, this left me with $16, how much did I start with?* To solve this, make the start n and use the line as below; n goes to $n − 23$, this is $16, so +23 to get back to n which is $39.
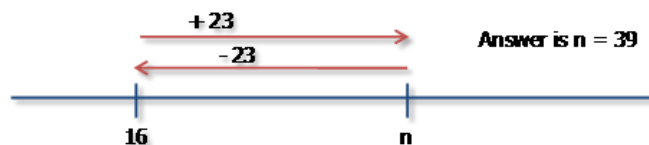
4.4 Later equivalence and equation activities

In the upper primary and junior secondary years, the skills learnt to model everyday activities and manipulate symbols can be widely used. It is a time when skills learnt in different chapters are brought together – balance rule and backtracking, equations and arrowmath symbols, expressions and equations.

In this section, we look at taking the balance rule from models to solving equations for unknowns, sequences and materials for moving from arithmetic to algebra (including substitution and simplification), and using expressions and equations to model the world.

4.4.1 Balance principle and solving equations for unknowns

This subsection looks at how the balance principle can be abstracted so it does not need to be connected to physical balances and lines – to connect it to mathematics as an abstraction.

Developing a concept from model to abstraction

The **mass model** for balance principle should develop as follows so that it goes from reality to abstraction across the years of schooling.

Real balance: no numbers

Can + bag = box + cup

Real balance: numbers
(only operations that make sense for balance)

$$3 + 4 = 7$$

Real balance acts out
number/variables and operations
that make sense for balance

$$2x + 5 = 11$$

Mathematical balance
(can do any operation)
numbers/variables

$$\frac{x}{4} + 7 = 2x - 1$$

Just equation

$$3x - 2 = 6 - x$$
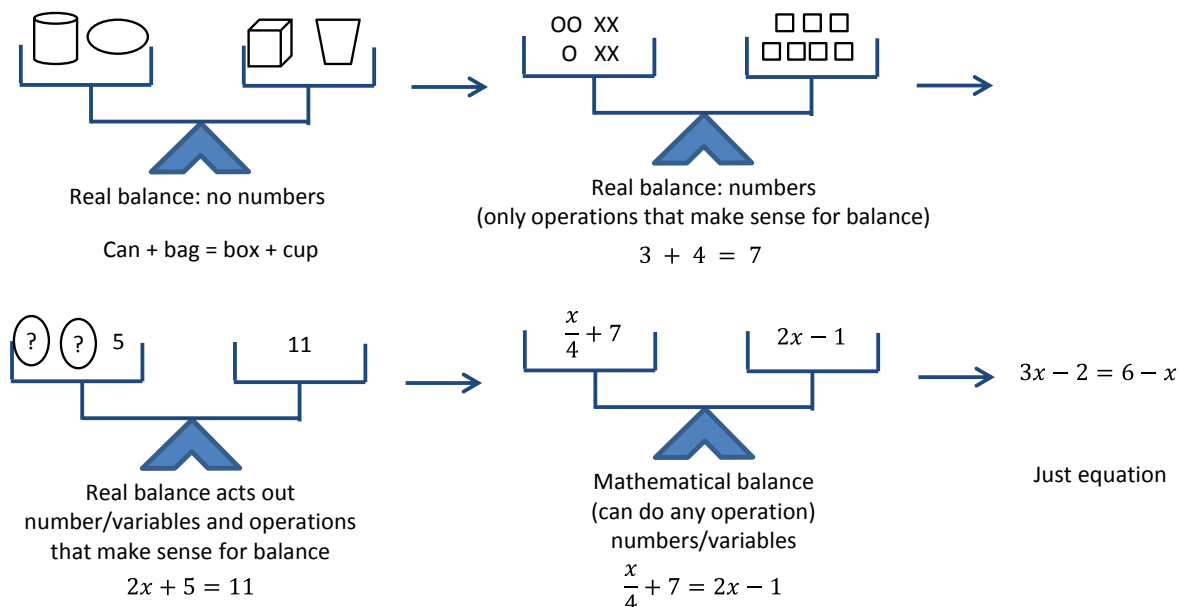
The **number line model** develops the same way:

No numbers
G = R + B

Unifix (real) numbers and operations
that make sense to use Unifix for

9 = 3 + 6

Only operations that make sense
for hopping along a line

$$2x + 3 = 11$$

$$\frac{x}{4} + 7 \qquad 2x - 1$$

Drawing variables/numbers
any operation

$$\frac{x}{4} + 7 = 2x - 1$$

$$\frac{x}{4} + 1 = 6 - x$$

Equation
variables/numbers
any operation
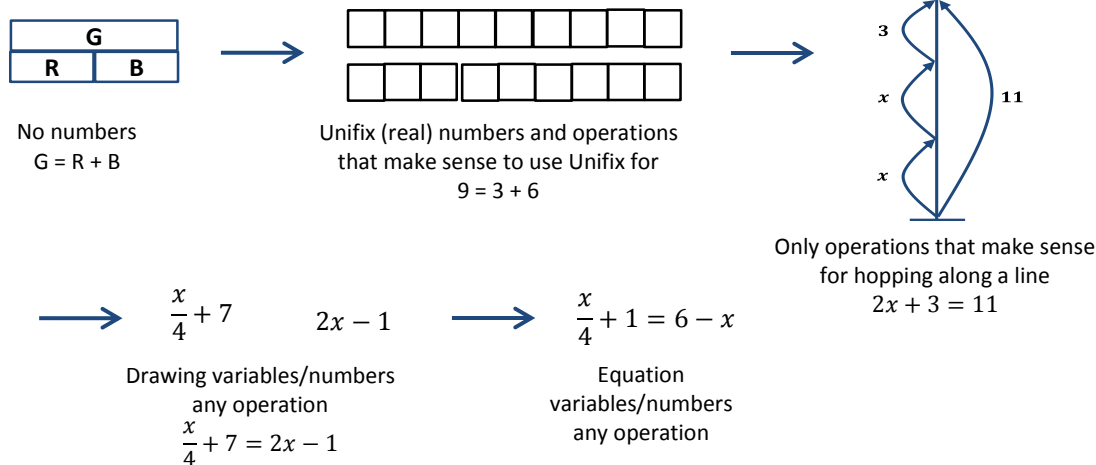
4.4.2 Analysing the balance principle

To use the balance principle means to understand it and how it relates to other ideas. The following should be worked through with students.

1. **Discuss the balance principle**. The balance principle states that if something is done to one side of an equation, the same has to be done to the other side to return to balance or to equal. Make sure students understand this. For example:

| $2x + 3 = 18$ | $2x + 3 + y \neq 18$ | $2x + 3 + y = 18 + y$ |
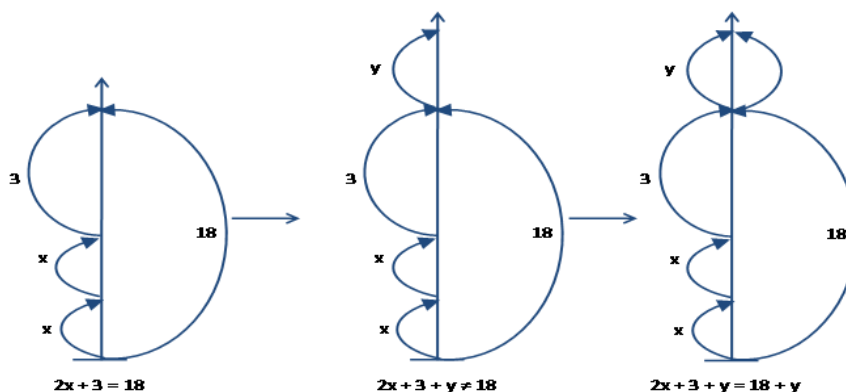
2. **"Same does not mean same"**. Ensure students understand that adding the same thing to both sides does not have to be in the same form; for example, you could +5 to the left-hand side and +7−2 to the right-hand side. Also in the abstract model any operation is acceptable.

3. **Relate equation and expression**. Keeping expressions and equations the same uses opposite strategies (like addition and subtraction use opposite strategies in compensation). The inverse or equivalence principle for an expression states that if the expression is to stay the same value then +0 and ×1 are the only possibilities and any change has to be compensated by the inverse change, for example:

$2x - y \neq 2x - y + k$ but $2x - y = 2x - y + k - k$ (because $+k - k = 0$) $= 2x + k - (y + k)$

The balance principle for equations and the equivalence principle for expressions, therefore, use the opposite strategy to ensure that any change has no effect. That is, to keep things the same, a change is repeated (on the other side) for an equation, while a change is undone or inversed (on the same side) for an expression.

4. **Solving for an unknown requires both expression and equation understandings**. The problem of solving for unknowns is that both expression and equation actions have to be done together and they can be mixed up. For example, to solve $2y + 3 = 35$ for y we need to look first at the expression and then the equation, as follows:

Expression:	$2y + 3$	$\rightarrow$	have to get y alone
		$\rightarrow$	y has been ×2 and +3
		$\rightarrow$	so −3 and ÷2 will get y alone
Equation:	$2y + 3 = 35$	$\rightarrow$	to keep balance, −3 and ÷2 on RHS as well as LHS
		$\rightarrow$	−3: $2y + 3 - 3 = 35 - 3$
			$2y = 32$
		$\rightarrow$	÷2: $\dfrac{2y}{2} = \dfrac{32}{2}$
			$y = 16$

However, some students get this confused and use the wrong rule in one of the parts, e.g. to keep balance do opposite; and some students get it doubly wrong but end up with the correct answer, for example:

$2y + 3 = 35$ $\rightarrow$ have to ×2 and +3 to get y (should be −3 and ÷2)

$\rightarrow$ equation means have to do opposite to RHS (should be the same)

$\rightarrow$ so, $y = 35 - 3 \div 2 = 16$ (gets right answer)

5. **Spend time discussing the two principles**. Ensure students understand the distinction. If there are problems, return the students to the models as in the example that follows.

Example: Consider $2x + 3 = 39$. The first stage is to consider the expression $2x + 3$ and use inverses to get x alone in the expression. This can be done with number line as on right.
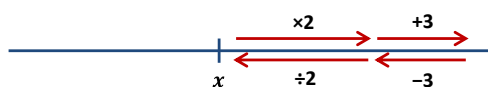
Then, the balance principle can be modelled on a balance as follows, first subtracting 3 and then dividing by 2 as the above number line indicates.

$2x + 3$		35

$2x + 3 = 35$

$\begin{array}{c} 2x + 3 \\ -3 \div 2 \end{array}$ | 35

$2x + 3 - 3 \div 2 \neq 35$

x | $35 - 3 \div 2$

$x = 16$
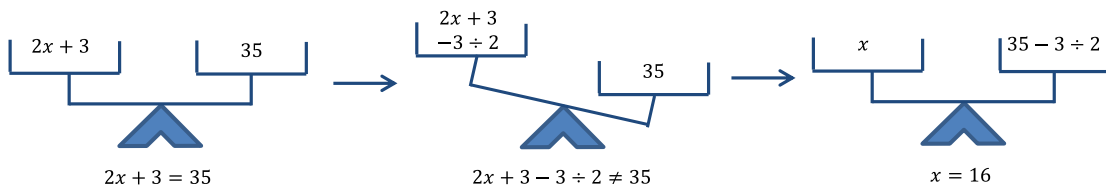
6. It is also necessary for students to realise that the inverse of an expression where there are more than two operations is not only the **inverse of each operation** but also **the inverse of their order.** For example, for this change:

$$1^{\text{st}} \text{ expression} \xrightarrow{\div 6} \xrightarrow{+11} 2^{\text{nd}} \text{ expression}$$

the inverse change is:

$$2^{\text{nd}} \text{ expression} \xrightarrow{-11} \xrightarrow{\times 6} 1^{\text{st}} \text{ expression}$$

4.4.3   Using sequences and materials to move arithmetic to algebra

To teach algebraic manipulation of expressions, including substitutions and simplifications, it is useful to follow the sequences below.

Sequence 1: Complex arithmetic activities as a step between arithmetic and algebra

The difference between arithmetic and algebra is that arithmetic expressions have separate processes, e.g. $2 \times 3 + 4$, and products (the answer), e.g. 10, while algebraic expressions have processes and products, e.g. $2x + 4$, as the same thing. Expressions can be closed (calculated out), e.g. $2 \times 3 + 4$ is $6 + 4$ is 10, in arithmetic but not in algebra. This means that, for most students, arithmetic expressions are always simple – a sequence of binary calculations to an answer, e.g. $2 \times 3 + 4 = 6 + 4 = 10$. However, algebra is mostly complex – consisting of expressions of two or more operations which cannot be calculated. This means that expressions have to be understood as processes involving many operations in algebra, but can be understood as a series of one-operation products (answers) in arithmetic.

Thus, we should not teach the traditional sequence from arithmetic to algebra which is really a large jump from arithmetic with one operation (which is understood as answers) to algebra with more than one operation (which is understood as processes), as in the diagram below:

Binary Arithmetic
e.g. 2×5, $5 + 3$

Complex Algebra
e.g. $2x - 6$, $(x + 3) \div 4$

We should teach it has two paths as in the diagram below (where we spend time with simple algebra and complex arithmetic before going on to complex algebra). By complex arithmetic, we mean understanding $2 \times 3 + 4$ as a process and not as 2×3 and $6 + 4$.

Binary Arithmetic
e.g. 2×5, $5 + 3$

Path 2

Complex Arithmetic
e.g. $2 \times 5 - 6, (5 + 3) \div 4$

Binary Algebra
e.g. $2x$, $x + 3$

Path 1

Complex Algebra
e.g. $2x - 6, (x + 3) \div 4$
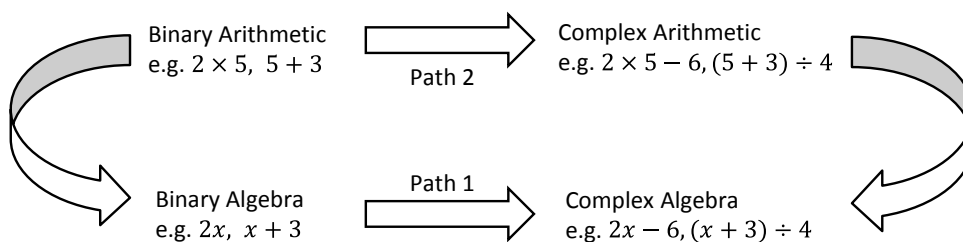
It is difficult to think of complex arithmetic examples because they rely on students not closing on the first binary part. An example of a good activity is as below. Arrowmath activities can also be good.

$$\text{“Work out } \frac{24+36}{6} \text{ without calculating } 24 + 36\text{”}$$

Sequence 2: Pre-algebra activities as a step from arithmetic to algebra

The first uses of letters are as unknowns. When used as unknowns, computation is predominantly arithmetic. This is called the pre-algebra stage. It is useful to go through this stage as in the table below.

