

YuMi Deadly Maths

Operations

Prep to Year 9

Prepared by the YuMi Deadly Centre
Queensland University of Technology
Kelvin Grove, Queensland, 4059

Prep to Year 9: Resource 3 – Operations





YuMiDeadly

*Growing community
through education*

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VERSION 3, 23/01/17

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<http://ydc.qut.edu.au>

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ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

DEVELOPMENT OF THIS BOOK

This version of the YuMi Deadly Maths Operations book is a modification and extension of a book developed as part of the Teaching Indigenous Mathematics Education (TIME) project funded by the Queensland Department of Education and Training from 2010–12. The YuMi Deadly Centre acknowledges the Department's role in the development of YuMi Deadly Maths and in funding the first version of this book.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at QUT which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

“YuMi” is a Torres Strait Islander Creole word meaning “you and me” but is used here with permission from the Torres Strait Islanders’ Regional Education Council to mean working together as a community for the betterment of education for all. “Deadly” is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

The YuMi Deadly Centre’s motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre’s vision: *Growing community through education*.

The YuMi Deadly Centre can be contacted at ydc@qut.edu.au. Its website is <http://ydc.qut.edu.au>.

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ABOUT YUMI DEADLY MATHS

From 2000–09, researchers who are now part of the YuMi Deadly Centre (YDC) collaborated with principals and teachers predominantly from Aboriginal and Torres Strait Islander schools and occasionally from low socio-economic status (SES) schools in a series of small projects to enhance student learning of mathematics. These projects tended to focus on a particular mathematics strand (e.g. whole-number numeration, operations, algebra, measurement) or on a particular part of schooling (e.g. middle school teachers, teacher aides, parents). They resulted in the development of specialist materials but not a complete mathematics program (these specialist materials can be accessed via the YDC website, <http://ydc.qut.edu.au/>).

In October 2009, YDC received funding from the Queensland Department of Education and Training through the Indigenous Schooling Support Unit, Central-Southern Queensland, to develop a train-the-trainer project, called the **Teaching Indigenous Mathematics Education** or **TIME** project. The aim of the project was to enhance the capacity of schools in Central and Southern Queensland Indigenous and low SES communities to teach mathematics effectively to their students. The project focused on Years P to 3 in 2010, Years 4 to 7 in 2011 and Years 7 to 9 in 2012, covering all mathematics strands in the Australian Curriculum: Number and Algebra, Measurement and Geometry, and Probability and Statistics. The work of the TIME project across these three years enabled YDC to develop a cohesive mathematics pedagogical framework, **YuMi Deadly Maths**, that covers all strands of the *Australian Curriculum: Mathematics* and now underpins all YDC projects.

YuMi Deadly Maths (**YDM**) is designed to enhance mathematics learning outcomes, improve participation in higher mathematics subjects and tertiary courses, and improve employment and life chances. YDM is unique in its focus on creativity, structure and culture with regard to mathematics and on whole-of-school change with regard to implementation. It aims for the highest level of mathematics understanding and deep learning, through activity that engages students and involves teachers, parents and community. With a focus on big ideas, an emphasis on connecting mathematics topics, and a pedagogy that starts and finishes with students' reality, it is effective for all students. It works successfully in different schools/communities as it is not a scripted program and encourages teachers to take account of the particular needs of their students. Being a train-the-trainer model, it can also offer long-term sustainability for schools.

YDC believes that changing mathematics pedagogy will not improve mathematics learning unless accompanied by a whole-of-school program to challenge attendance and behaviour, encourage pride and self-belief, instil high expectations, and build local leadership and community involvement. YDC has been strongly influenced by the philosophy of the Stronger Smarter Institute (C. Sarra, 2003) which states that any school has the potential to rise to the challenge of successfully teaching their students. YDM is applicable to all schools and has extensive application to classrooms with high numbers of at-risk students. This is because the mathematics teaching and learning, school change and leadership, and contextualisation and cultural empowerment ideas advocated by YDC represent the best practice for **all** students.

YDM is now available direct to schools face-to-face and online. Individual schools can fund YDM in their own classrooms (contact ydc@qut.edu.au or 07 3138 0035). This Operations resource is part of the provision of YDM direct to schools and is the third in a series of resources that fully describe the YDM approach and pedagogical framework for Prep to Year 9. It overviews the teaching of: (a) addition and subtraction of integers; (b) multiplication and division of integers; (c) the four operations with common and decimal fractions; and (d) operation applications with percent, rate and ratio. It includes concepts, principles, strategies, computation, problem solving and extension to algebra. Because YDM is largely implemented within an action-research model, the resources undergo amendment and refinement as a result of school-based training and trialling. The ideas in this resource will be refined into the future.

YDM underlies three projects available to schools: YDM Teacher Development Training (TDT) in the YDM pedagogy; YDM AIM training in remedial pedagogy to accelerate learning; and YDM MITI training in enrichment and extension pedagogy to build deep learning of powerful maths and increase participation in Years 11 and 12 advanced maths subjects and tertiary entrance.

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1 *Purpose and Overview*

As stated at the start of the purpose and overview of the YuMi Deadly Maths (YDM) Number book, number and geometry are the bases of the structure of mathematics. This Operations book continues the number aspect; it directly follows on from topics of the Number book and covers addition and subtraction of whole numbers, multiplication and division of whole numbers, all operations for common and decimal fractions, and operation applications for percent, rate and ratio. It is the second YDM teaching book.

The purpose of this book is to provide background information to support professional development workshops on how to teach the operations of addition, subtraction, multiplication and division. This how-to-teach will comprise powerful classroom teaching ideas that have two purposes: (a) to facilitate students' learning at the required year levels; and (b) to reveal the structure of mathematics as this is done. This combination is not easy: the structure for operations is very distant from everyday reality and yet the classroom activities cannot be.

The operations of addition, subtraction, multiplication and division, and the arithmetic processes they become when used with numbers, underlie most mathematics strands – they are generalised to algebra and applied in measurement, probability and statistics. The ideas that relate to operations, such as basic facts, computation, estimation, problem solving, and equations, rely on a student's understanding of operations. Because they are applied so widely, many problems students have with other strands are caused by limited understanding of operations.

This limited understanding is because it is easy to see operations in terms of answers instead of concepts, processes and strategies, and to see procedures instead of underlying structure. This is particularly relevant for operations because generalisations of operations and number (which is arithmetic) give algebra, the most powerful form of mathematics because it is based on structural knowledge where the meaning is in the relation between parts and not the context of the parts. Thus, the most important purpose of this book is to reveal the importance of mathematics structure in teaching and learning mathematics.

To begin this process, this chapter covers connections and big ideas (section 1.1), sequencing (section 1.2), teaching and cultural implications (section 1.3), and overview of the book (section 1.4).

1.1 Connections and big ideas

This section looks at operations as a connected whole – what is the same, what is different, where concepts come from in earlier years, and where they lead to in future years. It overviews the role operations plays in the structure of mathematics, describing how operations is **connected to the other strands** within the structure of mathematics for Prep to Year 9 and how it is based on a series of **big ideas** that recur across Prep to Year 9.

1.1.1 Connections, rich schema and mathematical structure

The purpose of this book is to provide a connected framework for the teaching of mathematics that will enable powerful learning to occur.

Rich schema

Powerful learning builds mathematics knowledge as rich schema – interconnected structures of knowledge that define, connect, apply and remember – that students can use in their everyday life and can see their world through. As stated in the YDM Number book, **rich schema** contains knowledge of **when and why as well as how**. Rich schema has knowledge as connected nodes, which facilitates recall and problem solving. We argue that knowledge of the structure of mathematics, particularly of connections and big ideas, can assist teachers to be

effective and efficient in teaching mathematics, and enable students to accelerate their learning. This is because it enables teachers to do the following.

- *Determine what mathematics is important to teach* – mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present.
- *Link new mathematics ideas to existing known mathematics* – mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem solving.
- *Choose effective instructional materials, models and strategies* – mathematics that is connected to other mathematics or based around a big idea commonly can be taught with similar materials, models and strategies.
- *Teach mathematics in a manner that makes it easier for later teachers to teach more advanced mathematics* – by preparing the linkages to other ideas and the foundations for the big ideas the later teacher will use. Thus it is essential that teachers know the mathematics that precedes and follows what they are teaching, because they are then able to build on the past and prepare for the future.

The basis of rich schema learning is to sequence and connect knowledge so that the underlying structure of knowledge is revealed and to reduce mathematics to big ideas so that its entirety can be grasped. Important components of this, such as big ideas, connections and sequencing, will be looked at in the next subsections but, first, some important purposes of this book in terms of teaching operations are discussed. These are purposes that are universal to all that we do in teaching operations.

Mathematical structure and operations

There are a series of number-sense principles and two major structures with regard to number and operations. These are summarised below and covered in more detail in later subsections.

1. The first are what we call the **number-size principles**. They are about how the operations calculate and what happens if larger or smaller numbers are used. In most examples, larger numbers give larger answers (e.g. $4 + 8 = 12$ and $7 + 8 = 15$). However, in important cases, this is not so (e.g. $8 - 4 = 4$ and $8 - 7 = 1$). This example is one instance of the inverse relation principle which describes how, for subtraction and division, it is possible for a number to increase and the answer to decrease.
2. The second is the **Field structure** (see section 1.1.4) which identifies that operations have identities (things that do not change other things, like adding zero) and inverses (e.g. $+3$ is undone by -3).
3. The third is the **equivalence class structure** that describes how equals acts (e.g. $5 = 2 + 3$ is as correct as $2 + 3 = 5$ because the meaning of “=” is “same value as” – see section 1.1.4).

Interestingly, these principles and structures mean that, **mathematically, only addition and multiplication are operations** – only these two operations obey all of the Field properties/principles (see section 1.1.4). Subtraction is, mathematically, addition of additive inverse (e.g. $5 - 2 = 5 + -2$) and division is multiplication by multiplicative inverse or reciprocal (e.g. $6 \div 2 = 6 \times \frac{1}{2}$). This is why the principles for addition and multiplication are similar, and subtraction and division are similar but different from addition and multiplication. It is also why subtraction and division have the property of inverse relation (increasing a number makes the answer smaller) that addition and multiplication do not have.

Sometimes in the teaching sequence, **reverting to the mathematically correct approach is useful** – for example, $\frac{2}{3} \div \frac{3}{4}$ cannot be done by sharing and grouping. While it is possible to demonstrate this with pictures and shading, it is still difficult to understand or see the pattern symbolically. It is more useful in this instance to consider the meaning of division as multiplication by inverse (e.g. $6 \div 2 = 6 \times \frac{1}{2}$). This helps to make sense of the process for division of fractions where $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3}$ so that students are able to represent this process symbolically and understand the reason for what they are doing.

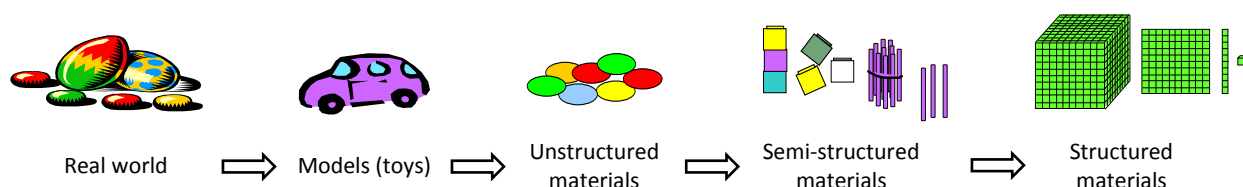
1.1.2 Connecting symbols, language, models and real-world situations

The end product of operations is a **symbol language** (e.g. $2 + 3$ and $2 + 3 = 5$). There are problems in this symbol language because a focus on answers and not relationships leads many students to believe that $2 + 3 = 5$ is acceptable and $5 = 2 + 3$ is not; for many students the equals sign has become a symbol for “put the answer here” or “do something” when its real meaning is “same value as”. Mathematically, *7 subtract 3 equals 4* is represented with symbols as $7 - 3 = 4$; it must be seen as $7 - 3$ **is the same value as** 4. This means that it is possible and equally correct to show $7 - 3 = 4$ as $4 = 7 - 3$. Similarly, students also need to be familiar with relationships that are not closed to a single answer such as $2 + 3 = 4 + 1$.

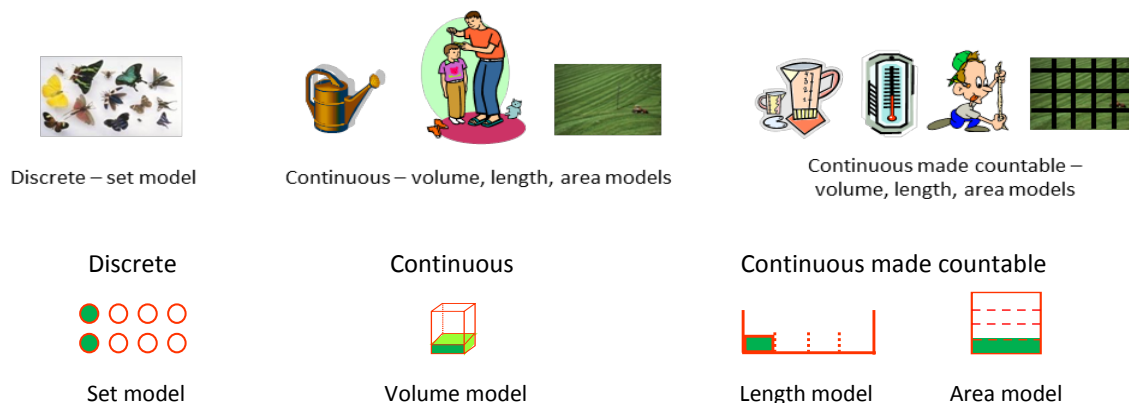
This symbol language has to relate to everyday language and real instances. Many forms of symbols are possible and all relate to stories as the addition stories below show. The interesting point is that treating the symbols like a concise language (so $7 + 4$ is 7 things joining 4 things), enables the symbols to tell stories and to describe the world, an outcome more powerful than answers. This is a major part of building the concepts of the operations (see sections 2.1 and 3.1).