Arithmetic	Pre-algebra	Algebra
e.g. $2 + 3 = 5$	e.g. $x + 5 = 16$	e.g. $x + 3 = 2x + 1$
$7 \times 4 = 28$	$3x + 7 = 25$	$x + 3y + 4x - 2y = 15$

In developing the algebra stage of sequence 2, we need, according to sequence 1, to build understanding of $x + 2$ and $3x$, and extend this to more complex examples. One way to do this is to use physical materials to model the algebraic expressions. Some materials and their uses are below.

Physical materials

Physical materials for algebra must have something for ordinary numbers and something that makes sense in terms of variable. In the use of the mass balance, we had a bag with a question mark on it into which we could put "any number" of weights. The following examples of physical materials have been used with success. (*Note:* There are also virtual forms of some of these materials available on the internet.)

Variable	**Numbers**	**Variable squared**
Envelope or box	Counters	
Cups	Counters	
Fixed length	A shorter length	A square tile

The way to use these materials is as follows.

$3x + 2$ 3 cups and 2 counters

$3(x + 2)$ 3 lots of 1 cup and 2 counters

$(2x + 1)(x + 3) = 2x^2 + 7x + 3$
(Use a distributive form of the materials as on right – see section 5.4.2.)

Thus, with materials, one can state a real-world situation, model it and give language and symbols. An activity to reinforce this is to get students to fill in the four columns of a table such as that below.

Story	**Material**	**Language**	**Symbol**
I bought 4 pies and a $7 cake		4 unknowns and 7 dollars	$4p + 7$ dollars

It is also important that this situation is reversed, that is, start with symbols, state language, model with materials and create a story. The creation of stories is powerful in teaching how to interpret stories.

Substitution and simplification

Once we are familiar with equations with variables, we can use these independently of real-world situations and the possible quantities the variables may represent. Other than solving for unknowns, the two important mathematical ideas are *substitution* and *simplification*. These are based on extension of the associative, commutative and distributive arithmetic principles to algebra.

1. **Substitution**. Substitution involves replacing the variable with a number and using arithmetic to work out the answer, for example:

$2x + 5$ when $x = 7$: $\qquad\qquad 2x + 5 = 2 \times 7 + 5 = 19$

$3x + y - 7$ when $x = 5, y = 9$: $\qquad 3x + y - 7 = 3 \times 5 + 9 - 7 = 17$

2. **Simple simplification**. Simple simplification involves using arithmetic understandings to calculate with variables – it is simply examples like $2y + 3y = 5y$ and $2 \times 3y = 6y$ for linear expressions, it is not factorisation of quadratics. These skills are necessary for the balance rule to be applied.

   As will be shown in subsection 5.4 of the next chapter, an effective way to build understanding of the rules of variable or algebraic calculation is to build the calculation rules from patterns seen in arithmetic (see below). However, it is important not to think that letters stand for objects; rather, they stand for numbers of objects. For example, if we have a box of apples and we use the letter a, it stands not for apples but the number of apples in the box. Thus, patterns for letters must come from numbers.

   > 3 apples + 2 apples = 5 apples
   > 3 eights + 2 eights = 5 eights
   > 3 hundreds + 2 hundreds = 5 hundreds
   > 3 any number + 2 same number = 5 of any number
   > $3x + 2x = 5x$

   This method of finding the pattern from arithmetic can be repeated to show simplifications such as $3x + 2y + 4x = 7x + 2y$, $3x = x + x + x$, 5×3 m = 15 m and even multiplication of x by itself = x^2, and $3a \times 4b = 12ab$.

4.5   Nonlinear equations, formula activities and modelling

As discussed in other sections, linear is the major focus of equations up until Year 9. However, there are nonlinear forms that can be considered (e.g. quadratics, cubics, exponentials). This section will look at these from two perspectives: (a) the role of the balance rule; and (b) solutions to quadratics beyond the balance rule.

4.5.1   The balance rule and nonlinear equations

In algebra, linear equations involve variables. However, these variables are simply x or y. They do not involve indices such as squares, cubes or exponentials, as can be seen below.

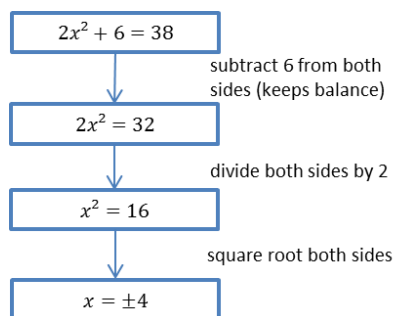| Linear equations	Nonlinear equations
$2x + 1 = 11$	$x^2 + 1 = 5$
$\frac{x}{4} - 2 = x$	$2x^3 - x^2 = 45$
$x + y = 8$	$x^2 - 3x = 2x^2 + 7x - 8$
$3x + 2 = 4x - 7$	$2x^3 - x^2 + 7 = 54 - x^3$
$2x + 3y - 4 = 2x - y + 8$	$2^x - 5 = 27$

However, some features are still the same for nonlinear equations:

1. Nonlinear equations are still two expressions equivalent to each other, e.g. $x^2 - 2x = 3x - 6$.

2. Nonlinear expressions are still symbolic of describing things in real life, e.g. x^3 is the volume of a cube. $6x^2$ is the surface area of the same cube, and $x^3 + 7x^2$ is the volume of a cube of side x plus a square prism with an end length of x and a height of 7.

3. Equivalence for nonlinear equations is still "the same value as" and still follows the rules of reflexivity, symmetry and transivity.
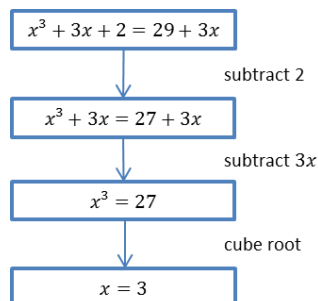
This means that the balance rule applies to nonlinear equations – whatever you do to one side, you should do to the other. There are, however, more possibilities in terms of what you can do. In linear equations we could add, subtract, multiply and divide. Now we can also do square root, cube, cube root, make exponentials and reverse the exponentials (logarithm) – see examples in right column of table on previous page.

We only have to check that the change is "well-defined" and only gives one answer. This is mostly true but for square roots, there are two possibilities: positive and negative. For example, using the balance rule to solve $2x^2 + 6 = 38$ gives two answers, +4 and −4, as the following shows:

$$2x^2 + 6 = 38$$

subtract 6 from both sides (keeps balance)

$$2x^2 = 32$$

divide both sides by 2

$$x^2 = 16$$

square root both sides

$$x = \pm 4$$

Keeping the above in mind, the balance rule can solve nonlinear equations in cases where all the unknowns are on one side and simple (only having one variable with one coefficient). Two further examples are provided below.

1. $x^3 + 3x + 2 = 29 + 3x$ can be solved by the balance rule alone:

$$x^3 + 3x + 2 = 29 + 3x$$

subtract 2

$$x^3 + 3x = 27 + 3x$$

subtract $3x$

$$x^3 = 27$$

cube root

$$x = 3$$

2. $x^2 + 3x + 2 = 29 + 2x$ cannot be solved by the balance rule alone:

$$x^2 + 3x + 2 = 29 + 2x$$

subtract 2

$$x^2 + 3x = 27 + 2x$$

subtract $2x$

$$x^2 + x = 27$$
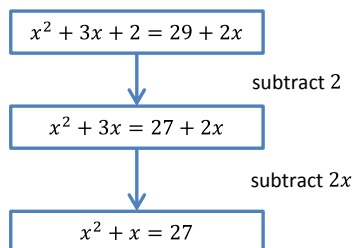
To be solved, this would need to be made equal to zero and the expression factorised or the rule for determining unknown for a quadratic used (see next subsection 4.5.2).

Challenge: Get students to make up nonlinear examples that can be calculated by the balance rule alone and solve them. For example:

(a) $x^3 + 2y - 4 = x^3 + y - 10$

(b) $3x + x^2 - 2 = x^2 + 15$

4.5.2 Going beyond the balance rule for quadratics (challenge)

One of the reasons the balance rule alone cannot solve quadratics is that the x^2 and the x often mean that there are two types of x that end up on one side of the equation, e.g. $2x^2 - 3x = 7$. However, the balance rule plus two other pieces of information can be combined to develop a way of solving quadratics. The two other pieces of information are:

(a) quadratic expressions are multiples of two linear expressions, e.g. $2x^2 - 3x = x(2x - 3)$ and $x^2 - 5x - 6 = (x + 1)(x - 6)$; and

(b) the product of two expressions can only equal zero if one of the expressions equals zero, e.g. $(x + 1)(x - 6) = 0$ means $x = -1$ or $x = 6$

Putting all of this together, we can solve quadratics by using the following steps to factorise the quadratic:

Step 1	Use the balance rule to make one side equal to zero.	$x^2 - 3x = 6x - 4 - x^2$ $\downarrow$ add x^2 $2x^2 - 3x = 6x - 4$ $\downarrow$ add 4 $2x^2 - 3x + 4 = 6x$ $\downarrow$ subtract $6x$ $2x^2 - 9x + 4 = 0$
Step 2	Determine a way to make the left-hand side equal to a product of two linear expressions.	$-4 \times -1 = 4$ $(2 \times -4) + (1 \times -1) = -9$ $\downarrow$ $2x^2 - 9x + 4 = (2x - 1)(x - 4)$ $\downarrow$ $(2x - 1)(x - 4) = 0$
Step 3	Work out the x's that will make the two linear expressions equal to zero. (Two answers are $x = \frac{1}{2}$ or $x = 4$)	$(2x - 1) = 0 \longrightarrow x = \frac{1}{2}$ $(x - 4) = 0 \longrightarrow x = 4$
Step 4	Check by substitution.	$x = \frac{1}{2}: x^2 - 3x = 6x - 4 - x^2$ $\frac{1}{4} - \frac{3}{2} = 3 - 4 - \frac{1}{4}$ $\longleftarrow$ correct $x = 4: 16 - 12 = 24 - 4 - 16$ $4 = 4$ $\longleftarrow$ correct

A formula has been constructed to find the two answers, no matter what the quadratic:

If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use either factorisation (as in the table above) or the quadratic formula to determine the expressions below:

(a) $3x^2 + 4x - 7 = 2x^2 + 7x - 9$

(b) $2x^2 - x + 6 = 6 - 2x^2 - 4x$

(c) $3x^2 + 5x - 7 = 6 - 4x - x^2$

Note: The algebra tiles (see section 4.1.4) can be used as a physical material to assist with the balance rule part of the above process.

4.5.3   Formulae activities

This subsection looks at (a) formulae as a way of introducing the unknown and variables, and (b) algebra applications with formulae.

Using formulae to introduce variable

So far in this book, we have seen how the following can be used to define variable and to introduce algebraic expressions mostly in linear form, e.g. $2n + 3$ or $\frac{n}{2} - 4$, but also in nonlinear form, e.g. $n^3 + 1$:

(a) using function machines and backtracking, and equivalence equations and the balance rule to introduce equation, equations with unknowns, pre-algebraic expressions and equations, and the notion of the unknown (i.e. n or x can be a symbol for an unknown number) as a precursor to variable; and

(b) using patterns (finding the position rule) and function machines (finding change rules) to introduce the notion of variable (i.e. n or x stand for any number) and algebraic expressions.

However, there is a third way to introduce unknown and variable, and algebraic expressions and equations:

(c) using formulae from measurement, geometry, probability and statistics.

For example, we can study rectangles drawn on square grid paper and see that a 4 by 3 rectangle has 12 square units. This enables us to see that the area of a rectangle is length × width. After meaning has been developed, the long written relationship is changed to a formula using letters, that is, A = L × W. In this way A, L and W are introduced as unknowns and variables.

Interestingly, formulae are not restricted to linear forms. For example, area of a circle is πr^2, volume of a cube is L^3, number of diagonals in an n-sided polygon is $\frac{n(n-3)}{2}$, and number of different outcomes for throwing a dice is 2^n.

Applications with formulae

Obviously formulae are used to determine the answers to the relationship being studied. This means that formulae are used in **substitution**. For example, the volume of a cylinder is $\pi r^2 H$. A common activity therefore is to work out volumes for given radii and heights. For example, if a radius (R) = 2 m and height (H) = 4 m, then the volume of the cylinder is $\pi \times 2^2 \times 4 = 16\pi$ cubic metres or m³.

However, the difficult problems in using formulae usually involve changing the subject of the formula. This requires using the balance rule. For example:

1. The tradesperson has to build a cylindrical tank with a diameter of 4 m to hold 120 m³. How high does the tank have to be?

Step 1	Write formula	$V = \pi R^2 H$
Step 2	Substitute	$120 = \pi \times 2^2 H$
Step 3	Change subject to H by using the balance rule	$\pi 2^2 H = 120$ $\downarrow$ divide by $\pi 2^2$
Step 4	Calculate answer	$H = \frac{120}{4\pi}$ m

It should be noted that this change of subject could be done without numbers; for example, change the subject of $V = \pi R^2 H$ to H.

Step 1	Write formula	$V = \pi R^2 H$
Step 2	Put H on the left-hand side of the equation	$\pi R^2 H = V$ divide by π
Step 3	Use the balance rule	$R^2 H = \dfrac{V}{\pi}$ divide by R^2
Step 4	Have a new formula	$H = \dfrac{V}{\pi R^2}$

2. The tradesperson has to build a cylindrical tank with a height of 5 m to hold 120 m³. What is the diameter?

Step 1	Write formula	$V = \pi R^2 H$
Step 2	Put R on the left-hand side of the equation	$\pi R^2 H = V$ divide by H
Step 3	Use the balance rule	$\pi R^2 = \dfrac{V}{H}$ divide by π $R^2 = \dfrac{V}{\pi H}$ square root $R = \sqrt{\dfrac{V}{\pi H}}$
Step 4	Calculate diameter (assuming only positive square roots)	$D = 2\sqrt{\dfrac{V}{\pi H}}$

4.5.4 Teaching formulae

It is important that the writing of formulae follows these stages:

1. Draw or construct examples from the context of the formula; for example, construct cubes from blocks and relate the length of the side to the number of blocks and from there to the volume of the cube. It will be seen that volume is L^3. Answers should be calculated using the first principles.