One way to represent this symbol language is to use **models**. Models connect and unify mathematics, whereas symbols tend to emphasise difference. For operations, there are three models: **set** (e.g. Unifix and counters), **length** (e.g. Unifix stuck together or number lines), and **array or area** (e.g. counters, Unifix, dot paper or graph paper), which is for multiplication and division (e.g. 3×4 is 3 rows of 4 or a 3 by 4 rectangle). These models do much more than just help with meaning – they show structure and apply across many topics. For example, the array model can help with: basic facts (e.g. 4×7 is 4 rows of 7 which is 4 rows of 5 plus 4 rows of 2); algorithms (e.g. 24×7 is 24 rows of 7 so it is 20 rows of 7 plus 4 rows of 7); fractions/decimals and percent (e.g. 0.2×0.4 is a rectangle 2 tenths by 4 tenths which gives 8 squares in 100 or 0.08). Multiplication is the inverse of division and $\times 1$ or $\times \frac{2}{2}$ or $\times \frac{3}{3}$ and so on leaves everything unchanged (e.g. $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$, and so on).

YDM focuses on knowing **connections between real-world situations, models, language and symbols**, and strategies that lead to meaning and generalisation rather than rote definitions and procedures (or algorithms) that lead to particular answers for particular numbers. In this approach, models can be used in the same sequence as for number and can be tailored to suit the level of number representation students are working at. Kinaesthetic acting out of operations should be completed first which can then be transferred to tangible concrete models and finally abstract number sentences (see figure below).



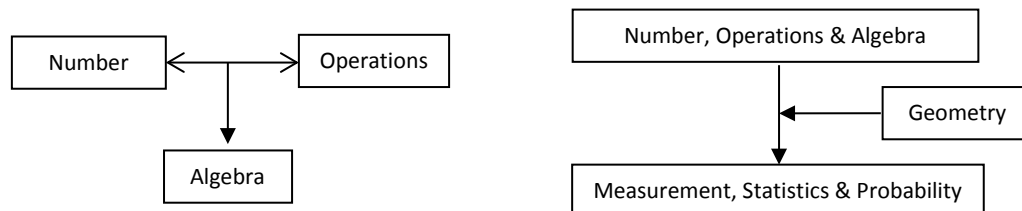
Models also assist us to link **discrete and continuous**; there are a range of models that cross that divide as the examples below show. The continuous made countable models are particularly important.



As number is taken from discrete objects and applied to continuous attributes, its nature changes. Numbers designate ends of units and zero represents the start of units and not nothing (although it represents no units).

1.1.3 Structural and topic connections

Connections are at their basis structural and structural connections are important because they show the place of operations within school mathematics. Operations concepts are used with number and lead into algebra. Through the inclusion of geometry, number, operations and algebra lead on to measurement and, more directly, to statistics and probability. This can be diagrammatically represented as below:



The important connections between operations and the other strands are as follows.

- *Operations and number* – an obvious connection as operations need numbers to act on. In particular, the strategies for computation relate to the numeration concepts, that is: (a) separation strategy relies on a place-value understanding of 2- to 4-digit numeration; and (b) sequencing and compensation strategies rely on a rank understanding of numeration.
- *Operations and algebra* – again an obvious relationship as algebra is generalisation of arithmetic activities. In particular, $2x + 3$ relates to an example like $2 \times 5 + 3$. The difference is that 5 is an actual number while x is a variable.
- *Operations and measurement* – measurement involves a lot of operations particularly with respect to formulae (e.g. perimeter, area).
- *Operations and statistics and probability* – both of these involve operations (e.g. in calculating mean and chance).

As well as between operations and other strands, there are connections between topics within the particular operations (addition, subtraction, multiplication and division) and between topics within each of these operations. The major connections are as follows.

- *Addition with subtraction and multiplication* – subtraction is the inverse of addition and one meaning of multiplication is repeated addition.
- *Subtraction with division* – one meaning of division is groups of or repeated subtraction.
- *Multiplication with division* – multiplication and division are inverses.
- *Concepts and problem solving* – the meanings of the operations are the basis of solving problems as they determine which operations relate to which situations.
- *Calculation and estimation* – estimation requires calculation but they also have strategies in common (the calculation strategies help to estimate).
- *Multiplicative comparison with applications of fractions, decimals, percent, rate and ratio* – all the latter are examples of comparison by calculation and can be solved/understood by the same models.

1.1.4 Big ideas underlying operations

The big ideas for operations are global and come from the principles of a *Field* and *equivalence class* (or extensions of these principles) – a Field is a mathematical structure that is followed by operations on numbers, while an equivalence class is a mathematical structure that is followed by equals. The major big ideas are as

follows. Some of the big ideas are considered in more detail in later sections in this book. These sections also consider the properties of operations in terms of number-size principles. They also show that subtraction and division are not real operations because they do not obey some of the principles.

Global principles

1. **Symbols tell stories.** The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. $2 + 3 = 5$ means caught 2 fish and then caught another 3 fish, or bought a \$2 chocolate and \$3 drink, or joined a 2 m length of wood to a 3 m length ... and so on).
2. **Relationship vs change.** Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. All relationships can be perceived as changes and vice versa. This is particularly applicable to operations; 2 plus 3 can be perceived as relationship $2 + 3 = 5$ or change $2 \xrightarrow{+3} 5$.
3. **Interpretation vs construction.** Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for $2 + 3 = 5$; construct a shape of four sides with two sides parallel).
4. **Accuracy vs exactness.** Problems can be solved accurately (e.g. find $5\,275 + 3\,873$ to the nearest 100 – rounding and estimation) or exactly (e.g. $5\,275 + 3\,873 = 9\,148$ – basic facts and algorithms).
5. **Part-part-total/whole.** Two parts make a total or whole, and a total or whole can be separated to form two parts – this is the basis of numbers and operations (e.g. fraction is part-whole, ratio is part-to-part; addition is knowing parts, wanting total).

Field properties for operations

1. **Closure.** Numbers and an operation always give another number (e.g. $2.17 + 4.34 = 6.51$ – for any numbers a and b , $a + b = c$ which is another number; and $2.17 \times 4.3 = 9.331$ – for any numbers a and b , $a \times b = c$, where c is another number).
2. **Identity.** The numbers 0 and 1 do not change things (+/– and $\times/\div$ respectively). Adding/subtracting zero leaves numbers unchanged (e.g. $9 +/– 0 = 9$, where 0 can equal $+1-1$, $+6-3-3$, $+11-14+3$, and so on). Anything multiplied by 1 = itself (i.e. for any number a , $a \times 1 = 1 \times a = a$). Anything multiplied by 0 = 0.
3. **Inverse.** A change that undoes another change. Addition is undone by subtraction and vice versa (e.g. $+5 - 5 = -5 + 5 = 0$, so $2 + 5 = 7$ means $7 - 5 = 2$). Multiplication's inverse is division and vice versa (e.g. $\times 5 \div 5 = \div 5 \times 5 = 1$, so $2 \times 5 = 10$ means $10 \div 5 = 2$). This principle holds for fractions and indices (e.g. for fractions, the inverse of $\frac{2}{3}$ is the reciprocal $\frac{3}{2}$ [or 1 over $\frac{2}{3}$] because $\frac{2}{3} \times \frac{3}{2} = \frac{3}{2} \times \frac{2}{3} = 1$; for indices, the inverse of 6^3 is 6^{-3} and vice versa because $3 + -3 = 0$ and $6^3 \times 6^{-3} = 6^{(3+-3)} = 6^0 = 1$).
4. **Commutativity.** Order does not matter for addition but does for subtraction (e.g. $3 + 4 = 4 + 3$, but $7 - 5 \neq 5 - 7$). Order does not matter for multiplication but does for division (e.g. $12 \times 4 = 4 \times 12$ but $12 \div 4 \neq 4 \div 12$; for any a , b and c , $(a \times b) \times c = a \times (b \times c)$). Also known as turnarounds.
5. **Associativity.** What is done first does not matter for addition and multiplication but does matter for subtraction and division (e.g. $(8 + 4) + 2 = 8 + (4 + 2)$, and $(8 \times 4) \times 2 = 32 \times 2 = 64$ and $8 \times (4 \times 2) = 8 \times 8 = 64$ but $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$).
6. **Distributivity.** Multiplication and division are distributed across addition and subtraction and act on everything (e.g. $3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$; $(21 - 12) \div 3 = (21 \div 3) - (12 \div 3)$). Distributivity does hold for all operations (e.g. $7 \times (8 - 3) = (7 \times 8) - (7 \times 3)$; $(56 + 21) \div 7 = (56 \div 7) + (21 \div 7)$; $(56 - 21) \div 7 = (56 \div 7) - (21 \div 7)$).

Extension of Field properties

1. **Compensation.** Ensuring that a change is compensated for so the answer remains the same – related to inverse (e.g. $5 + 5 = 7 + 3$; $48 + 25 = 50 + 23$; $61 - 29 = 62 - 30$).

2. **Equivalence.** Two expressions are equivalent if they relate by adding or subtracting 0 and multiplying or dividing by 1; also related to inverse (e.g. $48 + 25 = 48 + 2 + 25 - 2 = 73$; $50 + 23 = 73$; $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$).
3. **Inverse relation.** The higher the second number in subtraction and division, the smaller the result (e.g. $12 \div 2 = 6 > 12 \div 3 = 4$; $\frac{1}{2} > \frac{1}{3}$). For division, the more you divide by, the less you have (e.g. $24 \div 8$ is less than $24 \div 6$). This principle does not apply to addition or multiplication.
4. **Triadic relationships.** When three things are related there are three problem types where each of the parts are the unknowns. For example, $2 + 3 = 5$ can have a problem for: $? + 3 = 5$, $2 + ? = 5$, $2 + 3 = ?$. This principle holds for all four operations: $a + b = c$ ($? + b = c$, $a + ? = c$, $a + b = ?$); $a - b = c$ ($? - b = c$, $a - ? = c$, $a - b = ?$); $a \times b = c$ ($? \times b = c$, $a \times ? = c$, $a \times b = ?$); $a \div b = c$ ($? \div b = c$, $a \div ? = c$, $a \div b = ?$).

Equivalence class properties for equals

1. **Reflexivity.** Anything always equals itself (e.g. $2 \times 4 + 7 = 2 \times 4 + 7$); any number a is equal to itself, that is, $a = a$.
2. **Symmetry.** If something equals another thing then the another-thing equals the something (e.g. $2 + 3 = 5$ means $5 = 2 + 3$). For all numbers a and b , if $a = b$ then $b = a$ (the order of an equation can be reversed or “turned around”). This is important for equations as it means that $2 \times 3 = 6$ and $6 = 2 \times 3$, $2 \times 3 = 12 \div 2$ and $12 \div 2 = 2 \times 3$ are all correct and true.
3. **Transitivity.** If something equals another thing and the another-thing equals a third thing, then the original something equals the third thing (e.g. $2 + 3 = 5$ and $5 = 9 - 4$ means $2 + 3 = 9 - 4$). For all numbers a , b and c , $a = b$ and $b = c$ means $a = c$. This is also important for equations because it means we can say: $4 \times 3 \div 2 = 12 \div 2 = 6$ and so $4 \times 3 \div 2 = 6$.

For YDM, **curricula should be taught so that big ideas and connections are emphasised**. Thus, our aim is to construct teaching frameworks that contain the knowledge of how to do this by specifying and sequencing topics and big ideas within each year and drawing attention to connections. This will take some time to finalise as connections and big ideas are developed across the years by looking at similarities between different forms of the same mathematics in different years and topics.

1.2 Sequencing

This YDM Operations book covers the teaching of addition, subtraction, multiplication and division in whole numbers and common and decimal fractions, applications in percent, rate and ratio, plus extensions into directed number (integers), indices and algebra. This section looks at how this is all sequenced.

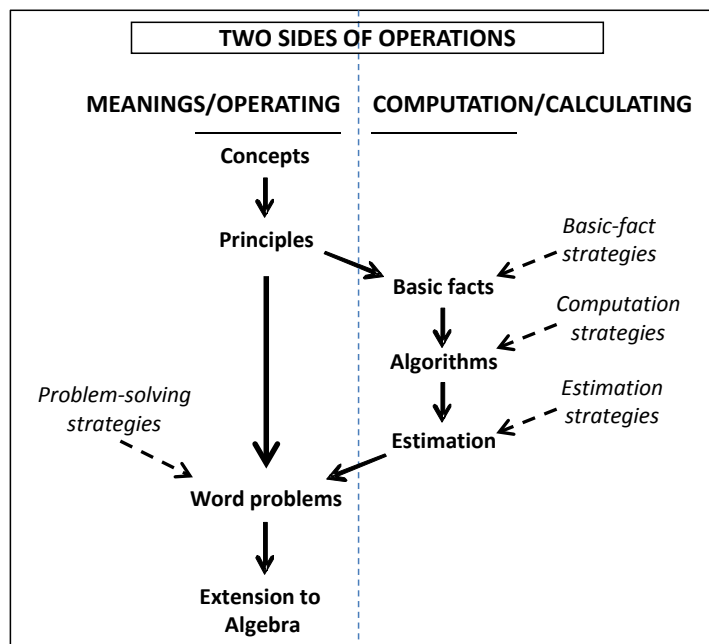
1.2.1 Sequence for each operation

The sequence for each operation is based on the work being divided into two columns or sides, **meanings** and **computation** (also known as **operating** and **calculating**); meanings-operating covers concepts, principles, word problems and extension to algebra, while computation-calculating covers basic facts, algorithms and estimation. Problem solving is based on meaning (i.e. concepts), not computation. Algebra depends on principles and concepts not computation. Basic facts, algorithms, word problems and estimation are based on strategies. Thus, there are two desired outcomes: (a) that students understand the concepts, strategies and principles that are behind operations; and (b) that students learn strategies and procedures to be able to compute in our everyday world. The figure below right describes this.