   OR Develop a new context from an existing formula and relate this to a formula that is already known. For example, the area of a square is L^2. So we can construct a prism on a square base using blocks 1 unit high, then 2 units, then 3 and so on.

2. Place examples in tables and look for relationship directly from examples (as in table below):

Length of side	1	2	3	4	(and so on)	L
Number of cubes	1	8	27	64	$\longrightarrow$	L^3

   OR Place examples in tables but relate to another formula (as in table below):

No. of levels	1	2	3	4	(and so on)	L
Area of base	L^2	L^2	L^2	L^2	$\longrightarrow$	L^2
Volume	$1L^2$	$2L^2$	$3L^2$	$4L^2$	$\longrightarrow$	$LL^2=L^3$

3. Make an effort to state and use formulae in words, so that it is easier to become used to:

   Area = length multiplied by width = 4 m × 3 m = 12 m², where length = 4 m and width = 3 m.

4. When formulae are well-known and their use is familiar, replace words with letters (A = L × W) and as this is done, A, L and W will be understood as unknowns and as variable because we are able to put anything in for L and W and this will always give A.

4.5.5 Modelling activities

One of the crucial skills to develop in students is the ability to translate real-world situations into equations and back again. This is done by teaching the students what the equation means as a story (Activity A) followed by activities interpreting the story in terms of symbols (Activity B is reversing Activity A).

Activity A

Equation	Story
$3x + 6 = 27$ $\longrightarrow$	• $3x$ means 3 × any number; +6 means to add 6; = means same value as and 27 is the balancing value • so we need 3 lots of the same thing plus 6 to balance the total 27 • there can be many situations – what about maxi taxis? • "Three maxi taxi loads of children were brought to the game. 6 children were already there. This made 27 children."

Activity B

Equation	Story
There are two times that are not given. They are the same. This is t + t or 2t. $\longleftarrow$ So, 2t + other time = 40 Thus, 2t + 26 = 40	"John waited for the lift, then spent 4 minutes travelling in the lift, then spent 15 minutes at his appointment and then waited the same amount of time as the first wait for the lift again. Then it was another 7 minutes before he left the building. He spent 40 minutes in the building."

We can use the balance principle on these equations to work out the unknowns, for example:

Activity A
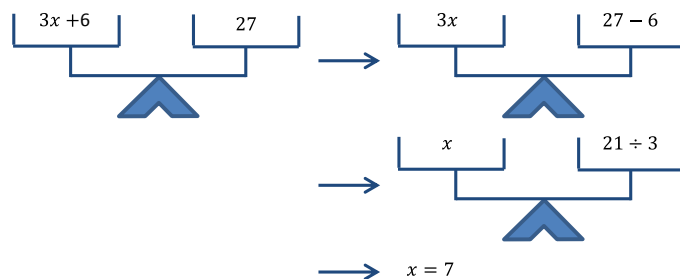
Activity B
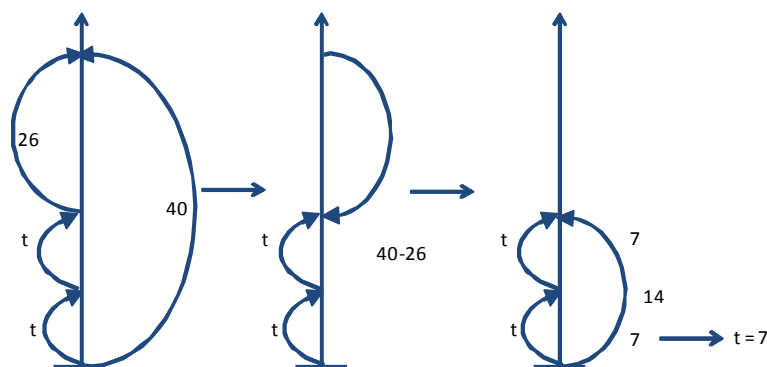
5 Arithmetic–Algebra Principles

Arithmetic and algebra have the same structures in terms of principles (also called properties or laws). In arithmetic, these principles are often invisible because of an overdue emphasis on product (getting answers) instead of process (the properties of operations and how these are used to get answers). Process and product are the same in algebra (the process of adding a variable and a number, e.g. $y + 3$, is also the answer to adding y and 3). This means that the structure of mathematics is more visible and more important in algebra because it is not masked by computation. Product is often dependent on numbers (e.g. 3 + 5 = 8 because of what 3 and 5 are) but process is based on principles that hold for all numbers (e.g. 3 + 5 = 5 + 3 no matter what we change 3 and 5 to). As algebra is the generalisation of arithmetic, these principles (as generalisations) come to the forefront.

The important principles that are the basis of arithmetic and that also apply to algebra are the equivalence and order principles and the operation or field principles (see 5.1 below). They are principles that apply to numbers, variables, expressions and functions. There is a third set of principles that have to do with the effect of number size, but these are covered in the YDM Operations book. As algebraic thinking is the most powerful form of mathematics thinking, it is important to develop these principles for arithmetic in the early years (to pre-empt later algebra use) and to then translate these across to algebra as unknown and variable are introduced. This leads to a sequence for teaching this chapter as shown below.

Equivalence principles (reflexive, symmetry, transitive)

↓ ↘

Order principles (reflexive, symmetry, transitive)

Early field principles
(identity/inverse, commutative, compensation, inverse relation)

↓

Later field principles
(associative, distributive, equivalence)

↓

Application to algebraic expressions
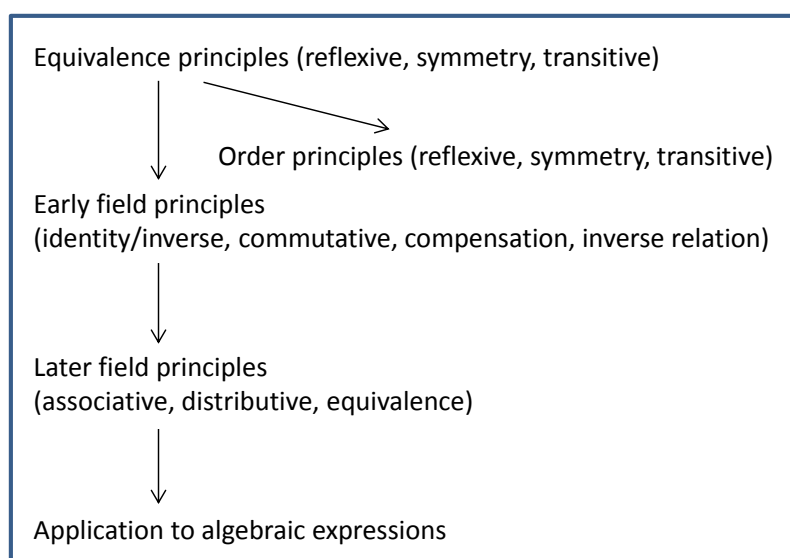
The chapter begins with descriptions of the principles in section 5.1 and then moves on to very early arithmetic principle activities (section 5.2), early to middle arithmetic principle activities (section 5.3), and later arithmetic to algebra principle activities (section 5.4).

5.1 Equivalence, order and operation structure

5.1.1 Equivalence and order

Arithmetic and algebraic expressions, whether they have one or many symbols, can be equal to each other (equivalence) or greater or less than each other (order).

When arithmetic examples of equivalence or order (e.g. 2 + 3 = 5 or 4 + 3 > 6) are studied, generalisations can be found. As shown in section 4.2, this is best done in unnumbered situations initially. These lead to structures made up of generalisations which we call principles.

The **equivalence** principles are as follows:

1. **Reflexivity principle**. Anything equals itself (e.g. 2 = 2).

2. **Symmetry principle**. Equals can be turned around (e.g. if 2 + 3 = 5 then 5 = 2 + 3).

3. **Transitivity principle**. Equals continues across equal relationships; the first in a sequence, equals the last (e.g. if 2 + 3 = 5 and 5 = 6 − 1 then 2 + 3 = 6 − 1).

The **order** principles are as follows:

1. **Well-ordered principle**. Two expressions have to be equal to, greater than or less than each other (e.g. 3, 5 → 3 < 5; 23 + 64, 72 − 15 → 23 + 64 > 72 − 15).

2. **Antisymmetry principle**. If two expressions that are less than are turned around then they become greater than (e.g. 6 < 9 is the same as 9 > 6).

3. **Transitivity principle**. Order continues across relationships; the first in sequence is greater than / less than the last (e.g. 6 − 1 < 8 + 2, 8 + 2 < 16 − 4, 16 − 4 < 48 ÷ 2 → 6 − 1 < 48 ÷ 2).

The methods for teaching the principles relate (as in other sections) to the use of models to show equals and order. The two most used are mass (balance) and length (strips of paper/number lines). Section 5.2 contains instructional activities for teaching the equivalence and order principles.

5.1.2   Field or operation structure

The most important structure in school mathematics is the field. It is a structure composed of expressions and two operations, addition and multiplication. Subtraction and division are not strictly operations as they do not obey all operation principles.

The **field** or **operation** principles are as follows.

1. **Identity principle**. There are two identities that leave everything unchanged, namely, 0 for addition and 1 for multiplication (e.g. 27 + 0 = 27 and 1 × ¾ = ¾).

2. **Inverse principle**. If a change is to be made it can be undone by an inverse change, namely, −3 for +3 and ÷7 for ×7. This means that − is the inverse of + and ÷ is the inverse of ×.

3. **Commutative principle**. Operations with + and × can be "turned around" without error (e.g. 6 + 7 = 7 + 6 and 23 × 4 = 4 × 23). However, this is not true for − and ÷ (e.g. 6 − 2 ≠ 2 − 6 and 12 ÷ 3 ≠ 3 ÷ 12).

4. **Associative principle**. Operations with + and × which have more than two expressions can be completed with any association in any order (e.g. 6 + 3 + 5 = 9 + 5 or 6 + 8 or 11 + 3).

5. **Distributive principle**. Addition means adding "like things" but multiplication means multiplying everything, that is, × distributes across + (e.g. 3 × (4 + 5) = (3 × 4) + (3 × 5); 23 × 2 = (20 × 2) + (3 × 2)). This means, for example, that 23 + 3 = 26 but 23 × 3 ≠ 29, and 23 × 3 = 69 but 23 + 3 ≠ 56.

There are additional principles as follows that emerge from these (but are well worth remembering in their own right).

1. **Compensation principle**. If there is a change in one number in + or ×, this is compensated by the inverse change in the other number. For example, 8 + 5 = 10 + 3 (8 + 2 = 10, 5 − 2 = 3); 12 × 3 = 4 × 9 (12 ÷ 3 = 4, 3 × 3 = 9). Because of their inverse nature, − and ÷ also compensate but not by using inverses; they compensate by doing the same to both numbers. For example, 8 − 5 = 12 − 9 = 3 (8 + 4 = 12, 5 + 4 = 9); 12 ÷ 3 = 24 ÷ 6 = 4 (12 × 2 = 24, 3 × 2 = 6).

2. **Inverse relation principle**. For + and ×, any increase in a number, increases the total (e.g. 8 × 4 = 32, 8 × 6 = 48). However, for − and ÷, increasing the second number decreases the total. For example, 8 − 3 = 5, 8 − 6 = 2 (3 increases so 5 decreases); 12 ÷ 3 = 4, 12 ÷ 6 = 2 (3 increases so 4 decreases).

3. **Equivalence principle**. Since + 0 and × 1 do not change anything, then two things are equivalent if one is + 0 or × 1 of the other (e.g. $\frac{2}{3} \times 1 = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$, so $\frac{2}{3}$ is equivalent to $\frac{4}{6}$ because $\frac{2}{2}$ is the same as 1; 404 − 186 = 404 − 186 + 0 = 404 + 14 − 186 − 14 = 418 − 200, so 404 − 186 can be solved by 418 − 200 because + 14 − 14 is the same as + 0.

Similar to before, these principles can be taught by the models that represent + and ×: the set and number line for + and the set, number line and array for ×. Section 5.2 below contains instructional activities for teaching the operation or field principles.

5.2 Very early arithmetic principle activities

Arithmetic principles reflect the structure of arithmetic and algebra. If they are learnt, then students have knowledge they can apply across all years of primary and secondary school mathematics – to whole numbers, fractions, variables, functions and calculus.

There are two major structures that the principles can be clustered into: the **equivalence and order** structure, and the **field** or **operation** structure. It is difficult to state where in the school years the principles should be developed because they need to be reinforced every time a new context (new types of numbers, variables, functions, etc.) appears.

The materials that are commonly used to teach these principles are those associated with the areas of equations and functions, namely, the **balance** and the **length** models, and those associated with the two operations of addition and multiplication, namely, the **set**, **number line** and **array** models. However, **calculators** and **finding patterns** are also excellent instructional activities for this area, and there are some active whole-body techniques that are very useful. It has also been found that activities in **unnumbered** situations (situations where there is no number) assist in seeing these structures.

The early years are when the balance and length models are introduced. If unnumbered contexts like groceries and coloured strips of paper are used for these models, the equivalence and order principles can be introduced.

5.2.1 Teaching the equivalence principles – unnumbered contexts

The equivalence principles are reflexivity, symmetry, and transitivity. They can be taught with the balance (mass) and length (strips of paper) models.

Balance or mass model

Here, equals is represented with groceries on a balanced beam balance. The use of balances begins as a real balance in early activities and then abstracts to a mathematical-balance drawing in middle activities and finally to an image of a balance in later activities. The drawing and the image balance can allow any operation.

The principles are easily shown physically with a balance. Have the students experience the principles with beam balances and groceries. Direct the students to state out loud the equation as you move your hand from their left-hand side of the balance to their right-hand side – "equals" is said as the hand goes past the balance point (centre) of the balance (it can be useful to stick "=" on the centre of the balance). Direct the students to record the balanced groceries as informal equations, e.g. "salt equals soap plus pasta". Discuss any generalities they find; encourage them to see the generalities below.

1. **Reflexivity**. This is fairly obvious (that two things that are the same will be equal) but it needs to be made explicit with equations. Students can investigate if same things always balance.

$$A + B = A + B$$

2. **Symmetry**. This can be seen by turning the balance 180°.