The overall sequence within operations is: concepts → principles → basic facts → algorithms → estimation → word problems → algebra. In reality, this sequence is integrated with basic facts, algorithms and estimation for smaller numbers leading to word problems for smaller numbers before the cycle is repeated for larger numbers.

Schools need to plan what they want from algorithms. Our view is that **all algorithm methods should be taught to all students as strategies**, because strategies are more important than getting answers which can be done with a calculator or, close enough, by estimation. We also believe that there

should be opportunities for students to **create their own ways of recording** and that they should also be able to **show their working** in some way. This requires students to examine their own thinking and, thus, develop metacognition. These creations need not be the same for each learner.



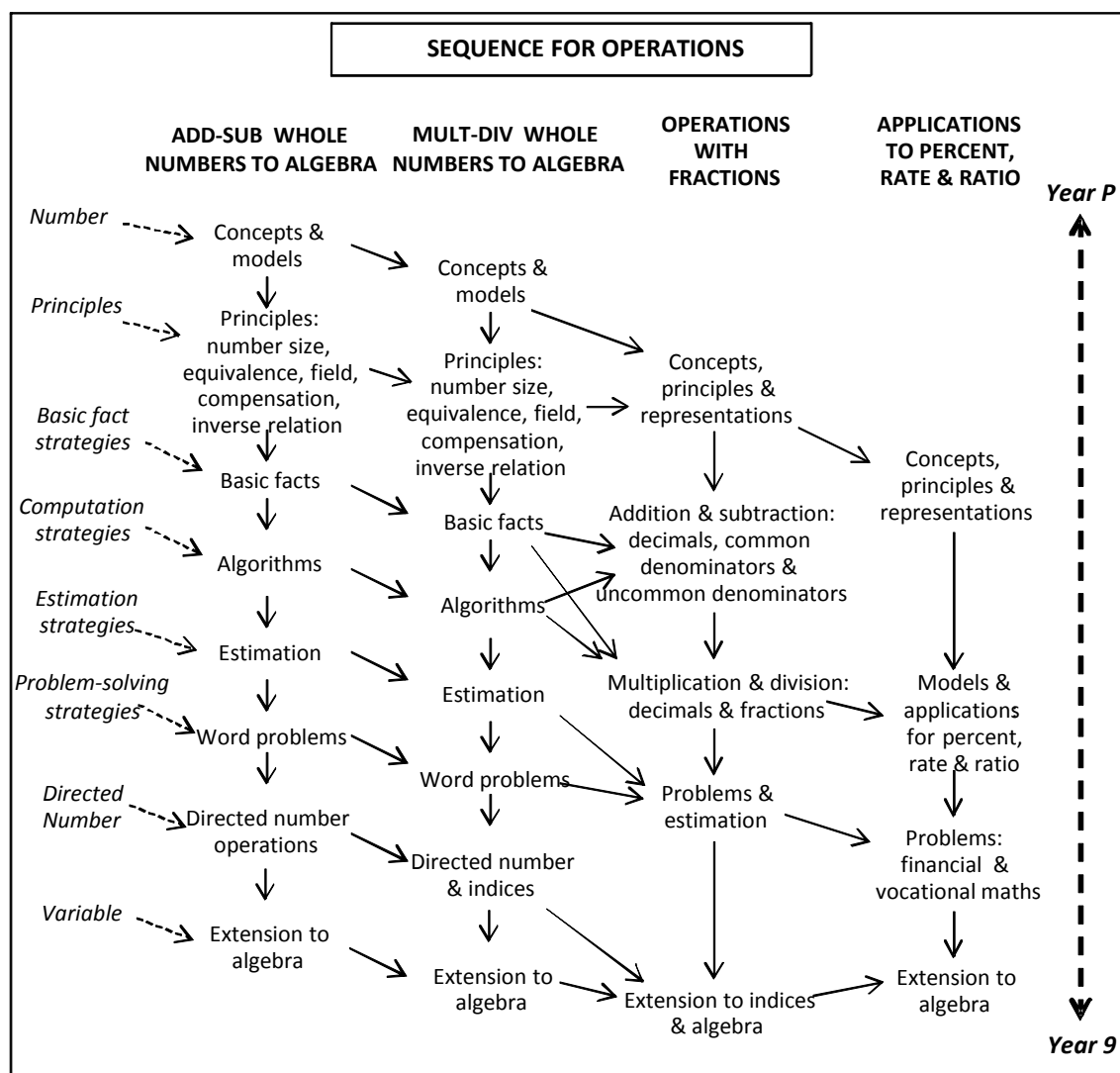
1.2.2 Overall sequence for operations

The figure on the next page provides the sequence for the topic of operations. As well as the two sections for addition and subtraction of whole numbers-integers and multiplication and division of whole numbers-integers, there are two more sections on operations with common and decimal fractions and applications to percent, rate and ratio. The sections are based on applying the ideas in the former sections into the latter. This is especially important for concepts and principles; for example, the models (pictorial/visual representations) for operations with whole numbers are set, number line and array (with array only for multiplication and division). The array model extends to an area model which is the basis for multiplications of common and decimal fractions and algebra, while the number-line model is applicable to fractions, percent, rate and ratio.

The figure on the next page diagrammatically summarises the sequence by showing how the topics relate to Years P to 9 (approximation only) for each of the four sections. The italics and dotted arrows on the left show how ideas come into the sequence. The vertical arrows in each column show the sequence across Years P to 9. The arrows between the columns show sequences across the four sections.

It should be noted that the diagram on the next page is not as simple as that for the YDM Number book. This is because number can all be put under five big ideas – part-whole, additive, multiplicative, continuous vs discrete (number line) and equivalence. Operations is not so simple because operations and number form arithmetic which is then generalised to algebra. Thus operations forms the structure that becomes part of algebra and there are many more than five of these.

The development of this overall sequence is a work in progress and is not complete. However, it is included because we believe that teachers of mathematics need a picture of the whole so they can pass this on to their students – **strong mathematics learning is based on seeing the whole**.



1.2.3 Components of the sequence

The components of the sequence are defined as follows.

- **Concepts** – these are meanings that define the operation in terms of the everyday world; they also cover the different models in which the meanings can be encapsulated (for addition and subtraction, there are two models – set and number line or length); they relate story, action, drawing, language and symbols (these are called representations).
- **Principles** – these are properties that hold no matter what the numbers are (i.e. they hold for whole numbers, decimals, fractions, and so on); for example, the commutative law, that first + second always equals second + first.
- **Basic fact strategies** – ways to find answers to addition and subtraction facts (albeit slowly); the major ones in this book are “count ons”, “near doubles”, “near tens”, “turnarounds”, “families”, and “think addition”.
- **Basic facts** – automated answers to addition and subtraction facts (one-digit numbers) from $0 + 0$ to $9 + 9$, from $18 - 9$ to $0 - 0$, and for multiples of 10 (e.g. $400 + 500$, $8000 - 6000$).
- **Computation strategies** – ways of finding answers to addition and subtraction algorithms; the major ones in this book are “separation”, “sequencing”, and “compensation”.

- *Algorithms* – calculating answers to addition and subtraction algorithms (more than one-digit numbers – mental, written and calculator).
- *Estimation strategies* – ways to find approximate answers to large-number computations; the major ones in this book are “front end”, “rounding”, “straddling”, and “getting closer”.
- *Estimation* – calculating approximate answers to large-number computations; uses principles to be accurate.
- *Problem-solving strategies* – these are general rules of thumb that direct towards the answer; for example, the major ones in this book are “identify given and wanted”, “act it out”, “make a drawing”, “restate the problem”, “solve a simpler problem”, “check the answer”, and “learn from the solution”. The strategies will also include Polya’s plan of attack (See, Plan, Do, Check).
- *Word problems* – being able to interpret problems given in words and determine the operation to use, and construct problems from equations.
- *Extension to algebra* – being able to repeat concepts for when there are variables (i.e. relate algebra sentences to actions and stories).

1.3 Teaching and cultural implications

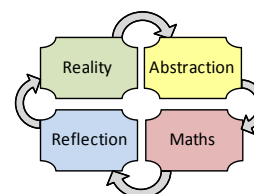
This section looks at the implications for teaching operations in terms of generic strategies and the culture of students. It is important to note that the mathematics taught in Australia is European in form and reflects that culture. Therefore, it can clash with other cultures (e.g. African, Polynesian, Middle East and East Asian), particularly the Aboriginal and Torres Strait Islander cultures.

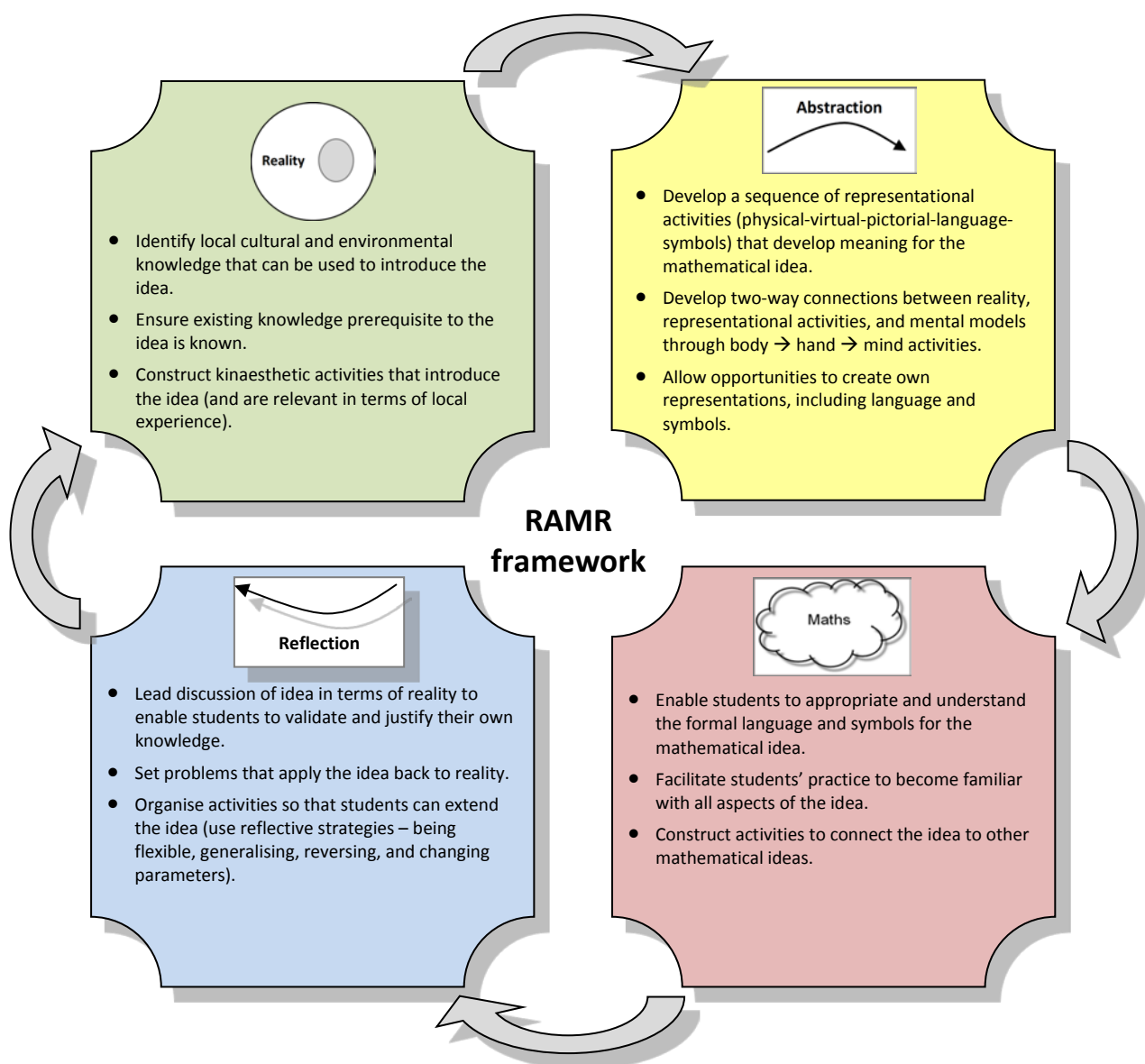
1.3.1 Generic teaching strategies

We see mathematics teaching as comprising three components – *technical* (handling materials), *domain* (the particular pedagogies need for individual topics) and *generic* (pedagogies that work for all mathematics). Interestingly, and fortunately, the domain section is not as complicated as it could be because mathematical ideas that are structurally similar can be taught by similar methods. For example, fractions and division are similar and both are taught by partitioning sets into equal parts – except that the set is seen as one whole for fractions and a collection of objects for division. There are also some generic teaching methods that hold for any topic.