"Turn the balance around"
A = B ⟶ B = A
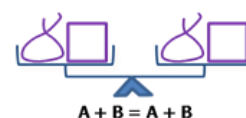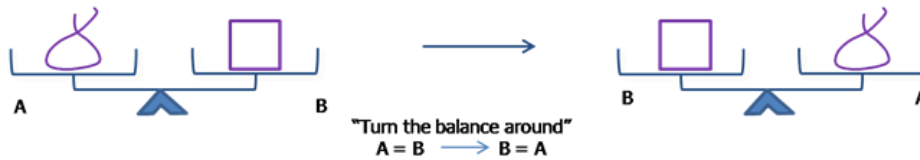
3. **Transitivity**. This requires the students comparing three things in three different ways to show that if A=B and B=C then A=C.


A = B   B = C + D   A = C + D
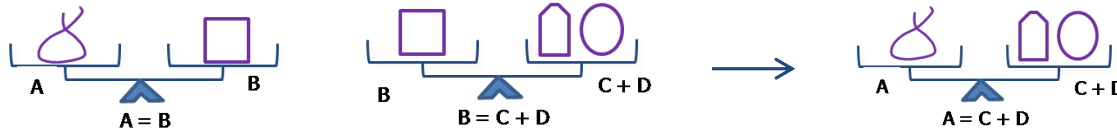
Length model

Here, equals and order are represented by same length and different length respectively. It begins by strips of paper, moves on to lines (the double number line) and finally to images in the mind. The equivalence principles are easily represented as in the three examples below. Again, effective strategies involve the students saying out loud the equalities, writing informal equations, and discussing generalities.

1. **Reflexivity**

A + B = A + B

2. **Symmetry**

C = A + B   A + B = C

3. **Transitivity**

P = Q   and   Q = R + S   P = R + S
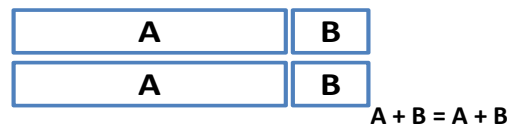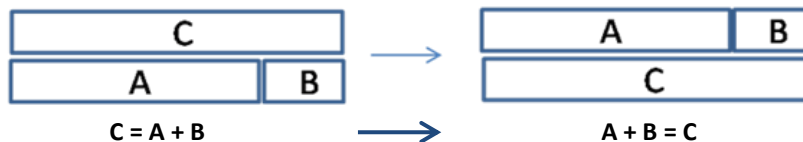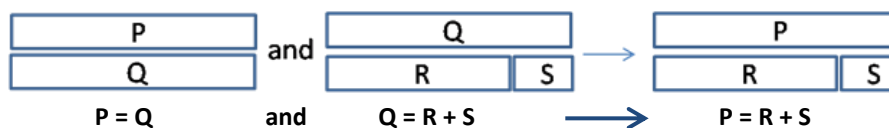
It is always best to start with the **human body** (e.g. hang plastic bags on arms and become a beam balance and walk different distances); and, as stated before, with **unnumbered situations** (e.g. compare groceries, use unmeasured strips of paper).

5.2.2 Teaching the order principles – unnumbered contexts

The order principles are well-ordered, antisymmetry and transitivity. As with the equivalence principles, they can be taught with the balance (mass) and length (strips of paper) models. For the balance, note that a heavier item pushes down more, so higher means lighter. Once again, effective strategies involve the students saying out loud the equalities or inequalities, writing informal equations, and discussing generalities. The following drawings show how both the balance or mass model with weights and the length model with paper strips can give students experiences with the order principles on which to base discussion.

1. **Well-ordered**. It is obvious that if you have two weights or two strips that they either have to be equal or one is larger (heavier or longer) than the other. However, this needs to be experienced and made explicit.

| A = B | or | A > B | or | A < B |

| P = Q | or | P > Q | or | P < Q |
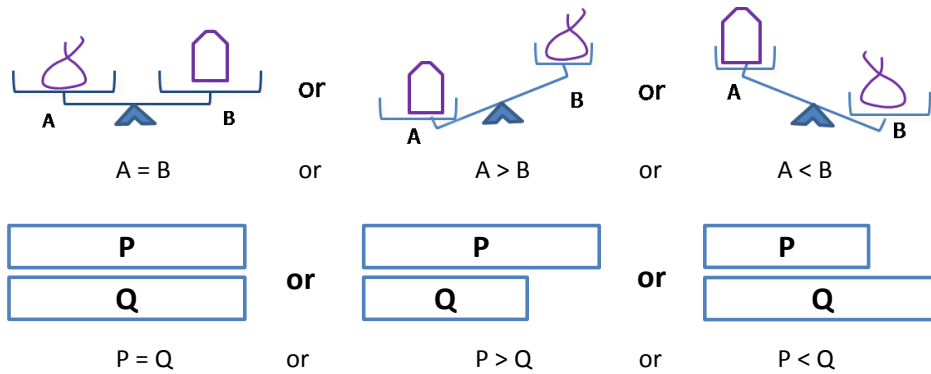
2. **Antisymmetry**. As for equivalence and symmetry, this can be seen by turning the balance 180°.

"Turn the balance around"
A > B   B < A

X > Y   Y < X
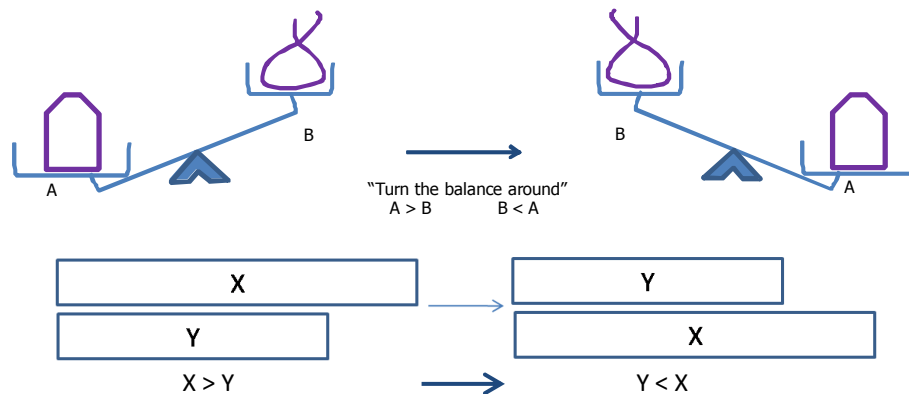
3. **Transitivity**. Once again the students compare three things in three different ways to show that if A>B and B>C then A>C.

A > B and B > C ────────➔ A > C

P > Q   and   Q > R   ➔   P > R
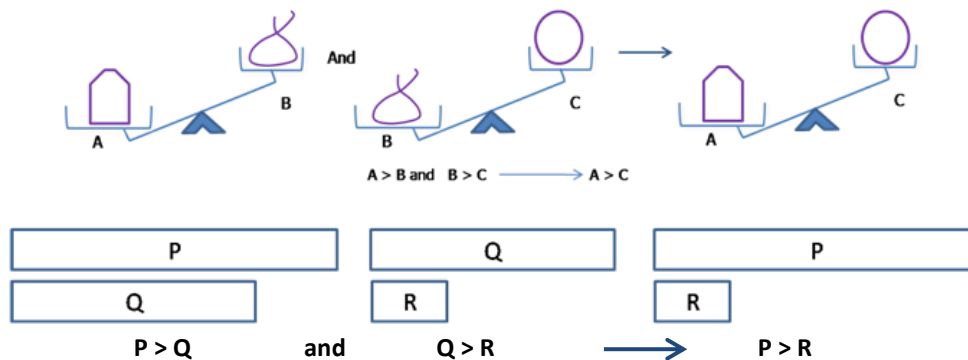
5.3   Early to middle arithmetic principle activities

The middle years are when the equivalence and order principles are reinforced in numbered situations and when the field/operation principles are introduced. The equivalence and order principles rely on balance and length models, while the field/operation principles are taught by the models that represent addition and multiplication: the set and number line for addition and the set, number line and array for multiplication.

5.3.1   Introducing the equivalence and order principles for numbered situations

In the middle years, the equivalence and order principles (from 5.2.1 and 5.2.2 above) are applied to numbers using balance beams with numbers of weights (e.g. coathanger balances with red, blue and green baked bean cans) and lines of blocks. Further work to reinforce these concepts should be undertaken with drawings of mathematical balances and vertical double number lines in the latter half of the middle years.

We will not attempt to show how to teach each principle in detail but provide diagrams of how activities with balances and number lines can be experienced and recorded as a basis for discussion and recognition of generalisations. We will not include reflexivity as it is obvious and transitivity will cover both equivalence and order.

1. **Symmetry**. This is important as it shows that an equation can be reversed, e.g. 3+4=7 and 7=3+4 are the same.

$$3 + 4 = 7$$ $$7 = 3 + 4$$

$$3 + 4 = 7 \longrightarrow 7 = 3 + 4$$
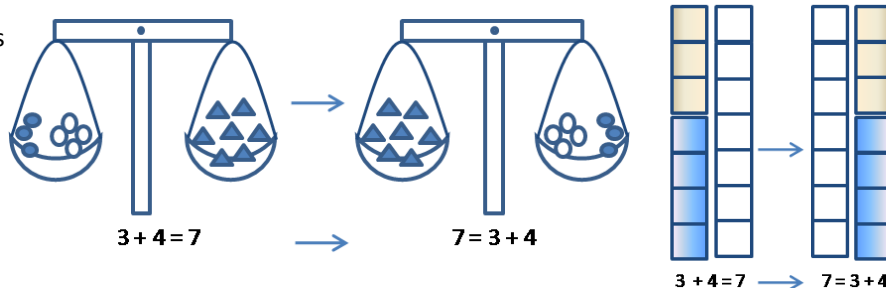
2. **Antisymmetry**. This extends symmetry in equivalence to order showing that when inequations are reversed the order changes from greater than to less than and vice versa.

$$3 + 4 < 8$$ $$8 > 3 + 4$$

$$3 + 4 < 8 \longrightarrow 8 > 3 + 4$$
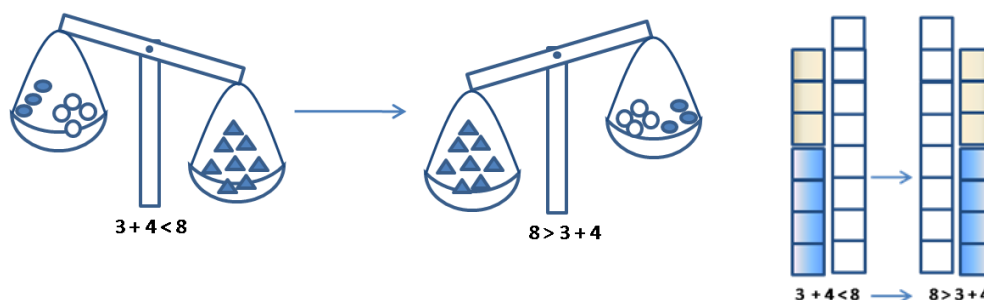
3. **Well-ordered**. This is making explicit the principle that two expressions are either equal or in order (greater than or less than).

4. **Transitivity**. This is showing that if one thing is equal to/less than/greater than a second thing and this second thing is equal to/less than/greater than a third, then the first is equal to/less than/greater than the third.

P = Q   and   **Q = R + S** $\longrightarrow$ **P = R + S**
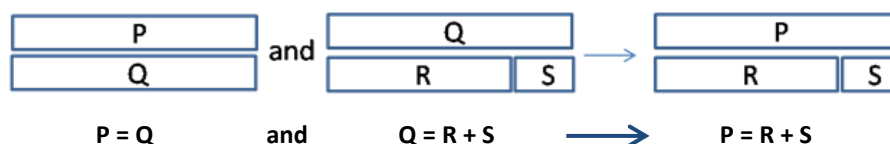
5.3.2 Introducing the field/operation principles

The field/operation principles comprise identity, inverse, commutativity, associativity, distributivity, compensation, inverse relation, and equivalence. There are two ways to teach the principles. The first is to use the set, array and number-line models to show the relationships. The second is simply to use a calculator to check many possibilities.

As for the equivalence and order principles, effective strategies for teaching the field/operation principles involve the students (a) experiencing activity with models, (b) saying out the relationships experienced on the way to the principles, (c) writing informally what these relationships are, and (d) discussing generalities. As well, any arithmetic experiences should be used to highlight the principles as they appear. The models used are the set, number line and array.

There is not the space to show in detail how to teach each principle, so the following examples will show how models may be used to teach them. In some cases, interesting and effective methods will be highlighted.

1. **Identity**. The aim is to show that adding 0 and multiplying by 1 do not change anything. For addition, the best idea is to add 2, then add 1 and finally add 0. For multiplication, we have 1 group, row or jump of the number or a number of groups, rows and jumps of 1 (both of which equal the number).


6 + 2 = 8   6 + 1 = 7   6 + 0 = 6

1 group of 6 (1 × 6 = 6)   4 jumps of 1 (4 × 1 = 4)   1 row of 5 (1 × 5 = 5)
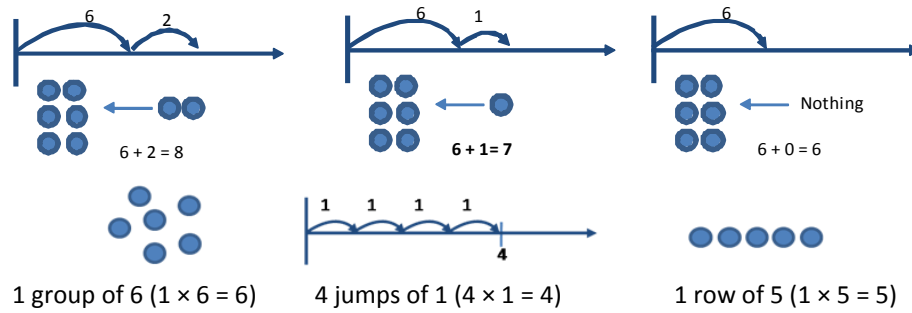
2. **Inverse**. The aim is to show that addition and subtraction, and multiplication and division, are inverses. Act out situations like *Share 12 amongst 3, what do we get?* [4]. *Make 3 groups of 4, what do we get?* [12]. Join and separate, rejoin and reseparate; make groups and share out groups, remake and reshare. Highlight that one action is the opposite and undoes the other (e.g. 7+2=9 and 9−2=7).

+ 6 − 6   1 row of 4 × 3 makes 12; 12 shared by 3 makes 4 each
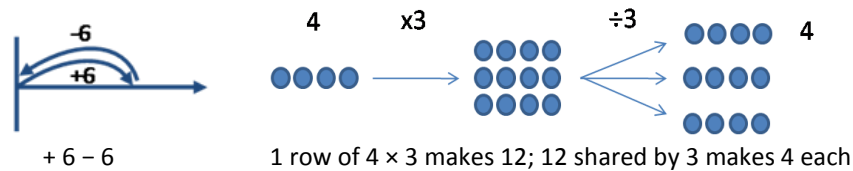
3. **Commutativity ("turnarounds")**. Show that order does not matter for addition and multiplication. This is achieved by showing that two numbers added gives the same regardless of the order, and (highly recommended) turning an array by 90 degrees to show, for example, that 3 rows of 4 is the same as 4 rows of 3.

7 joined by 2   2 joined by 7

3 x 5   Rotate 90   5 x 3
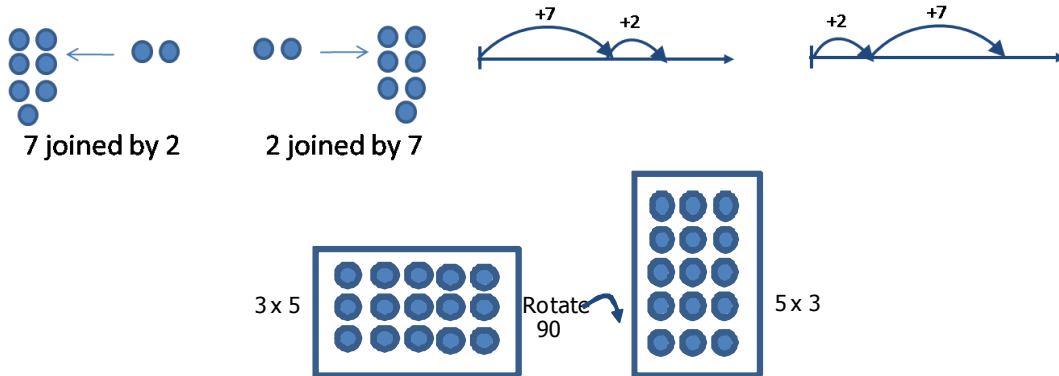
4. **Associativity**. The aim is to show that order does not matter for three numbers for the same operation of addition or multiplication. This is straightforward for addition (joining 3 sets so which sets join first is irrelevant – see below), but not so easy for multiplication because you need, for example, (3 groups of 4) groups of 5 objects to equal 3 groups of (4 groups of 5). This is more easily seen with calculation (and a calculator) than from the models.


3 + 4 + 5 = (3 + 4) + 5 = 3 + (4 + 5)
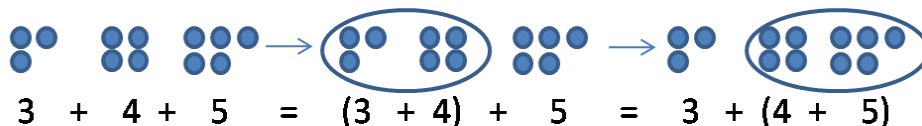
5. **Distributivity**. This is best done by dividing arrays, or the more abstract rectangles, into two parts.

7 by 4   5 by 4   2 by 4

	23	
4	4 × 23	

	20	3
4	4 × 20	4 × 3
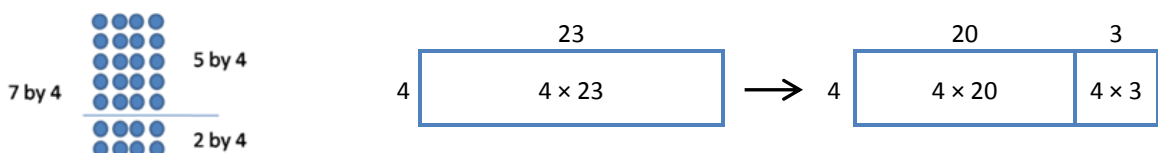
6. **Compensation**. This is where we show that any two numbers have the same addition or multiplication if a change in one number is undone in the other number by use of inverse. It is difficult to show with models. In fact, it is sometimes more easily seen by discussion, e.g. *Look at 2+3=5, what happens if 2 goes to 4, what if we want the sum to remain as 5?* However, models can be used as below.

$$4 + 7 = 4 + 2 + 5 = 6 + 5$$

$$4 \times 6 = 2 \times 2 \times 6 = 2 \times 12$$
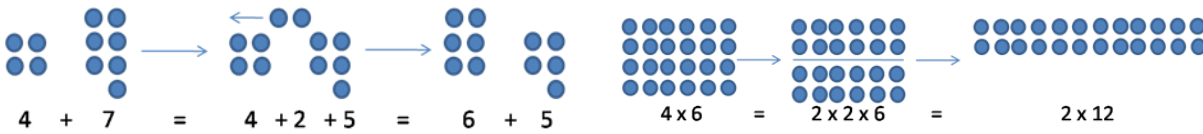
Compensation is one area where using kinaesthetic or whole-body activity is useful. Students have difficulty looking at materials and pictures and seeing that, if increasing the 8 in 8 + 5 = 13, it is necessary to decrease the 5 to keep 13 as the answer. However, one effective way to overcome this difficulty is to consider addition as a relay race in which one member does more than their share, and to act this out. Get students to form into pairs, mark out a relay walk (as on right) and a baton change and direct the pairs to walk the relay.

1st Person | Baton change over | 2nd person
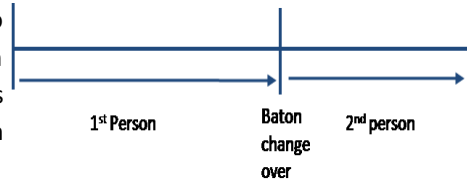
Discuss what would happen if the first person walked further (as on right) – what happens to the second person? Students can see that the second person has to walk less by the amount the first person walked more.

1st Person walks further | Baton change over | 2nd person walks ??

This method of teaching appears to make it easier for students to understand the compensation principle.

It should also be noted that although the field/operation principles apply to addition and multiplication, there is also compensation for subtraction and division. However, it can be confusing for students because compensation for addition or multiplication is the **inverse** of the first change, while compensation for subtraction or division is the **same** as the first change. For example, 8 + 5 = 10 + 3 because, for addition, 8 increased by 2 means that 5 needs to decrease by 2; but 8 − 5 = 10 − 7 because, for subtraction, 8 increased by 2 means 5 also needs to increase by 2. Similarly, 12 × 4 = 6 × 8 because, for multiplication, 12 divided by 2 means that 4 needs to be multiplied by 2; but 12 ÷ 4 = 6 ÷ 2 because, for division, 12 divided by 2 means that 4 also needs to be divided by 2 to maintain equivalence.

To avoid confusion either teach compensation for addition and multiplication, and subtraction and division separately, or place compensation under the inverse principle. Since addition and subtraction, and multiplication and division, are inverses, then it is reasonable that they would do the opposite with regard to the compensation principle. Since addition and multiplication compensation require the opposite change, it is reasonable that subtraction and division compensation require the opposite of opposite which is the same change.

7. **Inverse relation**. Subtracting more and dividing by more both have the effect of decreasing the answers to the computations, as can be seen in the examples on right and below.

11 − 5 > 11 − 8
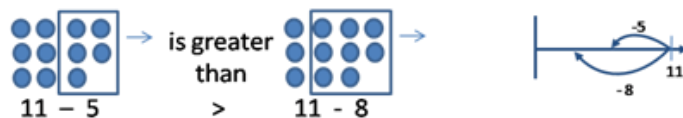
The more that is taken away, the less that is left

12 ÷ 3

12 ÷ 6

More shares, less each share
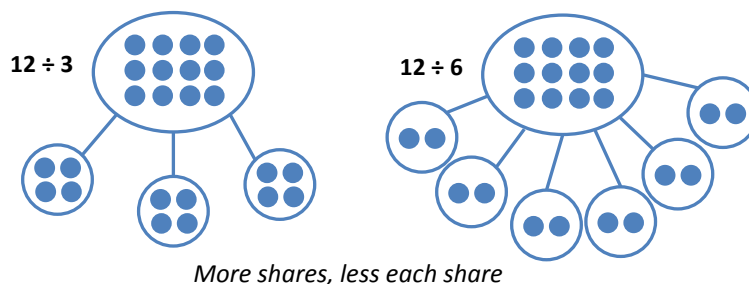
8. **Equivalence**. This principle says that as long as we add 0 or multiply by 1, the answer stays the same (the expressions are equal). This needs to be experienced and for students to become flexible with what 0 and 1 could be. For example, for 28 + 15, 0 = + 2 − 2, which means that 28 + 15 = 28 + 15 + 2 − 2 = 30 + 23, while for $^2/_5$, 1 = $^2/_2$, which means that $^2/_5$ = $^2/_5$ × 1 = $^2/_5$ × $^2/_2$ = $^4/_{10}$.

5.3.3   Reinforcing the equivalence and order principles with more abstract representations

At the end of the middle years, the equivalence and order principles can be further reinforced with mathematical balances (drawings with expressions on each side) and vertical double number lines (with operations drawn as arrow movements on left and right of the vertical line). We provide a few illustrations of the kind of activities students can experience.

1. **Symmetry**

$$2 \times 3 + 5 = 20 - 9 \qquad 20 - 9 = 2 \times 3 + 5$$

2. **Antisymmetry**

$$2 \times 3 + 5 < 20 - 3 \qquad 20 - 3 > 2 \times 3 + 5$$
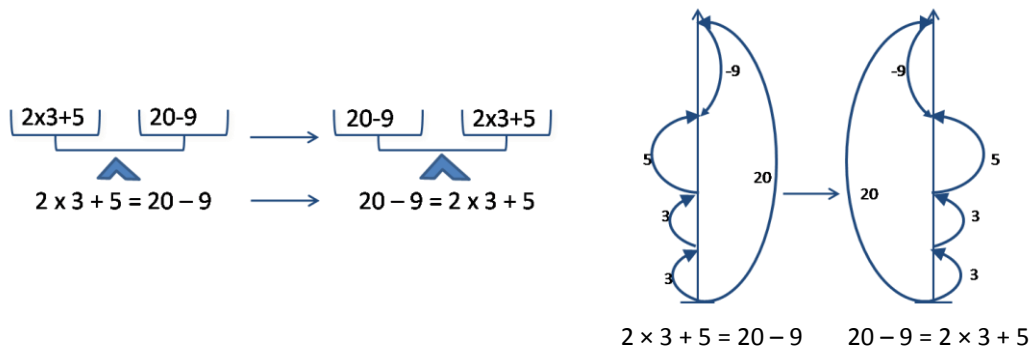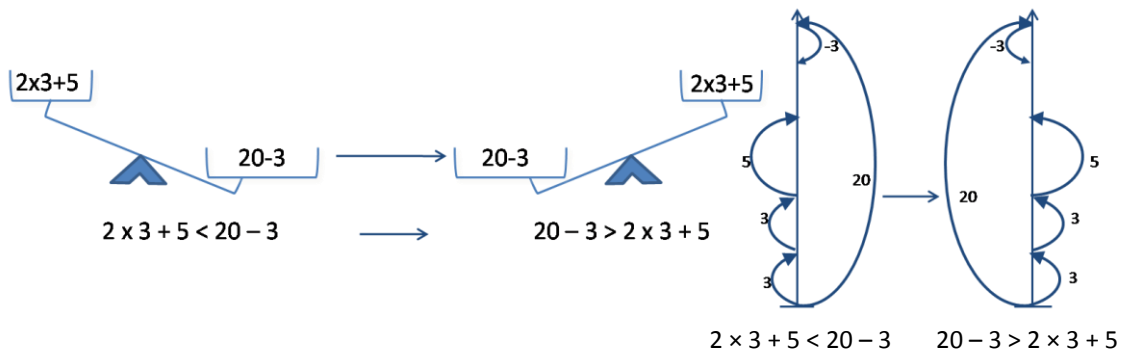
3. **Well-ordered**
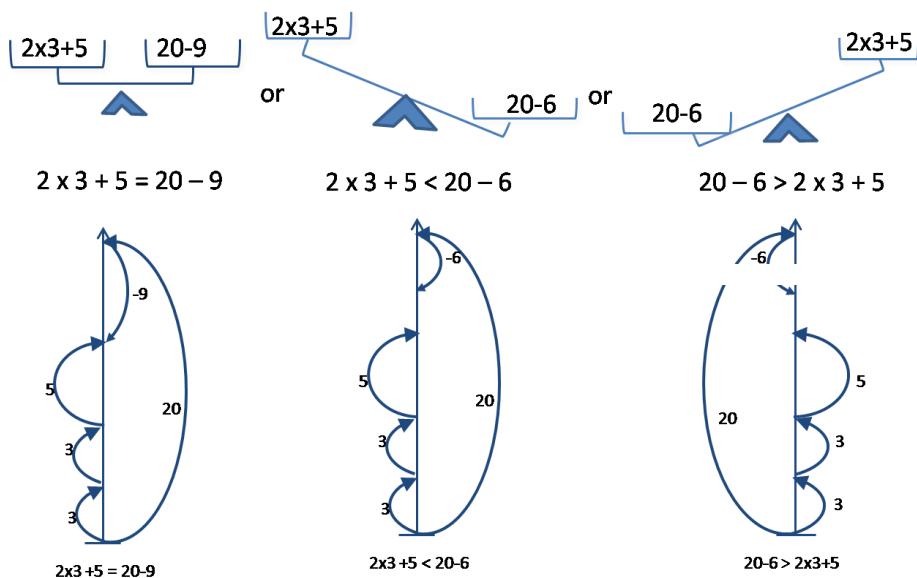
4. **Transitivity**
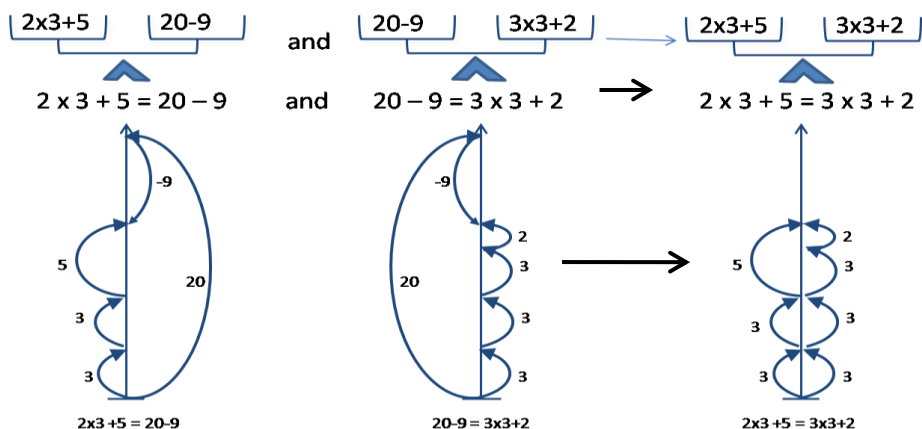
5.4 Later arithmetic to algebra principle activities

In the later years, the principles need to be reinforced for number and then applied to variables. This should be viewed as an extension of arithmetic. The principles are then extended to various algebraic skills (e.g. expansion, simplification and numerical factorisation). Some of these extensions have already been given in Chapters 2, 3 and 4.