The **Reality–Abstraction–Mathematics–Reflection (RAMR) framework** (see figure on the next page) is very useful for operations because of the generic teaching ideas contained in the framework. For a start, it grounds all mathematics in reality and provides many opportunities for connections, flexibility, reversing, generalisations and changing parameters, as well as body → hand → mind. The idea is to use the framework and all its components throughout the years of schooling and this will help prevent learning from collapsing back into symbol manipulation and the quest for answers by following procedures.

Within the chapters of this book, activities for teaching operations are sometimes provided using the RAMR framework on the next page. This enables these activities to act as exemplars of the RAMR framework. Other activities in this format are included throughout this book but not with every stage of the framework detailed, and some are just described without the stages. Activities that are given in RAMR framework form are identified with the symbol on right. *The activities are written in italics to help distinguish them from the main text.*





1.3.2 Cultural implications of operations teaching

Operations can be confronting for students from Aboriginal and Torres Strait Islander and low socio-economic status (SES) cultures if taught as a celebration of the success of Western middle-class knowledge and technology. Teachers should try to separate the Western and middle-class cultural imperatives from the operations activity and teach operations as something to be mastered to succeed within mainstream Australian society, something that will empower the learner and not something to be revered for its effects on Western society.

Extensive and dense number systems are an invention of number-oriented cultures such as the European heritage of mainstream Australia. They were developed to be applied to discrete objects, in particular, possessions of people in the culture (e.g. how many sheep, or how many gold coins). Because of beliefs about possessions (status depends on what you have), number and arithmetic became strengths of number-oriented societies, underpinning development of products and technology, and conquest of the world where most dominant societies are number-oriented (and Eurocentric). However, number and arithmetic are also weaknesses of number-oriented cultures as they produce people addicted to development. Thus, number-oriented societies have trouble living in harmony with the world through a tendency to want to dominate and control nature.

Indigenous cultures from around the world followed a different path from number-oriented cultures in the development of mathematics; for them people were seen as more important than number so their mathematics

specialised in other areas. This different focus could be seen as emanating from their cultural beliefs with regard to group rather than individual ownership. It means that Indigenous peoples developed different technologies (particularly weapons) and had different relationships to land than people from number-oriented cultures. These technologies were relevant to, and adequate for, Indigenous lifestyles but seen as primitive by number-oriented societies. However, the different focus also means that Indigenous cultures are expert in living in harmony with the world, and may provide the answers to the problems of poverty and global warming generated by number-oriented cultures. The teaching of operations can bring Australian mainstream Eurocentric school teaching into conflict with Indigenous students, both Australian and immigrant. It must be taught with care, as a part of a European culture that Indigenous people need to understand and should not be celebrated as something that raises some people above others.

For low SES, non-Indigenous students in Australia, the situation is different (but the outcome is the same). These people are part of Australia's Eurocentric number-oriented culture, but at the lower unsuccessful end. The number systems created as part of Eurocentric mathematics have always benefited high SES people at the expense of low SES people, yet promulgated the idea that bigger numbers (e.g. money, house cost, cars) are better. The way numbers function within Eurocentric societies achieves two outcomes simultaneously: (a) benefits one class of people at the expense of the other; and (b) puts blame for their lack of benefit on the actions of the class that is not benefited. Thus, similar to Indigenous peoples, mathematics must be taught with care to low SES students because its teaching can designate these students as failures.

There are two other conflicting consequences in relation to mathematics teaching and Indigenous and low SES students. First, the learning style and thinking strengths of Aboriginal and Torres Strait Islander and low SES students are holistic-intuitive and visual-spatial rather than verbal-logical, meaning that they have an affinity for mathematics taught as a structural connection to the world. Second, traditional Australian teaching of mathematics is procedural and rote and focuses on number and arithmetic as algorithmic procedures. Studies of Aboriginal, Torres Strait Islander and low SES classrooms by YuMi Deadly Centre staff over the last 10 years have identified number as the weakest link in students' knowledge and the cause of most of their problems in operations, measurement, probability and statistics handling.

1.4 Overview of book

Chapters

This book comprises the following chapters:

Chapter 1: Purpose and overview – covering connections and big ideas, sequencing, and teaching and cultural implications;

Chapter 2: Addition and subtraction of whole numbers – focusing on the two operations of addition and subtraction with respect to concepts, principles, basic facts, algorithms, estimation, problem solving, directed numbers and algebra;

Chapter 3: Multiplication and division of whole numbers – focusing on the two operations of multiplication and division with respect to concepts, principles, basic facts, algorithms, estimation, problem solving, directed numbers and indices, and algebra;

Chapter 4: Operations with common and decimal fractions – focusing on all four operations as they apply to common and decimal fractions; and

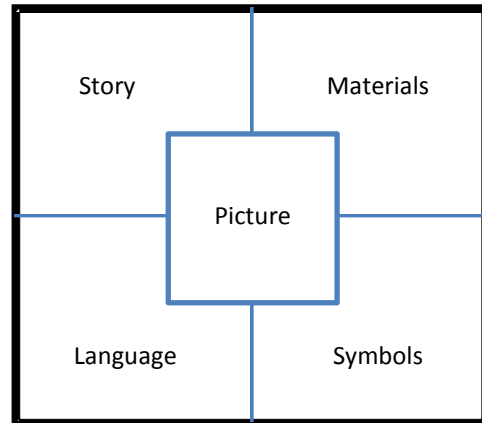
Chapter 5: Applications with percent, rate and ratio – focusing on applications of all four operations to percent, rate and ratio.

Particular teaching foci

1. As stated earlier in section 1.1.2, it is important to make connections between **reality** (real-world situations), **models**, **language and symbols** for operations. The Payne and Rathmell triangle is powerful for making connections with operations as for number (see YDM Overview and Number books). Another powerful approach to teaching operations is the **thinkboard** – an A3 sheet which provides five areas for connections:

stories $\leftrightarrow$ acting out/modelling with materials
 $\leftrightarrow$ pictures (drawing) $\leftrightarrow$ language $\leftrightarrow$ symbols

Teachers using thinkboards often photograph them when students have finished filling them out (but not removed the materials) so that a permanent record of the students' work can be displayed (the materials section on the board is not permanent).



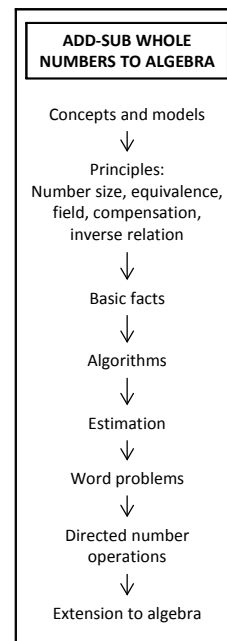
2. It is important to develop **visuals** for the operations so that students can make drawings.
3. Students should work from stories through to symbols on the board, and then **reverse** from symbols back to stories.
4. It is also important to use the computation side of operations to focus on **strategies**. Strategies such as “when faced with complexity, break into parts, do parts separately and then recombine” can be extended from whole numbers to decimal numbers, mixed numbers, metric measures, time, and algebra.
5. Give importance to **problem solving** and **estimation**. The easy access to a calculator means that the important work in operations is working out what operations and numbers to press and to be able to quickly estimate the cost of things.

2 Addition and Subtraction

This is the first of the operations chapters. It covers addition and subtraction for whole numbers, moving from meanings and models for one-digit addition and subtraction through to algebraic representations. The chapter is built around the **two components** for operations:

1. **Meaning/operating.** What does addition and subtraction mean (what are their concepts), where are they used in the world, and what are their properties?
2. **Computation/calculating.** What is the answer to the addition and subtraction operation, what ways can we work it out, and how accurate does this have to be? Like number, addition and subtraction operate on the world in two ways – on individual objects, and on measures.

The **sequence** for the chapter is based on the figure on right: concepts, principles, basic facts, algorithms, estimation, word problems, directed number, and extension to algebra. Problem solving and understanding of algebraic applications of addition and subtraction are based on meaning (i.e. concepts and principles), not computation. Obviously, algorithms are computation along with basic facts and estimation.



2.1 Concepts and models for addition and subtraction

Concepts and models describe the meanings of addition and subtraction – they include the ability to move between different representations of the concept, that is, real-world situations, models, language and symbols. Addition and subtraction concepts and models reflect everyday meanings such as joining and separating.

The complexity of the operations does not lie in what they do but in the many ways they can be understood. We believe in giving students the full set of concepts but spread out across the year levels.

Addition has four meanings: joining, part-part-total (including inaction), change and comparison, and inverse of taking away.

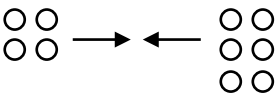
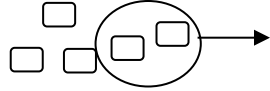
Subtraction has four meanings: separating (taking away), part-part-total (including inaction), change and comparison, and inverse of joining.

In this section we will look at the first two meanings for each operation, namely, joining/separating and part-part-total. The latter two meanings of each operation, namely, change/comparison and inverse, will be covered later with problem solving.

2.1.1 Joining and separating

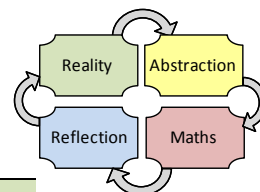
Addition and + are the language and symbol for joining. People joining each other, money and other objects being joined together, and lengths being joined are just examples of the many ways in which things join all the time in all societies. Thus, 2 people meeting 3 people becomes $2 + 3 = 5$ in mathematics.

Subtraction and – are the opposite actions (or inverses) of addition, that is, -2 undoes $+2$. So subtraction is separation (or take away). Again this is a very common action in the everyday world. For example, 2 people leaving the group of 5 people, \$2 being spent from \$5, and 2 m being cut off the end of a 5 m piece of wood, are all examples of $5 - 2 = 3$.

	ACTION	NUMBER SENTENCE
	4 and 6 items are joined	$4 + 6 = 10$
	5 items and 2 are taken away	$5 - 2 = 3$

2.1.2 Teaching joining and separating

The RAMR activities below are to teach addition as joining, and subtraction as separating or taking away, using the set and number-line models.



Reality

Act out problems using the students:

- Addition – there were 3 students sitting at a table and 2 more students joined them. How many students altogether?
- Subtraction – 7 students were eating their lunch together. 3 students had to go to a meeting and walked away. How many students were left?

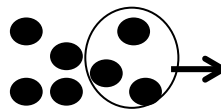
Abstraction

Set model

Start with acting out a problem with the body – identify parts and whole.

Encourage students to model the acting out with counters:

John has 7 pencils, he gives
3 to Jane, he has 4 left.



Have students draw a picture with their own representation for the story (can include own images for standard symbols for later discussion):

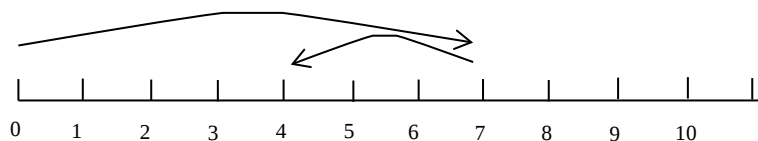
John has 7 pencils, he gives
3 to Jane, he has 4 left.



Number-line model

Start with acting out a problem (and finding the answer if important). This time students model the acting out with a number line:

John walks 7 blocks and then walks 3 back, how much further is he from where he started?



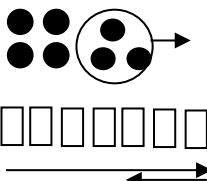
Mathematics

Introduce language and formal symbols. For example: How many pencils did John have? [7], how many did he give Jane? [3], and how many were left? [4]. This would be “seven subtract three is four” and $7 - 3 = 4$. Use the students’ own symbolism and forms of recording as a halfway house to this end.

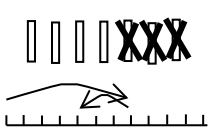
Reflection

Begin with any of the representations and complete the others, that is, relate all five representations: act out $\leftrightarrow$ models $\leftrightarrow$ pictures $\leftrightarrow$ language $\leftrightarrow$ symbols. Using the thinkboard, fill in one area (any of the areas), and ask for the other areas to be completed. For example, see thinkboard below.

Thinkboard:

Story John has 7 pencils, he gives 3 to Jane, he has 4 left; John walks 7 blocks, back 3, how far from start?	Materials/acting 
Language Seven subtract three is four	Symbols $7 - 3 = 4$

Drawing



It is important to ensure that:

- all models are covered – set and number line;
- all connections are both ways – students can write a story for language or symbols, and can interpret a drawing in a story or symbols;
- stories are used for a variety of situations – shopping, sporting, fishing, driving, TV stories, and so on; and
- students understand that operations are generic – $3 + 4 = 7$ means that 3 fish and 4 fish give 7 fish, \$3 and \$4 give \$7, 3 m and 4 m gives 7 m and so on. Thus, $3 + 4 = 7$ holds for every set of objects and every measure in the world.

2.1.3 Part-part-total

The four operations involve situations with parts that are known and parts that need to be found. Addition and subtraction situations involve two parts and a total.

- **Addition** is a situation when parts are known and total is wanted.
- **Subtraction** is a situation when total and one part is known and the other part is wanted.

Subtraction is not a real operation in Western mathematics as it is presently structured, because it is not commutative (e.g. $2 - 3 \neq 3 - 2$) and it is not associative (e.g. $(5 - 2) - 1 \neq 5 - (2 - 1)$). However, subtraction is treated as an operation in school, vocational and everyday mathematics and taught in conjunction with addition as it is a very common action in the everyday world.

Understanding the meanings of the operations allows students an **integrated and a single method** (part-part-total) to determine whether a problem is addition or subtraction – see the table in subsection 2.1.4 below.

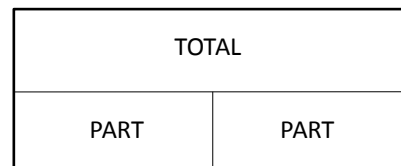
The part-part-total method encompasses all the other methods because it does not see the operations of addition and subtraction as actions or change. The other way to see addition and subtraction not as an action or a change is the **inaction** meaning. This meaning just looks at different sets and then the number in the superset that contains both. Examples of this meaning are:

- addition is in terms of the superset: there were 3 Holdens and 5 Fords, this made 8 cars; and
- subtraction is in terms of the subset: there were 7 cars and 3 were Holdens, this left 4 Fords.

2.1.4 Teaching and using part-part-total

To teach part-part-total, do the following:

- act out addition and subtraction situations, like $4 + 5 = 9$, with materials identifying the group of 9 by the name “total”, and the groups of 4 and 5 by the name “part”;
- show how the stories of addition and subtraction are the reverse of each other (addition as $P + P \rightarrow T$; and subtraction as $T \rightarrow P + P$);
- generally do the same activities as for teaching joining and taking away but continuously use the terms part and total instead of the language or symbols; and
- use a part-part-total diagram like that on the right – place information from the problems onto it – the numbers and a question mark for the unknown.



Note: Free resources are available on the YDC website giving lesson plans for introducing part-part-total – these are the result of an Australian Research Council Linkage funded project to train Indigenous teacher aides. Five booklets for addition and subtraction and five for multiplication and division are available under [Professional Learning resources](#).

To use part-part-total, use a storyboard or thinkboard to represent different problems that have an element missing, as in the summary below, and to identify part, part and total and which is unknown. Use the type of thinking that is given below.

OPERATION	MEANING	PROBLEM	THINKING
Addition	Know parts – want total	Straightforward problem: <i>I had \$11 892 in my account, I put in \$5 238, how much do I have now?</i>	“The \$11 892 and \$5 238 are parts. The wanted amount is the total. So, the operation is addition.”
		Complex problem: <i>I took \$5 238 from my account, this left \$11 892, what was in the account to start with?</i>	“The \$5 238 and \$11 892 are parts. The wanted amount is the total. So, the operation is addition.”
Subtraction	Know total – want a part	Straightforward problem: <i>I had a grant of \$9 561, I spent \$7 832, how much do I have left?</i>	“The \$9 561 is the total. The \$7 832 is a part. The wanted amount is a part. So, the operation is subtraction.”
		Complex problem: <i>I added the grant to my \$7 832 account, this gave me \$9 561, how much was in the grant?</i>	“The \$7 832 is a part. The \$9 561 is the total. The wanted amount is a part. So, the operation is subtraction.”

2.1.5 Role of equals

As discussed in section 1.1.2, for many students the equals sign has become a symbol for “put the answer here” or “do something” when its real meaning is “same value as”. Thus although 7 *subtract* 3 *is* 4 is represented with symbols as $7 - 3 = 4$, it must be seen as $7 - 3$ **is the same value as** 4. This means that it is possible and equally correct to show $7 - 3 = 4$ as $4 = 7 - 3$. In this way, many forms of equations are possible and all relate to stories, as the following shows.

STORY	SYMBOLS
There were 7 pencils, 3 were taken away.	$7 - 3$
There were 7 pencils, 3 were taken away. How many pencils were left?	$7 - 3 =$
There were 7 pencils, 3 were taken away, this left 4 behind.	$7 - 3 = 4$
There were 4 pencils. This was a result of 3 pencils being removed from the 7.	$4 = 7 - 3$
Jack had 7 pencils. He gave 3 to Jane. This meant he had the same number of pencils as Bill who had 4.	$7 - 3 = 4$
Jack had 7 pencils. He gave 3 to Jane. Bill had 2 pencils. He got 2 more from his teacher. Therefore Jack and Bill had the same number of pencils.	$7 - 3 = 2 + 2$ or $2 + 2 = 7 - 3$

The important point here is that the teaching must focus on:

- the line in $\frac{7}{-3}$ and the equals sign as meaning “same as value as”;
- the sequence, stories $\rightarrow$ acting out/modelling $\rightarrow$ pictures $\rightarrow$ language $\rightarrow$ symbols;
- the reverse of this sequence; and
- the RAMR cycle, reality $\rightarrow$ abstraction $\rightarrow$ mathematics $\rightarrow$ reflection.

The wide variety of problems given to students should give precedence to equals as “the same value as” because it is the long-term meaning used in algebra.

2.2 Early number-sense principles

This section looks at the properties of addition and subtraction that hold regardless of the numbers or variables being added and subtracted. It begins by looking at the number-size principles, then looks at the relationship principles (called Field principles), and finally the equals principles.

2.2.1 Number-size principles

The number-size principles are properties of addition and subtraction that hold for all numbers (whole numbers, decimal numbers, common fractions, measures, and so on).

Nature of principles

We can work out the principles by looking at the two simple examples on the right: $7 + 4 = 11$ and $7 - 4 = 3$ and the activities (a) to (l). For these examples, a number and an upward arrow means that the number is increasing; a number and a downward arrow means that the number is decreasing; and a number and an equals sign means that the number stays the same. For examples (a) to (d), the

$7 + 4 = 11$	(a)	$7 \uparrow$	11	3
$7 - 4 = 3$	(b)	$7 \downarrow$	11	3
	(c)	$4 \uparrow$	11	3
	(d)	$4 \downarrow$	11	3
	(e)	$7 \uparrow$	$11 =$	4
	(f)	$7 \uparrow$	$3 =$	4
	(g)	$7 \downarrow$	$11 =$	4
	(h)	$7 \downarrow$	$3 =$	4
	(i)	$4 \uparrow$	$11 =$	7
	(j)	$4 \uparrow$	$3 =$	7
	(k)	$4 \downarrow$	$11 =$	4
	(l)	$4 \downarrow$	$3 =$	4

What relationships/differences can you see between addition and subtraction?

activities require that answers be given for addition (middle) and subtraction (right). For examples (e) to (l), only the number on the right needs an answer.

Let us look at activity (a) – the 7 and arrow upwards means that the 7, in both the addition and subtraction examples, is increasing; the 11 and 3 on their own are asking what happens to the 11 in the addition and the 3 in the subtraction in this case. [The students have to work out that, the 11 and the 3 both increase – so an upward arrow is put beside both.]

Let us look at activity (e) – the 7 upward arrow and the 11 equals sign means that the 7 increases and the 11 has to stay the same in the addition; the 4 on its own is asking what happens to the 4 in this case. [The students have to work out that the 4 must decrease – so a downward arrow is written beside the 4.]

Overall, there is a relationship/difference between addition and subtraction that is important – the second number acts differently for subtraction than it does for addition, and what keeps the answer the same is different in addition than what it is in subtraction.

Important principles

The principles are as follows for changing the size of numbers (* shows differences between addition and subtraction):

CHANGE	ADDITION	SUBTRACTION
First number increases/decreases	Answer increases/decreases	Answer increases/decreases
Second number increases/decreases	Answer increases/decreases	Answer decreases/increases *
First increases/decreases, answer equals	Second decreases/increases	Second increases/decreases *
Second increases/decreases, answer equals	First decreases/increases	First increases/decreases *

Two overall principles are:

1. Addition and subtraction **act differently** for change in second number (subtraction change is called **inverse relation principle**).
2. Addition and subtraction **act differently** to keep answer the same (called **compensation principle**).

Are there more principles? [Hint – What if the answer increases? What could happen to the other numbers?]

Teaching the principles

The main teaching approach is to keep following the RAMR cycle – reality → abstraction → mathematics → reflection.

First, try to think of reality/kinaesthetic activities from the world of the students – a good one is a relay race – if two students each run 5 km, then total is 10 km; if the first runner runs 6 km and the second runs 5 km as normal, then the total is 11 km; if the total remains as 10 km and the first runner runs 6 km, the second only runs 4 km. Get the students outside in pairs with batons – set up walking short-course relay races – have the students change how far they walk.

Second, act out/model what is happening with counters and on number lines. **Third**, record all findings. **Fourth**, reflect back into the world and look for extensions.

Another powerful way to teach the principles is to look at addition and subtraction differences as an investigation – let students simply investigate addition and subtraction for properties and record all they find. A worksheet like the one at the beginning of this subsection with activities (a) to (l) may also be a way to do this.

2.2.2 Relationship or Field principles

The relationship principles or the Field principles are properties of numbers and operations. Although we are looking at addition and subtraction, only addition is a real operation in mathematics because subtraction does not obey all these principles.

Nature of principles

The relationship principles are as follows. (***) denotes principles for which subtraction does not hold.)

1. **Closure.** Numbers and an addition operation always give another number (e.g. $2.17 + 4.34 = 6.51$; for any numbers a and b , $a + b = c$ which is another number).
2. **Identity.** There exists a number (0) that does not change the number when added (e.g. $5 + 0 = 0 + 5 = 5$; for any number a , $a + 0 = 0 + a = a$).
3. **Inverse.** For any number, there exists a number which when added gives the identity (e.g. the inverse of 7 is -7 because $7 + (-7) = (-7) + 7 = 0$; for any number a , there exists a $-a$ so that $a + (-a) = (-a) + a = 0$).
4. **Associative.** Numbers can be associated/added in any way and give the same answer (e.g. $5 + 3 + 6 = 8 + 6 = 14$ and $5 + 3 + 6 = 5 + 9 = 14$; for any numbers a , b and c , $(a + b) + c = a + (b + c)$). ***)
5. **Commutative.** Numbers can change order without changing the answer (e.g. $7 + 4 = 4 + 7 = 11$; for any numbers a and b , $a + b = b + a$). ***)

Note: The full Field properties have multiplication as another operation with the same properties as above and a final principle involving both addition and multiplication called the distributive principle.

Teaching the principles

There are two ways of teaching these principles. The first is the same way as for the number-size principles: the RAMR cycle (reality → abstraction → mathematics → reflection) with attention to set and number-line models, and kinaesthetic activities. The second is to use patterns and calculators, for example:

1. *Do these with calculators:*

(a) $2.52 + 3.8 = \underline{\hspace{2cm}}$; $3.8 + 2.52 = \underline{\hspace{2cm}}$; (b) $34.9 + 7.86 = \underline{\hspace{2cm}}$; $7.86 + 34.9 = \underline{\hspace{2cm}}$; and so on.

2. *Do these without calculators:*

(a) $46.4 + 3.92 = 50.32$; $3.92 + 46.4 = \underline{\hspace{2cm}}$; (b) $8.56 + 7.84 = 16.4$; $7.84 + 8.56 = \underline{\hspace{2cm}}$; and so on.

The students are asked to see the pattern in the calculator activities that enables them to do the non-calculator activities without a calculator.

Examples of RAMR stages with the principles

Commutative principle

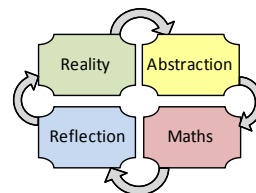
Abstraction. Show with objects that, for example, 3 joining 5 is the same as 5 joining 3.

Show with number line that, for example, a hop of 5 followed by a hop of 3 is the same as 3 followed by 5. Validate this with calculator for any two numbers, for example, $456 + 327 = 327 + 456$. Explore whether this is true for subtraction.

Mathematics. Do all these activities with materials and record with symbols.

Associative principle

Abstraction. Show with objects and number lines that, for any three numbers, say 3, 5 and 6, it does not matter whether 3 and 5 or 5 and 6 are joined or hopped first, $(3 + 5) + 6$ always equals $3 + (5 + 6)$. Validate with calculator for large numbers. Discuss whether this works for subtraction.



Mathematics. Do all these activities with materials and record with symbols as you go.

Identity principle

Reality. Discuss actions that leave things unchanged: do nothing, rotate 360 degrees, drive in a circle, and so on.

Abstraction. Show that $+0$ leaves numbers unchanged through sets and number lines. This is difficult for some students to understand – you may have to use patterns, e.g. $5 + 3 = 8$, $5 + 2 = 7$, $5 + 1 = 6$, $5 + 0 = ??$.

Mathematics. Do these patterns with materials and recordings.

Reflection. Discuss that the following equations are allowed and what they mean, and discuss how to relate equations to stories and vice versa: $2 + 3 = 5$, $5 = 2 + 3$, $8 - 5 = 3$, $3 = 8 - 5$, $4 + 2 = 7 - 1$ and $7 - 1 = 4 + 2$. Stress the need to have understandings that cover stories with two equal components as well as stories that reach an answer (e.g. “2 cats on a fence, 3 jump up to join them, this made the same number of cats that were on another roof where there had been 8 cats and 3 had left” as well as “2 cats on a fence, 3 jump up to join them, there are now 5 cats on the fence”). Write equations such as $3 + 4 = 7$ and $4 + 3 = 7$ where students write their own meaning for the rule, and so on.

2.2.3 Equals principles

The equals principles are the properties that always hold for equals – there are three of them:

1. **Reflexive.** Any number a is equal to itself, that is, $a = a$.
2. **Symmetric.** For any numbers a and b , if $a = b$ then $b = a$ (order of an equation can be reversed or “turned around”); this is important for equations as it means that $3 + 4 = 7$ and $7 = 3 + 4$; $3 + 4 = 8 - 1$ and $8 - 1 = 3 + 4$ are all allowed and are all correct and true.
3. **Transitive.** For any numbers a , b and c , $a = b$ and $b = c$ means $a = c$; this is also important for equations because it means we can say: $4 + 8 - 3 = 12 - 3 = 9$ and so $4 + 8 - 3 = 9$.