There are important features of arithmetic and algebra that must be considered when looking at the concepts and skills of expansion, simplification and factorisation:

(a) they hold for division and subtraction as well as multiplication and addition;

(b) subtraction is the inverse of addition but this does not affect activities based on distribution as the crucial component is multiplication (and division); and

(c) division is the inverse of multiplication and so the effect is that expansion for division is similar to factorisation for multiplication. For example, $3 + 9a = 3(1 + 3a)$ is factorisation which is similar to expansion for division $(3 + 9a) \div 3$; in both we have to find the common factor.

Thus, this section will focus on multiplication and addition and rely on inverses to give insight into what happens for division. That is, if something works for multiplication and addition, it also works for adding a negative (subtraction as inverse of addition) and multiplying by reciprocal (division as inverse of multiplication).

5.4.1 Reinforcing and extending the principles from arithmetic to algebra

This subsection looks at how the arithmetic principles can be extended to algebra, by looking at patterns in arithmetic. Much of this subsection will be based on the distributive principle. This will be shown with the area model for multiplication across addition. The ideas also hold for division and subtraction. Subtraction situations can be undertaken similar to addition. However, division is more difficult and can be approached in two ways:

1. Division is actually the **inverse of multiplication**. Expansion with division may need factorisation before the division can be completed. For example, $\frac{6x+9y}{3}$ cannot be immediately factorised. The numerator must first be factorised as follows: $6x + 9y$ factorises to $3(2x + 3y)$. Then the division can be completed so that $\frac{6x+9y}{3} = \frac{3(2x+3y)}{3} = 2x + 3y$. Division examples should be explored after students are comfortable with multiplication and addition/subtraction.

2. Division can also be taught as **multiplication by reciprocal**. In this way the multiplication ideas can be extended simply to cover division. For example, $\frac{6x+9y}{3} = \frac{1}{3}(6x + 9y) = \frac{1}{3} \times 6x + \frac{1}{3} \times 9y = 2x + 3y$.

Reinforcing principles in arithmetic

An effective way to reinforce the principles for numbers is to give students calculators and encourage them to explore principles (e.g. *Is the first number plus the second always equal to the second plus the first? Is a number multiplied by 1 always equal to itself?*). There are two steps to this process as the examples below for the distributive principle and the symmetry principle show. (*Note:* A more structured way to do this is given for the distributive law in subsection 5.4.2).

1. Provide students with **examples to check** with the calculator. For the **distributive principle**, this would mean activities like below.

   Are these the same? $11 \times 13 + 11 \times 24 =$ _____; $11 \times (13 + 24) =$ _____
   Check with a calculator. $23 \times 27 + 23 \times 56 =$ _____; $23 \times (27 + 56) =$ _____
                                    $34 \times 78 - 34 \times 23 =$ _____; $34 \times (78 - 23) =$ _____
                                    and so on

   For the **symmetry principle**, students could be given examples to check as below. Examples should involve all operations.

   Are these both correct/true? Yes/No $245 \times 23 < 67\ 231$ _____; $67\ 231 > 245 \times 23$ _____
   Use your calculator.   and so on

2. Give students **examples to solve** that require using the principle being checked. For the **distributive principle**, this would mean activities like below. Students should be allowed to check by adding or subtracting the numbers in the brackets.

   Calculate these without adding or $54 \times (76 + 28) =$ _____
   subtracting the numbers in the brackets. $186 \times (259 + 543) =$ _____
   You can use a calculator. $74 \times (168 - 89) =$ _____
                                            and so on

   For the **symmetry principle**, students could be given examples as below. Once again, the calculator should be used to check.

   Use your calculator to determine the order in the first 49×53 _____ 2733; 2733 _____ 49×53
   activity of each set. Do the second without calculator. 54×58 _____ 29×109; 29×109 _____ 54×58
                                                              and so on

For Step 2 in other field principles, students could be asked to calculate as follows:

(a) Commutative – calculate $345 + 672$ another way, that is, without entering [345], [+] and [672] on their calculator in that order [answer – reverse the order].

(b) Associative – calculate $158 + 436 + 277$ another way, that is without entering [158], [+], [436], [+] and [277] in that order [answer – add the last two numbers first].

Extending principles to algebra

An effective way to show that the principles apply to algebra is to build the principles from arithmetic as follows for two examples, identity and distributive. The method starts by looking at arithmetic examples, replaces the numbers with "any number", and finishes with a letter representing a variable. (*Note:* A more structured way to do this is given for the distributive law in subsection 5.4.2).

1. **Identity**

   $7 + 0 = 7$ $8 \times 1 = 8$
   $23 + 0 = 23$ $47 \times 1 = 47$ and so on
   any number $+ 0 =$ any number   any number $\times 1 =$ any number
   $x + 0 = x$ $y \times 1 = y$

2. Distributive

3 tens + 4 tens = 7 tens (3 + 4 = 7)	3 twos and 3 fives = 3 sevens (two + five = seven)
3 eights + 4 eights = 7 eights ...	3 elevens + 3 nines = 3 twenties (eleven + nine = twenty) ...
3 any no. + 4 same no. = 7 same no.	3 any no. + 3 some other no. = 3 (same any no. + same other no.)
$3x + 4x = 7x$	$3x + 3y = 3(x + y)$

Continuing in the same manner can show that:

$4x + 4y = 4(x + y)$
$12x + 12y = 12(x + y)$ ...
thus, $px + py = p(x + y)$ for any number p

Note: This also holds:
the associative principle
for multiplication

3 lots of 4 eights = 12 eights (12 = 3 × 4)
8 lots of 6 twelves = 48 twelves
4 lots of 5 anythings = 20 anythings, ...
p lots of q anythings = $(p \times q)$ anythings
$p \times (qx) = (p \times q)x$

5.4.2 Extending the principles to cover expansion

This subsection looks at expansion, which is based on the distributive principle. The distributive principle is that multiplication (and division) acts across all components of addition (and subtraction); that is, $a \times (b + c) = (a \times b) + (a \times c)$ and $\frac{p-q}{r} = \frac{p}{r} - \frac{q}{r}$. This can be seen in the difference between 43 + 2 and 43 × 2 (as on right).

```
  43        43
 + 2       × 2
 ----      ----
  45        86
```

The distributive principle is best seen on the area model of multiplication. The area model is an extension of the array model (see below).

Arrays 7 × 4 — Two versions of the distributive principle

Area 7 × 4 — Two versions of the distributive principle
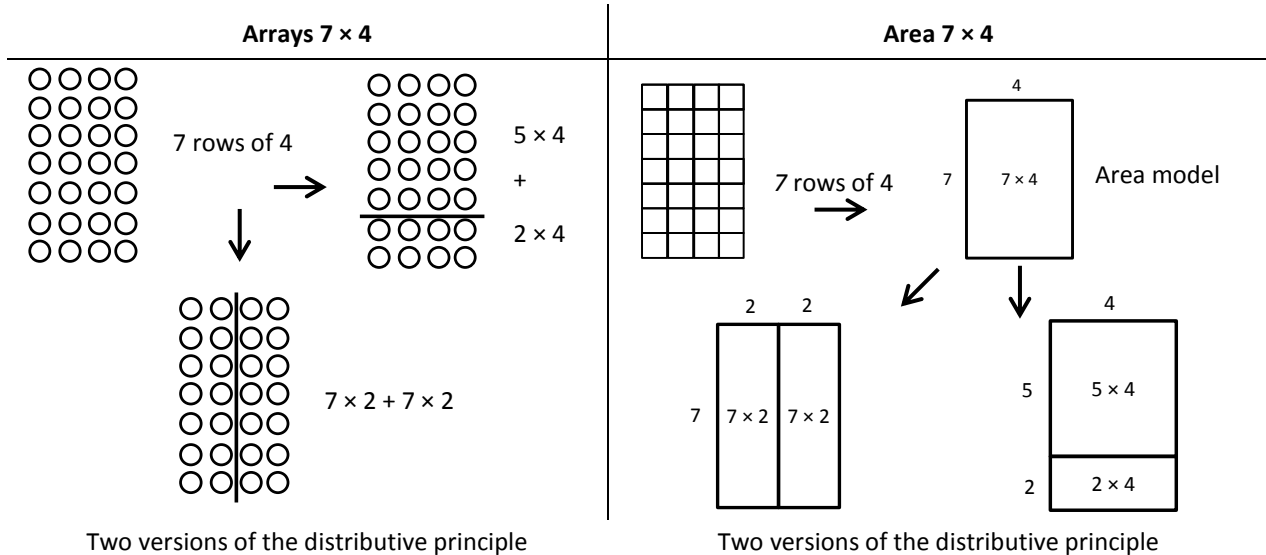
The steps to build expansion are as follows.

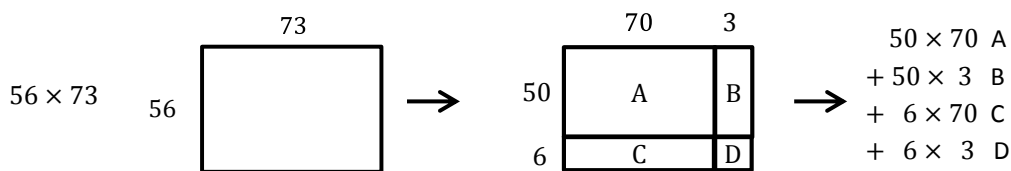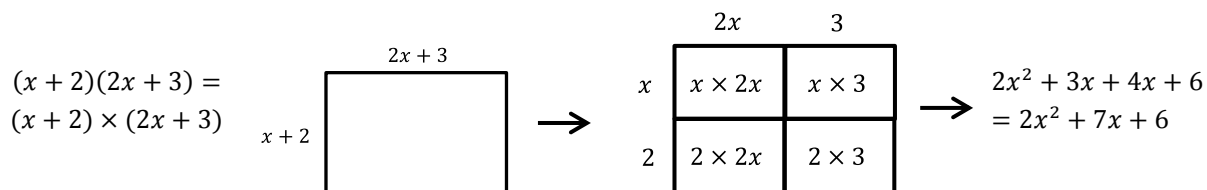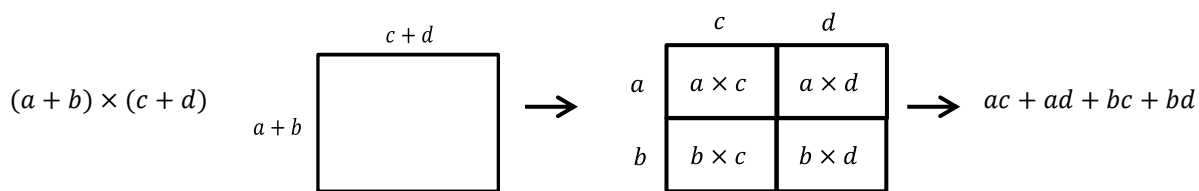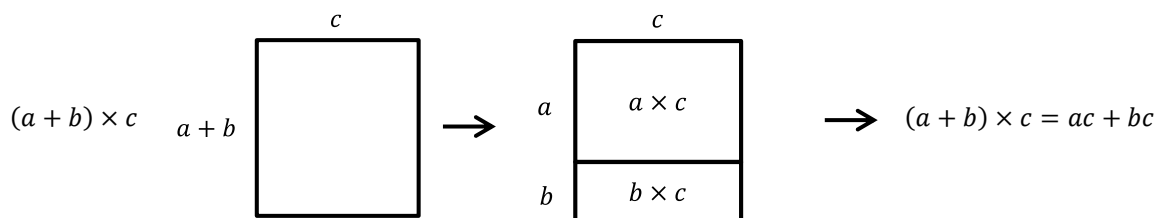1. Return to arithmetic and look at an example of expansion:

Basic facts

7×8 → $5 \times 8 + 2 \times 8 = 56$
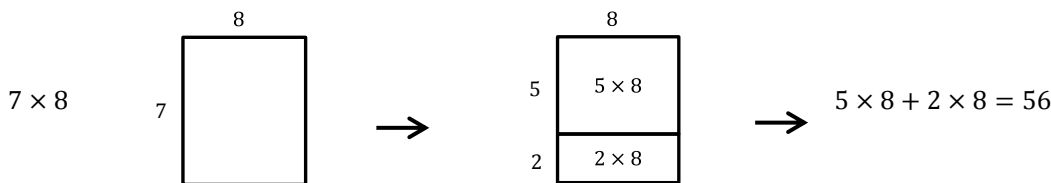
Algorithms 2 digit × 1 digit

3×24

3×20 A
$+ 3 \times 4$ B
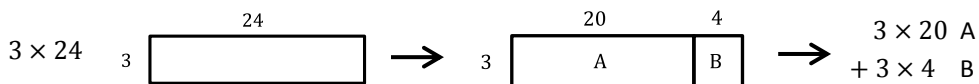
Algorithms 2 digit × 2 digit

56×73

50×70 A
$+ 50 \times 3$ B
$+ 6 \times 70$ C
$+ 6 \times 3$ D

2. Relate this to algebra – use the area model (also tiles):

$6p = 6 \times p$

$4 \times p + 2 \times p = 4p + 2p$

$a \times a + 2$

$a \times a + a \times 2 = a^2 + 2a$

$(a + b) \times c$

$(a + b) \times c = ac + bc$

$(a + b) \times (c + d)$

$ac + ad + bc + bd$

$(x + 2)(2x + 3) =$
$(x + 2) \times (2x + 3)$

$2x^2 + 3x + 4x + 6$
$= 2x^2 + 7x + 6$
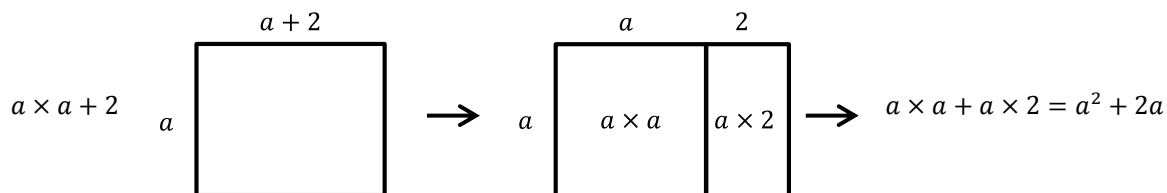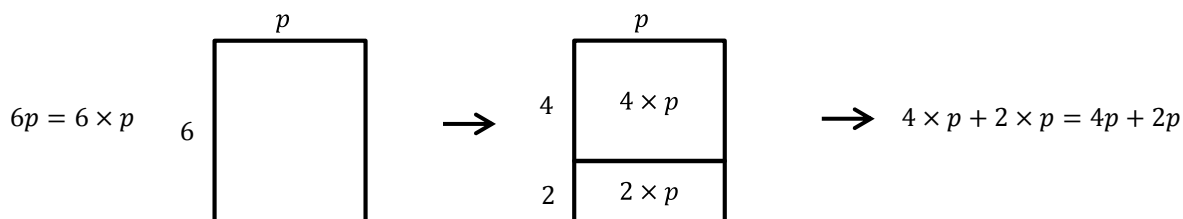
3. To reinforce the above show the expansion in vertical format and relate to algorithm (see next page).

```
      24                                      a+b
    × 3                                     ×   a
      12   3×4                                  ab   a×b
      60   3×20                                 a²   a×a
      72                                      a²+ab


      24                                      x + 2
    × 37                                    × 2x + 3
      28   7×4                                   6   3×2
     140   7×20                                 3x   3 × x
     120   30×4                                 4x   2x × 2
     600   30×20                               2x²   2x × x
     888                                  2x² + 7x + 6
```

This shows that algebraic expansion is an extension of the expansion used in the traditional algorithm, which is also based on the distributive principle.