Once again, the way to teach students is to use the RAMR cycle and the set and number-line models. However, the best model for these principles is the mass model – to think of equals as two sides being balanced.

2.3 Basic fact strategies

Once the concepts and principles of the operations are introduced, it is time to teach ways to calculate the answers more quickly than representing the operation with counters and counting to get the answer. This skill is called knowing basic facts and, for addition and subtraction, is being able to make unthinking, immediate calculations with numbers less than 10 for addition and the inverse for subtraction, that is: $0 + 0$, $0 + 1$, $0 + 2$, ..., $0 + 9$; $1 + 0$, $1 + 1$, $1 + 2$, ..., $1 + 9$; $2 + 0$, $2 + 1$, $2 + 2$, ..., $2 + 9$; ...; $9 + 0$, $9 + 1$, $9 + 2$, ..., $9 + 9$; and $0 - 0$, $1 - 0$, $2 - 0$, ..., $9 - 0$; $1 - 1$, $2 - 1$, $3 - 1$, ..., $10 - 1$; $2 - 2$, $3 - 2$, $4 - 2$, ..., $11 - 2$; ...; $9 - 9$, $10 - 9$, $11 - 9$, ..., $18 - 9$; plus the multiples of 10 or extended facts such $200 + 300$, $6000 + 7000$ and $1100 - 600$.

The aim for the teaching of basic facts is that they be “known off by heart” as automaticity with facts is essential to free up working memory space for more complex mathematical thinking and multi-step problem solving. However, it is not recommended to use rote learning and drill and practice to achieve understanding about the operation concepts and the calculation of basic facts. It is recommended to build basic facts through the use of strategies because: (a) strategies can help develop students’ understanding of the operation concepts as well as the basic facts themselves; and (b) automaticity in basic facts is best achieved by speeding up strategy steps rather than making direct connections. After introduction through strategies, repeated practice can lead to automaticity. Therefore, this subsection looks at basic fact strategies for addition, subtraction and multiples of 10, and briefly at diagnosing and practising these facts.

2.3.1 Addition facts

The major addition fact strategies are to: use counting, use doubles, and use tens.

Use counting

Counting on is a strategy used for addition facts where one number is 0, 1, 2 or 3, e.g. $6 + 2$. The idea is to change from counting both numbers all together (called “sum”) to where only the 0, 1, 2 or 3 are counted and the other number is the start. Counting all is inefficient and students need to develop an ability to count on rather than count all. Counting on more than 3 is also inefficient and there will be other strategies which will work better for basic facts beyond $+0$, $+1$, $+2$ and $+3$. Students need to be able to subitise (recognise a quantity by sight without counting) and trust that this number does not change to be able to count on. A student that cannot recognise 5 or a collection of 5 objects as “five” or does not trust that this is “five” will feel the need to count to check. This is inefficient. A student who trusts 5 as “five” will be able to count on to work out $5 + 2$ by using the counting on strategy from 5, for example “five; six, seven”.

To develop this strategy, use a collection of objects. Make the larger number with materials (e.g. for $6 + 2$ make a collection of 6). Ensure the student trusts that this is six. Cover the six objects with a hand or a container, recall its number and then count on the 0, 1, 2 or 3. For example: *Put 6 counters into your left hand. Put 2 counters in your right hand. Say “six” showing the left hand and then drop in the counters one at a time from the right to the left hand, saying “seven, eight”.* A tin labelled with a symbol into which marbles can be dropped for counting on is also good for this.

This strategy is also used when subtracting 0, 1, 2 or 3 (counting back). For example, $7 - 2$ is “seven, six, five”. This can be taught by dropping counters out of a hand or a container: *Put 5 counters in your left hand, show hand and say “five”, drop out 3 counters one at a time into the right hand saying “four, three, two”.*

This strategy is based on the *associative* principle.

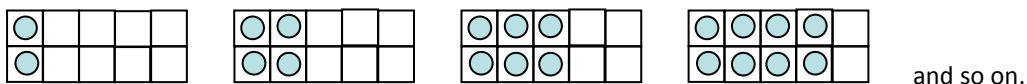
Use doubles

Doubles are addition facts where two of the same number are added, e.g. $4 + 4$. Using mental images of real situations that involve doubles can be helpful. For example:

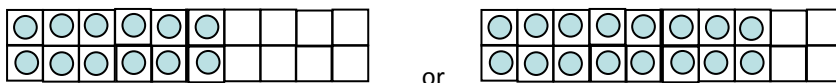
NUMBER TO DOUBLE	MENTAL IMAGE
1	2 feet or 2 hands on a person
2	4 tyres on a car
3	6 stumps in wickets for cricket
4	8 legs of a spider
5	10 fingers on our hands
6	12 eggs in an egg carton
7	14 days in a fortnight (use a calendar to reinforce)
8	16 legs in two octopi (or two spiders)
9	the 18 dots in two Channel 9 symbols

If students have their own images or ideas of mental images for doubles these may also be useful. Allow for creativity and flexibility through discussion and drawings.

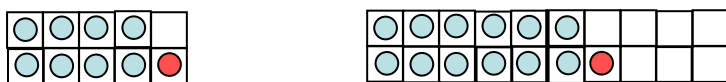
Doubles with totals to ten can be modelled using a ten frame where counters are added in 1:1 correspondence.



Doubles with totals to 20 can be modelled using two ten frames end to end.



The Use Doubles strategy also applies to facts that are 1 or 2 from doubles (e.g. $4 + 5$ is “double four, eight; plus one, nine”, and $6 + 7$ is “double 6, twelve; plus one, thirteen”).

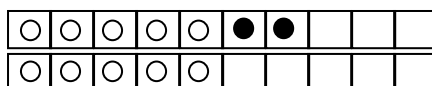


$4 + 5$ as double $4 + 1$

$6 + 7$ as double $6 + 1$

The students can see a double as the two rows of counters and the extra (shown above in red).

Doubles +2 can be modelled similarly. The example below shows $7 + 5$ as double 5 + 2. The students can see the double five and the two extras and count or visualise that this is 12, e.g. “ten; eleven, twelve”



so $7 + 5 = (5 + 5) + 2$ i.e. $7 + 5$ is double $5 = 10$; 11, 12

It should be noted that counting on is not the only way to use this strategy. Some students *count back* (e.g. $6 + 8$ is “double eight, sixteen; back two, fifteen, fourteen”), while some students *level pairs*, for the “two” case (e.g. $6 + 8$ is “seven plus seven by adding one to six and taking one from eight, is double seven, fourteen”). This strategy is based on the *associative* principle.

The related subtraction doubles facts relate to halving. Once students are familiar with the addition doubles they will be able to relate to the halves of known doubles. For example, once a student knows that double 7 is 14 they will also be able to work out that half of 14 is 7 or that $14 - 7 = 7$. This strategy is also described in subsection 2.3.2 (Subtraction facts – Think addition) where students work out subtraction basic facts by using the related addition facts.

Use ten

The Use Ten strategy focuses on all the facts that total 10, e.g. $5 + 5$, $7 + 3$, as well as the ones that are close to these, e.g. $5 + 6$, $7 + 4$. As our number system is based on 10 this set of facts is very important and will become valuable when calculating with larger numbers using computation strategies involving separation, sequencing or compensation (see subsections 2.4.1, 2.4.2 and 2.4.3).

The first thing to be taught is the difference between each number 1 to 9 and the number 10. This can be done on the fingers: *Show 10 fingers on your two hands. Drop your first seven fingers. How many left? How many to the 10? Repeat for four, six and eight fingers.* It can also be done using ten frames. Students place a number of counters in the ten frame and then fill the frame with another colour. The students know the frame holds 10 so they see the basic facts that make 10 clearly.



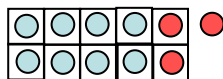
$8 + 2 = 10$

$6 + 4 = 10$

$5 + 5 = 10$

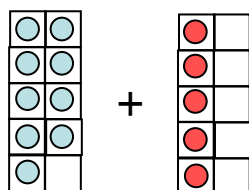
$9 + 1 = 10$

Students can also use 10 to work out facts that are 1 or 2 from the 10 facts, e.g. $8 + 3$ is $8 + 2$ and 1 more:

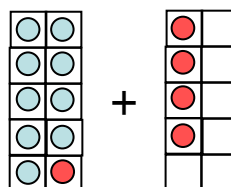


$$8 + 3 = 11$$

This understanding can be used in “build to ten” mode for basic facts involving numbers close to 10, e.g. 8 or 9. For example, the basic fact $9 + 5$ can be thought of as “nine plus one to make ten plus another four, fourteen”. This can be modelled using ten frames as well. In this thinking, the 5 has been partitioned into 1 and 4 for convenience as it makes a 10.



$$9 + 5$$



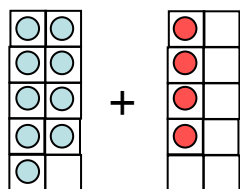
$$9 + 1$$

$$+ 4$$

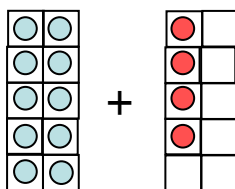
$$= 10 + 4 = 14$$

This understanding can also be used to model the compensation situation where students can understand that $9 + 5 = 10 + 4$. The partition strategy and the compensation strategy are valuable for computation beyond basic facts. Understanding of these strategies can begin with basic facts if these are taught using a strategy approach.

Another “near ten” strategy utilises student understandings of 10. This strategy is called “add ten”. Students identify basic facts that involve a number near 10 and they choose to add 10 instead and then compensate for the extra by subtracting. If students have been introduced to teen numbers using two ten frames, e.g. 14 is a full ten frame and 4 more (1 ten and 4 ones), they can use this number-sense understanding to solve basic facts involving 9 or 8. So $9 + 4$ can be thought of as $10 + 4$ and then subtracting 1.



is thought of as



$10 + 4$ is 14 then subtract 1 = 13

This strategy can be modelled on a number board (100 or 99 board) where $+9$ is seen as adding ten (one row down) and then going back 1. This strategy is based on the *identity* and *inverse* principles.

The Use Ten strategy also relates to the subtraction basic facts where a number is subtracted from 10. Students can use knowledge of the tens addition basic facts to work out the related subtraction facts. For example, if students know that $6 + 4 = 10$ they will also be able to work out that $10 - 6 = 4$. This can be modelled using ten frames by taking a full ten frame and removing or covering a quantity of counters showing what remains. Using the Use Ten strategy for subtraction also relates to the Think Addition strategy described in subsection 2.3.2.

Use a rule

The *identity* principle which is about leaving things unchanged applies to addition in that the addition of zero will leave any number unchanged. This is easy to conceptualise. When zero is added there are no more and therefore the number stays the same. The Use Counting basic fact strategy covered the addition of zero as count on zero. It is worth considering this special basic fact as using a rule as well.

2.3.2 Subtraction facts – Think addition

The Think Addition strategy is used for subtraction facts. The idea is not to do subtraction but to think of the facts in addition terms. When students want to work out a subtraction basic fact they can think of the subtraction as

difference and relate the subtraction fact to an addition fact and strategy. For example, $8 - 3$ can be thought of as “what is added to 3 to make 8”. To use this strategy, students need to understand that subtraction and addition are inverses of each other. This is another situation where ten frames can assist student understanding. By using the frame the empty cells left after counters are removed are still visible. For example, using a ten frame to show that $6 + 4 = 10$ (and that $4 + 6 = 10$) can also show that $10 - 6 = 4$ and that $10 - 4 = 6$. Students can see that when they have 4 counters they will need 6 more to make 10.

This situation can also be modelled using numbers other than 10. For example, take 7 counters and 4 counters. Combine them to make 11. Separate them back to 7 and 4. Repeat this for 3 and 6 counters and 5 and 8 counters. The notion of adding on to get a subtraction can also be directly modelled: Put out 11 counters. Below them put 7 counters. Add counters to the bottom group until both groups are the same. Repeat for 8 and 13.

This strategy is to reinforce “think addition” and to relate $+$ and $-$. It is based on the *inverse* principle. For each addition/subtraction fact, there are four members of the fact family, e.g. $3 + 5 = 8$, $5 + 3 = 8$, $8 - 5 = 3$, and $8 - 3 = 5$. Families for $4 + 7$ and $15 - 9$ are:

$$4 + 7 = 11, 7 + 4 = 11, 11 - 4 = 7, 11 - 7 = 4$$

$$9 + 6 = 15, 6 + 9 = 15, 15 - 9 = 6, 15 - 6 = 9$$

2.3.3 Addition and subtraction extended facts

Extended facts are where the strategies used to develop and compute basic facts are used for other computations beyond the basic facts. Generally, extended facts relate to *multiples of ten* so the basic fact strategy is used for the tens or hundreds (and so on) as if they were ones. For example, the basic fact of $6 + 4 = 10$ can be used to work out $60 + 40 = 100$. The basic fact strategies can also be used for *extensions* which are other numbers where the number of tens/hundreds can alter and the basic fact strategy can be applied. For example, the Use Doubles strategy used for $4 + 4 = 8$ can also be used for $54 + 4$ or $540 + 40$.

Multiple-of-ten facts

Students can be shown how strategies for basic facts (single digit numbers) also work for the related multiples of ten. Students can think tens for computations that use multiples of ten. For example, the count-on basic fact of $5 + 2$ also works for $50 + 20$. Students can think of this as 5 tens + 2 tens and the basic fact strategy will work. This also works for Use Doubles facts, for example, $6 + 6 = 12$ so $600 + 600 = 1200$ (double six hundred).

Complete activity sheets with related computations like below and discuss the strategies used, highlighting the transferability of known strategies to the larger numbers:

$4 + 2 = \underline{\hspace{2cm}}$	$40 + 20 = \underline{\hspace{2cm}}$	$400 + 200 = \underline{\hspace{2cm}}$
$7 + 6 = \underline{\hspace{2cm}}$	$70 + 60 = \underline{\hspace{2cm}}$	$7000 + 6000 = \underline{\hspace{2cm}}$
$11 - 8 = \underline{\hspace{2cm}}$	$110 - 80 = \underline{\hspace{2cm}}$	$1100 - 800 = \underline{\hspace{2cm}}$

Discuss patterns that would enable multiple-of-ten facts to be determined from basic facts. (*Note:* this can also be completed using the “Do these with a calculator – Do these without” approach.)

Extensions

Extensions are computations beyond basic facts, other than straight multiples of ten, where basic fact strategies can be applied. The basic fact will be evident in the digits of the computation and the other digits can be considered as multiples of ten. For example, the basic fact $5 + 3 = 8$ which could be solved using a count-on or a near-double strategy can be used to assist in completing computations like $55 + 3$ or $550 + 30$. Students can practise related computations and discuss the patterns they find in the answers and in the strategies that they use.

$$\begin{array}{lll}
 5 + 6 = \underline{\hspace{2cm}} & 45 + 6 = \underline{\hspace{2cm}} & 185 + 6 = \underline{\hspace{2cm}} \\
 9 + 9 = \underline{\hspace{2cm}} & 89 + 9 = \underline{\hspace{2cm}} & 490 + 90 = \underline{\hspace{2cm}} \\
 15 - 9 = \underline{\hspace{2cm}} & 175 - 9 = \underline{\hspace{2cm}} & 150 - 90 = \underline{\hspace{2cm}}
 \end{array}$$

Discuss patterns that would enable extension facts to be determined from basic facts. (*Note:* this can also be completed using the “Do these with a calculator – Do these without” approach.)

2.3.4 Automaticity

The goal of learning basic facts is for students to develop automaticity. Students who have automated instant recall of basic facts will have these facts available in task and problem situations without needing to waste their thinking on working them out. This way automaticity does not take away any thinking from the task – automated facts have no cognitive load. We believe that developing basic fact automaticity should be the focus after learning of the strategies and investigation of the patterns and connections within the basic facts has occurred. A student who has learnt the various multiplication and division basic fact strategies will have a fall-back method to work out facts they can’t remember where a student who has only had learning focusing on rote memorisation will not have the same support. The added benefit to learning the strategies is their applicability to computations beyond basic facts.

Diagnosis

If teaching upper primary students, the first step with basic facts is to diagnose what facts are not known and to set up regular speed practice for the unknown facts. To do this, give students a list of random basic facts to complete, keeping all students together on the facts by reading each fact with a short time to write the answer. Mark the results and place on an addition and subtraction table as below.

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

-	18	17	16	15	14	13	12	11	10	9	8	etc
0												
1												
2												
3												
4												
5												
6												
7												
8												
9												

	Counting on		Use ten
	Doubles		Think addition
	Near doubles		Use a rule

These grids can be used to determine both the facts with which students make errors and the strategies needed to help students with their errors. For instance, the counting on, near doubles and near ten facts can be placed on an addition grid using different colours. Then, if a student’s errors are placed on the grid, the position of the errors will determine which strategy or strategies are needed. However, many addition and subtraction basic facts can be completed using more than one of the strategies. For example $2 + 2$ is a double but it could be answered using a count-on 2 strategy. Students can choose which strategy they prefer for these. (*Note:* Generally counting strategies have less transferability to other computations than other strategies.)

Practice

Once students have been taught the various basic fact strategies, practising facts that use these strategies will assist in strengthening their understanding and application of the strategies as well as boosting their automaticity and recall of the basic facts. Activities that focus on speed can be effective for students with knowledge of the strategies. The following appear to work:

- To practise a particular strategy, provide multiple examples of facts that use this strategy.
- To practise speed and overall automaticity, provide a mix of basic facts so different strategies are used.
- Use student tracking worksheets to aid students with the process and enable them to gauge their own progress – and to make students complicit in the process. Get each student to mark facts and graph the correct number of answers each day to compare with previous days. Get students to also record any errors they make.
- Set up a regular daily practice program – 10 minutes per day (e.g. 4 minute mile, flash cards, bingo) for random speed practice and a later time where students can practise their errors recorded at the earlier practice.

2.4 Strategies for computation

Computation is the calculation of answers to the application of operations and the strategies used to complete these calculations. Calculation has always been a significant component of school mathematics programs. The National Research Council back in 1989 noted that “the teaching of mathematics is shifting from a preoccupation with inculcating routine skills to developing broad-based mathematical power” (p. 82). A key element of this mathematical power is the ability to calculate exact answers efficiently and with understanding. Australian curriculum documents in recent years have shown a shift in emphasis from written algorithms for computation to mental computation, the use of computation strategies and electronic tools like calculators and computers (Australian Curriculum, Assessment and Reporting Authority, 2011; Australian Education Council, 1991; Queensland Studies Authority, 2004).

Threlfall (2000) described mental computation strategies as requiring students to “construct a sequence of transformations of a number problem to arrive at a solution as opposed to just knowing, simply counting or making a mental representation of a paper and pencil method” (p. 30). Three strategies are focused on in this book. Each of these strategies applies to the four operations. We recommend that students learn and be exposed to a range of computation strategies and methods (mental, written and calculator). We advocate that mental computation strategies be used for numbers up to three digits and operations on larger numbers be carried out using calculators.

Three computation strategies are described in this section. Each of these strategies provides a means for students to transform calculations to make them more manageable. Each of the strategies applies to each of the operations. The three strategies are: **separation**, **sequencing** and **compensation**. Each subsection describes these strategies and provides sample activities for using the strategy for addition and subtraction.

Important note: We advocate always doing calculation in **problem situations** – do not give students worksheets with exercises in numeral form – **always give worksheets with written problems** where the student has to both create the setting out and undertake the calculation.

2.4.1 Separation strategy

The separation strategy works on transforming the numbers in a calculation by partitioning them or breaking them into parts. The parts are then worked on and the result is found by recombining the parts. There are three ways that numbers in a calculation can be separated: (a) into place-value parts, (b) into compatible parts, or (c) into a combination of place-value and compatible parts.

In the separation strategy both (or all) of the numbers in the calculation are separated. Students need to understand the structure of our number system and how place values are multiplicatively related so that when numbers are separated they are considered in terms of their value, not as digits. The separation strategy is the basis of the traditional written algorithm. The traditional algorithm has a tendency for students to work with digits in columns rather than numbers that have value. Because of this it is valuable to teach and practise separation strategies using Place Value Charts (PVCs) and size materials such as bundling sticks, MAB and money placed on top of these PVCs. MAB and money can be used similarly to the bundling sticks. It is crucial that students experience addition with real bundling sticks, MAB and money before moving to do the virtual bundling stick, MAB and money activities. The separation strategy is used in a variety of contexts – it is useful for whole numbers (e.g. 346×8), decimal numbers (e.g. $4.65 + 0.8$), measures (e.g. $3 \text{ m } 342 \text{ mm} \times 5$), mixed numbers (e.g. $3 \frac{1}{6} \times 4$), and algebra (e.g. $2a \times (3a + 2b)$).

Separation strategy for addition

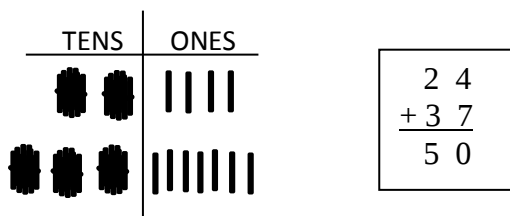
The separation strategy sees both (or all) numbers in a computation separated into their place-value parts. An example of this strategy for addition is $32 + 32$ as $30 + 30 + 2 + 2$ (see figure on right). The values of the tens in each number are added followed by the values of the ones. This strategy can be completed where the ones are combined first, as is the focus in the traditional written algorithm, but there is a risk that students will consider the tens as $3 + 3$ rather than $30 + 30$.

$$\begin{array}{l}
 32 + 32 \\
 30 + 30 = 60 \\
 2 + 2 = 4 \\
 64
 \end{array}$$

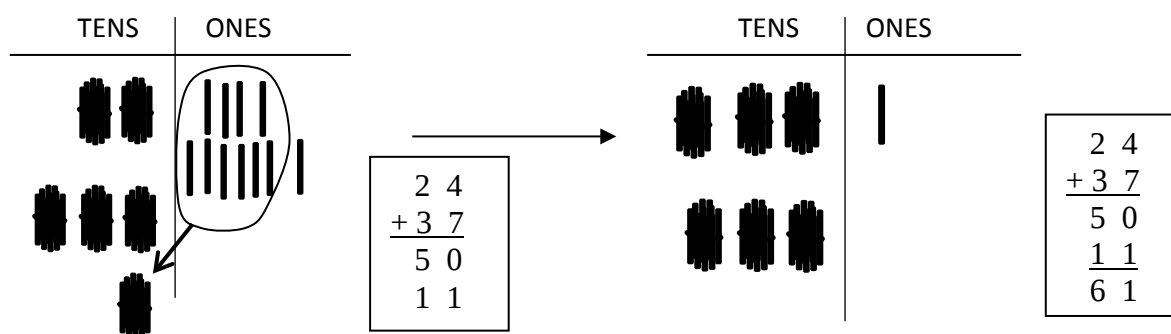
Using set materials

An example of the separation strategy for addition using bundling sticks and a PVC shows how the concept of regrouping to make a ten is needed in this version of this strategy. By using the bundling sticks the tens are clearly visible and not likely to be treated as ones. The number of tens (5) is recorded as 50 in the first step. This could be done horizontally just as easily as vertically.

Put out 24 and 37 in tens and ones.



Combine the tens and ones separately. This example combines the tens first, then combines the ones.



Trade 10 ones for 1 ten and move to tens place-value position. Record while manipulating the materials and write the answer at the end (see symbolic representations to the right of each diagram).

This example could also be recorded as $24 + 37 = 50 + 11 = 61$. Students who understand place value and the concept of addition can manage the need to regroup as an addition.

Separation strategy for subtraction

The separation strategy for subtraction is more complex and research has shown that many students do not manage the separation of both numbers in a subtraction as well as they do for addition (Klein, Beishuizen, & Treffers, 1998). This is the case particularly when there needs to be regrouping. The students separate both numbers into their place-value parts and while the subtraction of the tens is straightforward, subtraction of the ones causes confusion. The figure below shows student work samples where this error is evident.

$51 - 25$ 24	$50 - 20 = 30$ $30 - 5 = 25$ $1 = 24$	$32 - 18$	$30 - 10 = 20$ $20 - 8 = 12$ $12 - 2 = 10$
-------------------	---	-----------	--

An alternative way of completing subtraction using a separation strategy is to partition the numbers in a different way that allows the subtraction of the ones to be completed. This is particularly useful for computations that require bridging (regrouping). For example, for the $51 - 25$ example on right, separating the 51 into non-place-value parts can make the computation more manageable, i.e. 51 as 40 and 11 allows the 20 and 5 to be subtracted more easily. This separating requires knowledge of place value and the relationship between place values. Completing computations using this strategy could enhance student understanding of place value.

51 - 25

Separate 51 into 40 + 11
 Separate 25 into 20 + 5

$$\begin{array}{l}
 40 - 20 = 20 \\
 11 - 5 = 6
 \end{array}
 \begin{array}{l}
 \nearrow \\
 20 + 6 = 26 \\
 \text{(sum of parts)}
 \end{array}$$

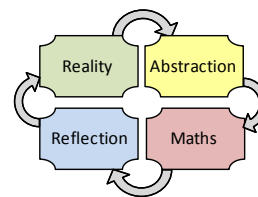
Effectively this replicates the traditional algorithm but this method requires the student to use number-sense understandings about 51 to choose how to separate it rather than a procedural approach. There are also many other ways the 51 could be separated; for example, $26 + 25$ (using knowledge that double 25 is 50) makes it very easy to subtract the 25 leaving the 26. Students with well-developed number-sense understandings can separate numbers in a wide variety of ways depending on the numbers in the computation.

The traditional written algorithm for subtraction uses regrouping of tens to manage the subtraction of the ones. The following RAMR activity shows this strategy using a set material – bundling sticks.

Teaching the separation strategy for subtraction with set materials

Reality

Find real situations within students' life experiences that are relevant. For example, You pay a \$37 bill with \$50; how much change should you get?



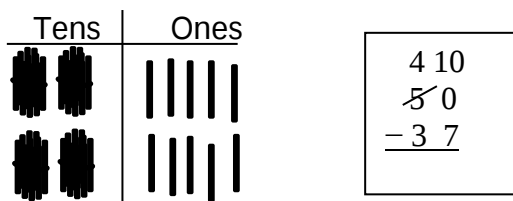
Abstraction

Put out the 50 (not the 37 as well) – 5 tens and 0 ones. Students need to realise they have to remove 3 tens and 7 ones.