4. This can also be done for negatives, if we allow, for instance, 2 tens and 8 ones to be 3 tens and −2 ones:

```
    2 8              3 −2             a+2              3b−4
  × 7              × 7              × 7              × 7
  1 4 0  (7×20)    2 1 0  (7×30)     7a  (7×a)        21b  (7×3b)
   5 6  (7×8)     −1 4  (7×−2)       14  (7×2)       −28  (7×−4)
  1 9 6            1 9 6            7a+14            21b−28
```

It can also work for examples like 3 7 × 2 9; it can be done as 4 −3 × 3 −1. This extends to (a−2) × (2a−3):

```
      37                 4 −3                            a −2
    × 29     =         × 3 −1                          × 2a −3
     600               1200   40×30                       2a²  a×2a
     270               − 40   40×−1                       −3a  a×−3
     140               − 90   −3×30                       −4a  −2×2a
      63                  3   −3×−1                          6  −2×−3
    1073               1073                           2a² − 7a + 6
```

Note: The negative can also be shown with graph paper and the area model.

5.4.3   Extending the principles to cover simplification

Expansion makes things longer. For example, $5x = 2x + 3x$ because $5 = 2 + 3$ and $(2 + 3)x = 2x + 3x$ by distribution, and $(x + 1)(x + 2) = x^2 + 3x + 2$ because it is $x(x + 2) + 1(x + 2)$. Simplification is the reverse of this, it makes things simpler. It is based on associative distribution principles as shown in the examples below.

Example 1: $2x + 3x = 5x$

This is the reverse of $5x = (2 + 3)x = 2x + 3x$ and this can be used to introduce it. However, there are other ways such as patterns from arithmetic as used in subsection 5.4.1. Two others are as follows:

1. **Use reality**. Think up a story for this. For example, All boxes have the same number of lollies, x. I bought 2 boxes of x lollies and then 3 boxes of x lollies. How many lollies did I buy altogether? Obviously it is five boxes worth of lollies which is $5x$.

2. **Use materials**. Think of a cup as variable (can hold an unknown number of counters) and counters as numbers. Then,

---

Example 2: $2 \times 3x = 6x$

This has been covered in subsection 5.4.1. However, the two other ways above still hold. For example,

1. **Use reality**. A story for $2 \times 3x = 6x$ is I bought 2 lots of 3 boxes of lollies, how many lollies did I buy altogether? Obviously there are 6 boxes of lollies equivalent to $6x$.

2. **Use materials**. $2 \times 3x = 6x$ is represented by cups as on right.

Example 3: $2x + 3y + 4x + 5y$

In this one we return to the meaning of addition which is to add like things. This can be seen as follows.

1. **Use reality**. A story for the above is *I bought 2 bottles of drink for $x each and 3 pies for $y each, then I bought 4 more bottles of drink for $x each and 5 more pies for $y each. How much did I pay?* Obviously this will be 2+4 $x and 3+5 $y, thus $2x + 3y + 4x + 5y = 2x + 4x + 3y + 5y = 6x + 8y$.

2. **Use materials**. We need two different cups to represent x and y, then $2x + 3y + 4x + 5y$ is

    Which is 6 ⊽ and 8 ☐   or 6x + 8 y
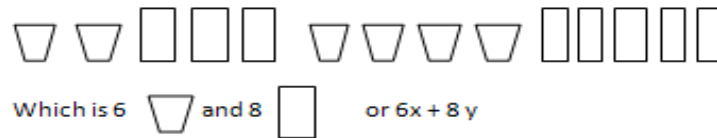
3. **Use algorithm setting out**. Compare to 23 + 45 and extend the vertical algorithm.

23		$2x + 3y$	
+45		$+4x + 5y$	
8	add ones	$6x$	add x's
60	add tens	$8y$	add y's
68		$6x + 8y$	

Note: The use of cups and counters can help with early algebra. For instance, many students confuse $3x + 2$ with $3(x + 2)$ but with cups and counters, as below, it is easy to see the difference:

The cups and counters for $3(x + 2)$ are   = 3x + 6

This compares with $3x + 2$ which is

5.4.4 Extending the principles to cover factorisation

Factorisation is the reverse of expansion. Up to Year 9 it is only necessary to factorise for numbers, for example, $2 + 4x = 2(1 + 2x)$. However, the inverse understanding is a powerful way to view all factorisations. Thus, there are two ways to teach factorisation.

1. **Indirectly**. Use examples like those below to become familiar with expansions and various multiplications and think about their inverse (only need examples ** for up to Year 9):

Expansion	Inverse
$2(a + b) = 2a + 2b$ **	$2a + 2b = 2(a + b)$ **
$3(3a + 5b) = a + 15b$ **	$9a + 15b = 3(3a + 5b)$ **
$5(2x + 3) = 10x + 15$ **	$10x + 15 = 5(2x + 3)$ **
$a(a + b) = a^2 + ab$	$a^2 + ab = a(a + b)$
$a(2a + 3b) = 2a^2 + 3ab$	$2a^2 + 3ab = a(2a + 3b)$
$p(x + y) = px + py$	$px + py = p(x + y)$

    In this way, students become familiar with factorisation as an inverse of expansion.

The first step is to identify what is common in these inverse examples – they have something in each part that is a factor (e.g. a in $a^2 + ab$, 3 in $9a + 15b$, and so on). Then reinforce this by setting up exercises to determine if there is a factor, for example:

Tick the ones that have a factor.
Write the factor.

Expansion	Yes/No	Factor
$3a + 4$		
$6a + 2$		
$3a + ab$		

The second step is to realise that the factor can be taken out of all parts (check by expanding), for example:

$$3x + 6 \rightarrow 3 \text{ is a factor} \rightarrow 3x + 6 = 3(x + 2) \text{ because } 3x = 3 \times x \text{ and } 6 = 3 \times 2.$$

2. **Materials**. Use cups and counters as follows (only useful for numerical factorisations).

Factorisation is not the difficult or important task it used to be. For example, there are formulae for solving quadratics and there are calculators and apps that can solve equations for you by just entering them. In fact, like algorithms, solving equations can be done with technology. This means the important algebraic skills to develop in students is how to model problems and how to know what equations they have to solve.

6 Teaching Framework for Algebra

The teaching framework organises the content for algebra into a framework of four topics: repeating and growing patterns, change and functions, equivalence and equations, and arithmetic–algebra principles. Each of these topics is partitioned into sub-topics. Each sub-topic is described and any concepts or strategies used in the teaching framework are listed. They are also related to big ideas. Topics and sub-topics are chosen so as to represent ideas that recur across all year levels. The resulting framework is given in Table 1. This overall framework can be compared to the Australian Curriculum to produce year-level frameworks.

Table 1. *Framework for teaching algebra*

TOPIC	SUB-TOPIC	DESCRIPTION AND CONCEPTS/STRATEGIES/WAYS	BIG IDEAS
Repeating and growing patterns	Repeating patterns	Following/copying patterns; continuing patterns; completing patterns; constructing patterns; identifying repeats	
		Relating position to item; finding rule for this relationship; numbers → language → variable	Relating position and item for repeating pattern
		Generalising repeat; extending to equivalent fractions and proportion	
	Linear growing patterns	Copying, continuing, completing and creating patterns, objects → numbers; visual analysis → table analysis	
		Relating position to term (visual → table); finding pattern rule (sequential and position – numbers → language → variables); one, two or more operations (+/– → ×/÷)	Relating growing part and constant part to pattern rule
		Generalising sequencing and pattern rules; relating pattern rule, pattern and graph; introducing variable and algebraic expressions	Relating pattern, pattern rule to slope, y-intercept and graph
	Patterns in other strands	Using patterns to find number and operation ideas and relationships	
	Nonlinear growing patterns	Copying, completing, continuing, finding and graphing pattern rule; exploring differences to rule and graph	Relating differences to pattern rule and graphs
Change and functions	Meanings and notation	Change; function machine; input–output tables; unnumbered → numbered; relating stories to change	Symbols tell stories
		Arrowmath notation; arithmetic excursions; relation to equations	
		Generalisation of change rules (numbers → language → variables)	
	Backtracking	Inverse; backtracking; one, two and more changes; one or more operations (+/– → ×/÷)	Backtracking
	Solutions	Consider actions of unknowns; using backtracking to solve equations	
	Graphing	Graphing change; relating graphs, equations, arrowmath notation, input–output and real-world stories	
Equivalence and equations	Exploring meanings of equals and order principles	Same–different/equals–unequals; mass and length models; unnumbered to numbered contexts	Equals and order principles (see arithmetic–algebra principles)
		Relation of real-world stories to equivalence and equations	
		Mass and length contexts → "mathematised" contexts → abstract contexts	
	Balance rule	Exploring matching changes to keep sides as the same value; unnumbered → materials → generalised	Balance rule
	Unknowns/variable	Representing unknowns and unknowns/variables in equations and inequations	
	Solutions	Solving for unknowns in equations and inequations	Balance rule and inverse

TOPIC	SUB-TOPIC	DESCRIPTION AND CONCEPTS/STRATEGIES/WAYS	BIG IDEAS
Arithmetic– Algebra principles	Equals	Exploring equivalence class and laws of equals	Reflexivity, symmetry, transitivity
	Order	Exploring greater than and less than in terms of order class and laws	Non-reflexivity, antisymmetry, transitivity, well ordered
	Number size	How numbers change in operations in relation to other numbers	Compensation, inverse relation, equivalence
	Field	Exploring the principles of both arithmetic and algebra – called the Field principles (or properties)	Identity, inverse, commutative, associative, distributive
	Manipulation of alegebraic expressions	Substitution, simplification, operations with algebraic expressions, and factorisation.	Area model, distributive law

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Appendix A: Mathematics as Story Telling (MAST)

This approach to teaching mathematics uses the "creating own symbols" part of abstraction to teach students the role of symbols in mathematics, that symbols are a language that tells stories. (*Note:* This was part of the Algebra project at Dunwich State School led by Chris Matthews.) There are seven steps.

1. **Symbols**. Students explore how symbols can be assembled to tell a story, first in Indigenous situations (e.g. Indigenous art) and then creating and interpreting symbols for simple actions (e.g. walking and sitting at a desk).

2. **Exploring simple addition**. Students act out a story (e.g. 2 students join 3 students to make 5 people). Discussion identifies story elements – objects (the 2, 3 and 5 people) and actions (joining, making).

3. **Creating own symbols**. Students create their own symbols to tell the story. They first do this free style (a drawing that represents the "joining" and the "making") and discuss results (e.g. are they linear – showing the action left to right, or are they more holistic?). Secondly, the students create symbols in a more structured and linear setup (students use objects for students and drawing for "join" and "make" or "same as"), as in the example on right.
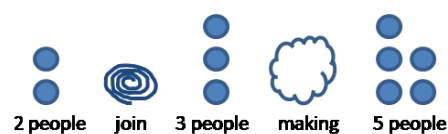
Here the "join" picture is a vortex that picks up the 2+3 and the "making" picture is a cloud that brings them down together as rain.

4. **Symbol showing**. Students share symbols and explain their symbols' meanings. Students then use other students' symbol systems to represent other stories (e.g. 4 people join 7 people), and make up stories where other students' symbols are used to represent addition.

5. **Story modification**. Teacher removes one counter from the left-hand side two counters and asks if story still true (see example on right). Most students will say no. Teacher then asks how it can be made true again. The normal answers are:

    (a) put the counter back (as on right);

    (b) remove a counter from right-hand side (the balance principle for equations) (as on right);

    (c) add a counter to the three counters as on right (the compensation principle); or

    (d) draw another vortex on the left-hand side and put a counter in front of it.
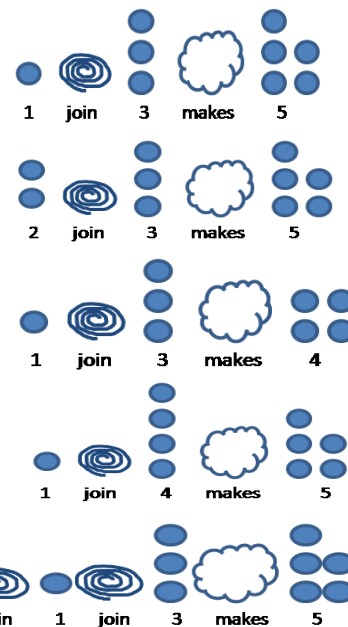
6. **Unknown**. Teacher sets up story: "unknown number of people joined 3 to make 5". Students create their own symbol for unknown as on right. Students use balance principle to find unknown.

7. **Formal symbols**. Teacher introduces students to common formal symbols for join (+), results (=) and unknown (x) e.g. $2 + 3 = 5$ and $x + 3 = 5$. Students begin to relate these symbols to everyday situations and to solve for the unknown.

Through these MAST activities, symbols are introduced as a shorthand language for telling stories, improving students' ability to solve word problems. As well as this, the big ideas of balance and compensation are also introduced.

Appendix B: The power of algebraic big ideas

In this appendix we will show how teaching a powerful algebraic big idea (in this case a principle) can enable a lot of particular mathematics ideas to be achieved across many years of schooling. The example is the inverse principle, one of the most powerful principles or big ideas in mathematics, algebra and arithmetic.

Models for teaching the inverse principle

The inverse principle is the understanding that each operation has a partner operation that reverses or returns it to where it started. For example +2 is reversed by −2 and ÷6 is reversed by ×6. This is true for any number, measure or expression, therefore the operations themselves can be seen as inverses, that is + is the inverse of − and × is the inverse of ÷. Inverse, along with identity (the number that does not change anything, e.g. +0 and ×1) is one of the most important principles. It can be introduced by many methods; three are provided below.

Undoing

In this method, the teacher begins by talking about actions and how they can be **undone** so you return to where you started. For example, turning to the right can be undone by turning to the left, and stepping forward can be undone by stepping back. The second step is for the teacher to act out a series of undoings: for example, step to right → step to left, move hands clockwise → move hands anti-clockwise, and so on. This notion of undoing can be used to introduce a lot of important inverses. (*Note:* the answers are in square brackets like this [ ].)

Take five counters and join four counters to them, to make nine. Discuss how we can undo this. [Take the four counters away and return to five.] This can be

understood as +4 being undone by −4. Further examples will enable students to see that + is undone by −.
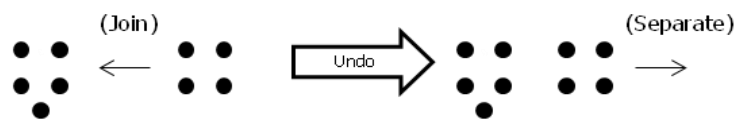
For two-step problems look at, for example, how putting on a sock and a shoe is undone:

Putting on requires putting the sock on (Step 1), and then putting the shoes on (Step 2).

Undoing requires taking the shoe off (undoing Step 2), and then taking the sock off (undoing Step 1), returning to a bare foot.
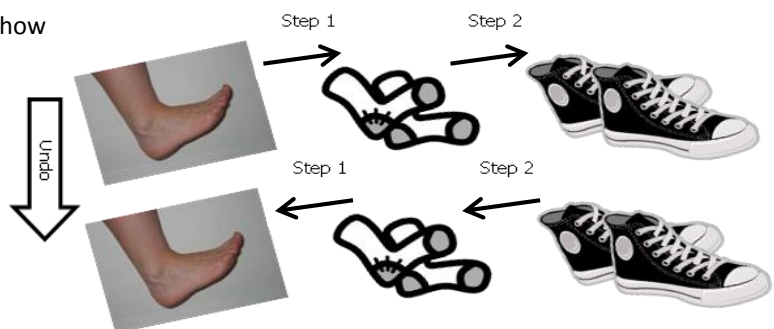
We can see that each action is undone but also that both actions are undone in the opposite order (i.e. socks on, shoes on is undone by shoes off, socks off). Experiencing further examples leads to seeing that, for example, +3 +2 is undone by −2 −3.
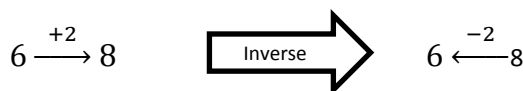
Function machines

In this method, the teacher builds a function machine "robot" from a large box, hangs an operation around its neck, and develops sets of cards (e.g. input cards 1 to 20, output cards 1 to 30). A student hides inside the box, other students put in different number input cards and the hidden student puts out the appropriate number output cards on the opposite side of the box. The final (output) number cards are calculated by following the operation on the front of the robot (see diagram below).
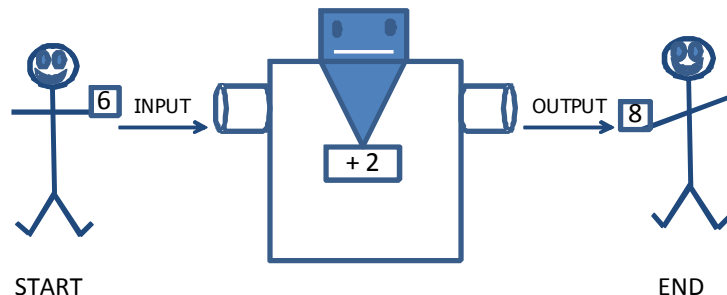
This is represented by arrowmath notation as a **change**; for example: $6 \xrightarrow{+2} 8$

The question is asked, *What if you get 10 at the end? What was put in at the start?* Most students will be able to say 8 with support and most classes will be able to say that to "go backwards" requires subtracting 2 (after discussion and examples). In arrowmath notation, this reversal of the operation can be represented as a change in the reverse direction (which inverts the operation), for example:

$$6 \xrightarrow{+2} 8 \qquad \boxed{\text{Inverse} \Rightarrow} \qquad 6 \xleftarrow{-2} 8$$

Multiple operations can be handled by two function machine robots:
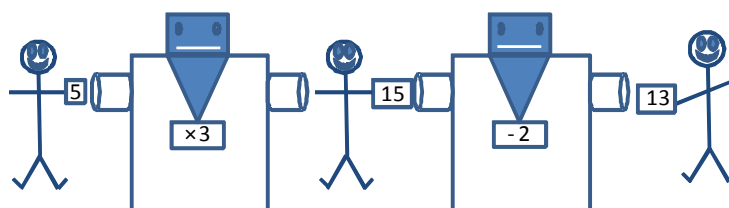
This enables inverses of sequences of operations to be studied:

Forward: $5 \xrightarrow{\times 3} 15 \xrightarrow{-2} 13$

Backward: $5 \xleftarrow{\div 3} 15 \xleftarrow{+2} 13$ (inverse)

In practice, it is effective to actually walk students forward past the function machine from left to right and backward from right to left, verbalising each function as they walk past. This process assists students with grasping change and reversals. The construction of function machines can be novel and may appeal to students' imaginations; using a large enough box so that a "wheel" can be attached, a student can turn the "wheel" to indicate something is happening, with another student sitting inside as the "machine" to indicate something is going to come out, thus showing change.

Number line

In this method, the teacher shows a number line (numbered) and discusses what happens as we move back and forth along the line. Says, *We are at 7, 2 is added, how do we get back to 7?*, and acts this out along the line showing inverse (as on right). Then, the teacher moves onto an unnumbered line. The teacher says, *Your Dad gives you money and you spend $8, what has to happen to get back to the*
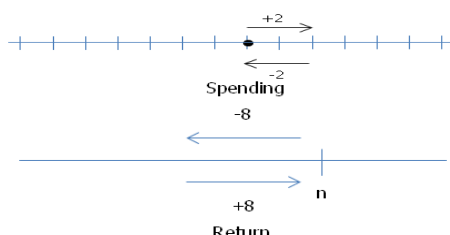
same amount of money you were given? Students act this out on the number line with n as the letter signifying the money that Dad gave. Teacher discusses how to get back to n. [Find someone to give you $8.] It shows that +8 is the inverse of −8.

If the number line is made "mathematical" so that all operations are possible, then we can use the line to do sequences of inverses.
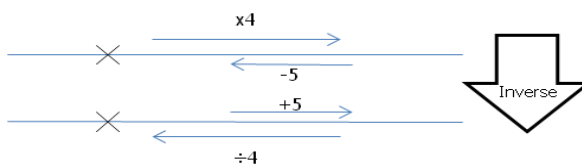
(*Note:* for younger students inverses for + and − can be "walked" on a number track, e.g. 3 goes to 7 by +4 and then 7 goes back to 3 by −4, as on right.)
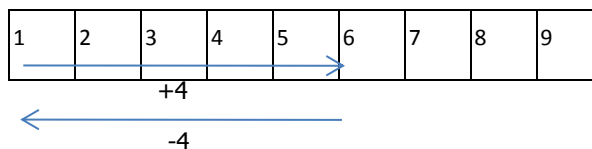
Applying the inverse principle

Once a learner has an understanding of inverse, it can be applied to particular mathematics topics. The important point here is that this one piece of knowledge, the knowledge of inverse, because it is general and across topics, can be used to understand and solve problems in a number of different topics. Separate rules or algorithms do not have to be taught for each topic because the inverse knowledge is enough. It is important to realise that the models used to teach the inverse principle also play an important role in the applications. Some examples of inverse applications are as follows.
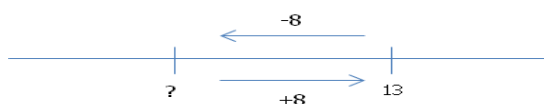
Basic subtraction facts

If addition facts (e.g. 8 + 5 = 13) are known, inverse can be applied to calculate the subtraction facts (e.g. 13 − 8 = 5). This is based on subtraction being the inverse of addition and using this to rethink subtraction in terms of addition (and, therefore, using our addition facts to calculate the subtraction facts). For example, using function machines and arrowmath notation, 13 − 8 can be thought of as on right (13 being changed by −8):

$$13 \xrightarrow{\;-8\;} ?$$

While, reversing the notation, the inverse is as on right (8 being added to something to make 13):

$$? \xrightarrow{\;+8\;} 13$$

Similarly, on a number line (see below), 13 − 8 is as follows in terms of inverse.

From both of these models, 13 − 8 = ? can be seen as the same as ? + 8 = 13 (or "what plus 8 equals 13"). Thus, 13 − 8 can be calculated using the already known addition fact, 5 + 8 = 13. This means 13 − 8 = 5. Thus we have a strategy for solving subtraction facts which comes from the inverse big idea; it is called the "think addition" strategy.

Subtraction computation

Once students have learnt addition computation, inverse can be used in two ways in subtraction. The first is to check the subtraction as below.

Subtraction:	52	Check by addition:	25
	−27		+27
	25		52

(This method can also be used to check additions, by using subtraction; multiplications by using division; and divisions, by using multiplication).

The second way is as a method that uses addition to do subtraction computations. It is best seen with the number line model. To do the method, students have to "think addition" (similar to basic facts) and solve subtractions such as 52 – 27 by thinking "what has to be added to 27 to make 52". This can be solved using an alternative jump method where you start from 27 and determine what jumps will get to 52, as in the diagram on right. The answer is 3 + 10 + 10 + 2 = 25.
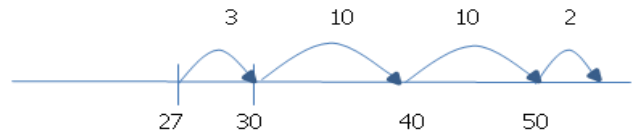
This additive subtraction method is also useful for subtracting money (e.g. working out change), mixed numbers, decimal numbers, and measures (time and length).

Solutions of linear equations

Situations like *I bought $3 pies for all of my friends and a $7 roll. I spend $25. How many friends?*, can be considered in terms of change, can be acted out on ×3 and +7 function machines, and can be represented by arrowmath notation and an equation, for example:

$$? \xrightarrow{\times 3} \xrightarrow{+7} 25 \qquad\qquad 3x + 7 = 25$$

The inverse or backtracking method that is part of learning inverse on function machines can be used to solve the problem. Since the change is ×3 and +7, it is reversed by −7 and ÷3. This reversing or backtracking provides the answer, for example:

$$3x + 7 = 25 \ \rightarrow \ 3x = 25 - 7 = 18 \ \rightarrow \ x = 18 \div 3 = 6$$

Solving % problems

Situations like, *I paid a 40% down payment of $120 on the dress, how much was the dress?* can also be thought of as change and solved by reversing, or finding the inverse of, the change. Both the function machine and number-line models can assist here and the use of each is given as follows.

Function machine or change model. The problem can be considered as a function machine that changes the cost of a dress to 40% of that cost, that is, that multiplies original cost by 0.4. This can be represented by an arrowmath diagram and solved by reversing or backtracking, as follows. The original cost of the dress is therefore found by dividing the 40% cost by 0.4, i.e. cost = $120 ÷ 0.4 = $300.
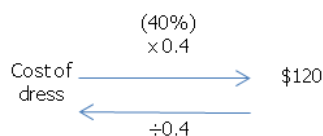
Double number-line model. The problem can also be considered as a change on a line. An excellent way to do this is to consider the line as having two sides (this is called a double number line) – one side as % and the other as $. In this situation, the 100% changes to 40% while the original cost changes to $120. The change on both sides is the same. From 100% to 40%, the change is to multiply by 0.4, so to go the other way, or undo the change, is to divide by 0.4. This means that the original cost (the ? in the diagram) is $120 ÷ 0.4 = $300.
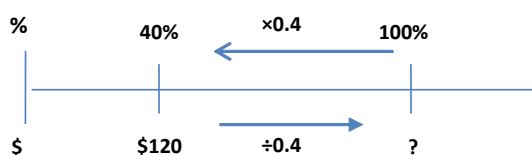
Solving rate problems

For problems like, *I bought the petrol for $1.40 per litre, how much petrol did I buy for $63?*, both the function machine and number-line models again apply.

Function machine or change model. Once again, the problem is considered as change, but change from litres to dollars by multiplying by the rate of change (i.e. × 1.40), for example:

$$\text{Litres of petrol} \xrightarrow{\times \ 1.40} \text{money in } \$$$

Thus, we use the arrowmath notation to set up the change and use the inverse (i.e. ÷ 1.4) to solve it:

$$? \ L \xrightarrow{\times \ 1.40} \$63$$
$$\xleftarrow{\div \ 1.40}$$

By using inverse or backtracking in this way, the number of litres is ? = 63 ÷ 1.4 = 45 L.

Double number-line model. Once again, we can use the double number line with one side L (litres) and the other side $ (dollars). In this situation, 1 L is $1.40 so the line becomes as below. To get from $1.40 to $63 is to multiply by 63 and divide by 1.4. Thus, the number of litres of fuel is ? = 1 × 63 ÷ 1.4 = 45 L.
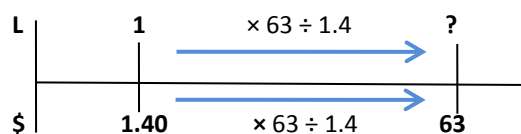
Overall, learning the inverse principle enables you to solve many problems in many areas of mathematics. The big idea, inverse, enables a whole collection of what is often seen as distinct mathematics situations to all be solved with the one idea.

It should also be noted that the five applications above are only a few of the uses of inverse. It is also useful for ratio, measurement (metric conversion), currency conversion, and scale problems.

YuMiDeadly

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