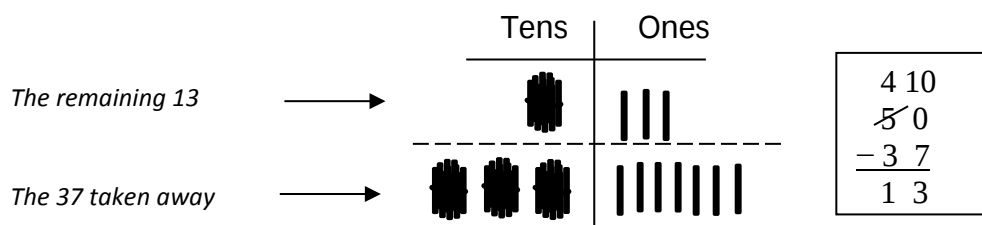
Tens	Ones

$ \begin{array}{r} 50 \\ - 37 \\ \hline \end{array} $

Check if there are sufficient quantities of tens and ones. Exchange or trade to have sufficient ones to subtract 7 ones.



Remove the 3 tens and 7 ones – slide them down the chart. What remains at the top is the answer (here 13) – or 1 ten and 3 ones. Check by joining – should get back to 50.



Altogether still make 50. Record the information along the way.

Mathematics

Students imagine materials in their mind and then complete algorithm without material – either pen and paper or mentally.

2.4.2 Sequencing strategy

The sequencing strategy is different from the separation strategy as follows: separation breaks both components into parts (usually on place value), while the sequencing strategy breaks one number up (can be place value but need not be) and keeps the other number whole. The parts are done with the whole number in sequence. In these examples, we will just show the steps through abstraction to mathematics. So the sequencing strategy works on the partitioning of numbers like the separation strategy. The operation being applied to the parts will vary depending on the operation of the overall computation. The sequencing strategy is often more efficient than the separation strategy as there are fewer parts when only one number is partitioned and so the entire computation will involve fewer steps.

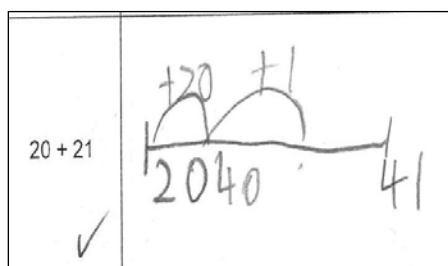
As with separation strategies, there are three ways that the number in a calculation being partitioned can be separated:

- (a) into place-value parts;
- (b) into compatible parts; or
- (c) into a combination of place-value and compatible parts.

The sequencing strategy works for each operation. This strategy is best taught from the rank understanding of number, that is, by using number boards and number lines. When the number line is used the operations are based on adding being to the right and subtracting being to the left (towards the zero).

Sequencing strategy for addition

Addition is commutative and therefore it makes little difference which number in an addition computation is partitioned and which number is left. The student work samples in the figure below show different ways students have partitioned one number to make an addition computation more manageable. In the first example the student has partitioned one number into place-value parts, in the second the student has partitioned into compatible parts so as to make a ten. This second example also shows the student's understanding of the number-sense concept of making ten and the Use Ten basic addition fact strategy.



$$9 + 6 = 15$$

9 + 1 + 5
breaking up

Using a number board

Partitioning one number into place-value parts and progressively adding the parts (sequencing) can be well represented on a number board (99 or 100 or other board that is 10 columns wide). The following example shows how this can be used to add $33 + 22$: 33 is the start number and the 22 is partitioned into place-value parts – 20 (2 tens) and 2 (2 ones). The tens jumps are completed first in this example. It is also possible to complete the ones jumps first.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The jumps down the number board are in multiples of ten so in this example there are 2 tens hence the two jumps down (+20). The jumps to the right add 1 one per jump. There are two jumps to the right (+2).

Computations involving bridging (regrouping) on a number board require jumps off the right-hand end of the board which can prove difficult to manage. Using a different number board that still has ten columns but starts on a different number can alleviate these problems. The following example shows how this is done for $89 + 22$.

34	35	36	37	38	39	40	41	42	43
44	45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72	73
74	75	76	77	78	79	80	81	82	83
84	85	86	87	88	89	90	91	92	93
94	95	96	97	98	99	100	101	102	103
104	105	106	107	108	109	110	111	112	113
114	115	116	117	118	119	120	121	122	123
124	125	126	127	128	129	130	131	132	133

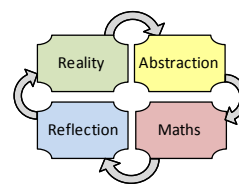
Using a number line

Another representation of this strategy is the number line. The initial number which is being left untouched is the start number and the other number is partitioned into parts – either place-value parts or compatible parts

and these parts are added sequentially as jumps along the number line. The number at the end is the answer. The following RAMR activity shows the sequencing strategy used with a number line.

Teaching the sequencing strategy for addition with number line

Sequencing involves one number being left as is and the other number being separated, so that parts of it are added in sequence. For example $27 + 48$, the 48 is separated into 40, 3 and 5 and these numbers are added in sequence $27 + 40 = 67$, $67 + 3 = 70$, and $70 + 5 = 75$.

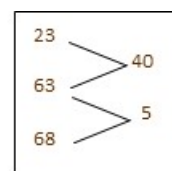
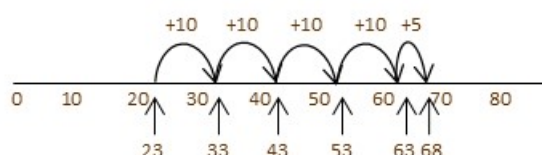


Reality

Look for real-world instances of addition and subtraction that have a length orientation so that the number line is appropriate as a model.

Abstraction

Act out the number line by walking the students – start at the first number and then move forward the second number – large steps for tens and small steps for ones. For the example $23 + 45 = 68$, use a number line from 1 to 100 with 10s marked in. Start on 23 and move 45 to the right, 4 tens forward then 5 ones forward to get to 68 – so $23 + 45$ is 23, 33, 43, 53, 63 (large steps), 64, 65, 66, 67, 68 (small steps).



Mathematics

A recording procedure as on the right in the above diagram can be used to imitate what happens on the line. Students should be working towards not needing to rely on a physical number line.

The sequencing strategy can be recorded as a set of steps. The following example (on right) shows the addition of two three-digit numbers using the sequencing strategy. The 455 has been left alone and the 278 has been partitioned into a combination of place-value and compatible parts.

455 + 278

Start with 455. Separate 278 and add progressively.

$$\begin{array}{r} 455 \\ + 200 = 655 \\ + 50 = 705 \\ + 20 = 725 \\ + 5 = 730 \\ + 3 = 733 \\ + 278 \end{array}$$

Sequencing strategy for subtraction

Subtraction is not commutative so the order of the numbers in a subtraction operation cannot be reversed. With the sequencing strategy for subtraction it is the subtrahend (the number being subtracted) which should be partitioned. This builds on the operation concept of subtraction as taking away and that there is a total and a part being subtracted to work out the other part.

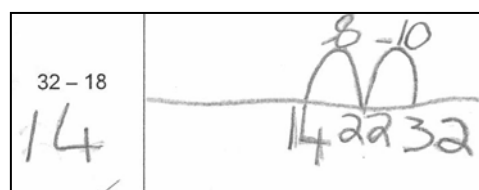
Using a number board

The use of number boards can facilitate representation of the sequencing strategy for subtraction as it does for addition. The partitions will be place-value based and the jumps around the board are reversed (a jump up subtracts 10, a jump to the left subtracts 1). The example below shows $44 - 21$ which is modelled using two jumps up (2×10 or 20) and one jump left (-1)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Using a number line

The number line also works for subtraction with the jumps represented from right to left rather than left to right. The figure on right shows a student work sample using a number line for subtraction using a sequencing strategy.

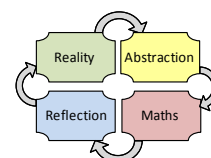


The sequencing strategy can be used to complete subtraction using an additive process. This requires students to understand subtraction as difference and that a move from the second number to the first number will enable us to find the difference between the two numbers. This makes contextual sense in relation to money to work out how much more is needed for a purchase or for counting back change. For example, $76 - 28$ can be thought of as how far is it from 28 to 76 or what do I need to add to 28 to get 76?

The following RAMR activity shows how a number line can be used to keep track of the jumps and the progressive total. This example is working out $620 - 332$. This strategy is particularly effective as it eliminates the need to deal with regrouping in computations that require bridging. The answer is found by adding the jumps. This represents the difference between the two numbers.

Teaching the sequencing strategy for additive subtraction

This is where we think of subtraction in terms of addition and move from the second number to the first number. For example, $76 - 28$ is thought of as how far from 28 to 76.

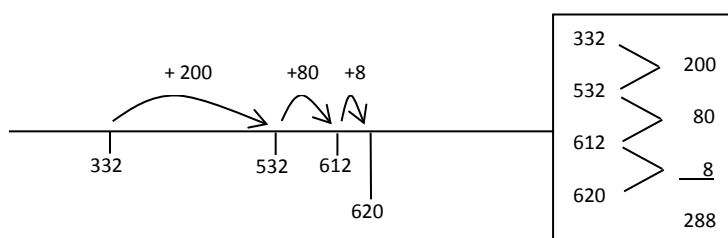


Reality

Again look for real-world instances, e.g. giving back change, and walk an additive subtraction from small to large number along a number line.

Abstraction

Use the number line as shown below. The idea here is to follow a path from small to large number and also to underestimate how much to increment in each leap (as you can simply do another leap). The answer is found by adding up the leaps. For example, $620 - 332$ is found by starting at the 332 and working towards the 600.



Use the recording procedure on the right in the diagram above.

2.4.3 Compensation strategy

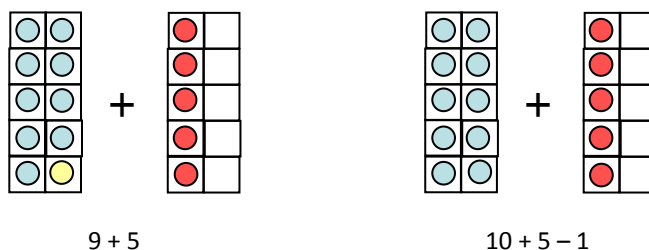
The compensation strategy does not partition the numbers involved in a computation but adjusts or changes the computation to use numbers that are more manageable. Sometimes there is a compensation for the change as an extra step after the change and sometimes the compensation is managed as part of the change. The focus is on making the computation more manageable. Often the changes that are made involve multiples of ten as it is easier to calculate with these numbers than others. The compensation strategy works for all four operations. Many of the compensation strategies use number-sense and basic fact understandings that students develop in early years of schooling. For example, knowing the numbers that add to make ten and the extensions of this Use Ten addition basic fact strategy are very useful with this strategy for larger numbers.

Compensation strategy for addition

It is easier to add multiples of ten than other numbers and numbers that add to form ten or multiples of ten can also be helpful when simplifying a computation. There are many different materials as well as good number sense that can assist to model this strategy.

Using ten frames

In the early years the compensation concept can be modelled using ten frames. With experience students soon learn that the frame has ten spaces and when full represents ten. By using these to model teen numbers (as was described in section 2.3, Basic fact strategies) students see that a full ten frame and a part full one is both a teen number and an addition. For example, the basic fact $9 + 5$ can be thought of as $10 + 5$ which is easier to work out ($=15$) but this is one too many. The answer to $9 + 5$ must be one less ($15 - 1 = 14$). The students can identify the counter that needs to be removed when it is modelled with this material. This shows an adjustment followed by another action for the compensation.



A further example of this compensation strategy using decimal numbers is provided on the right.

Students can also be introduced to the idea of a double adjustment which changes the computation in such a way that there is no need for compensation. This works on the balance concept. The computation is modelled in the ten frames using the numbers given (e.g. $9 + 5$). One of the counters making the 5 is moved to the other ten frame to complete the ten. The total has not changed but the computation now is easier to work out because one of the numbers is now 10.

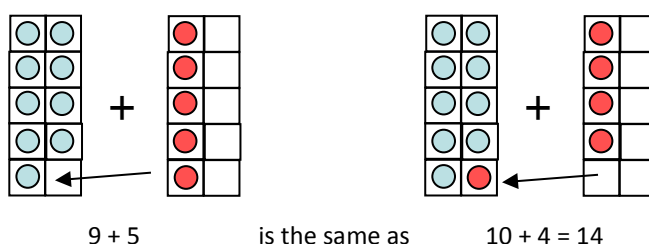
35.8 + 34.3

Change 35.8 to 36 (by adding 0.2)

$36 + 34.3 = 70.3$

Compensate by -0.2

$70.3 - 0.2 = 70.1$



This strategy can be extended to larger numbers and decimals and can be generalised to state that you can change any addition computation by doing the opposite to each number. In the example above one number was changed by +1 and the other was changed by -1. The example on right shows this strategy being used with larger numbers. This example also shows how there can be more than one change made. It will only take a few changes to make the computation very easy.

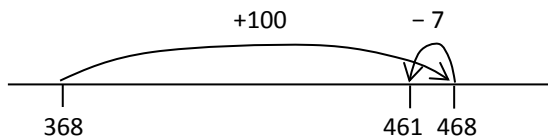
1878 + 674

Change to $1880 + 672$ (+2 and -2)
 Change to $1900 + 652$ (+20 and -20)
 Change to $2000 + 552$ (+100 and -100)
 = **2552**

Using the number line

Addition can be modelled as jumps along a number line. One of the numbers in the computation is the start number and the other is the amount to be added which can be done in jumps of any size. In the sequencing strategy one of the numbers was partitioned and added in parts to the first number. The number line was used to record the jumps. With the compensation strategy the number line is used the same way but the number being added is changed to a nearby multiple of ten and an over jump is completed, because it is easier, and then the compensation is the jump back so the answer is accurate. It should be noted that if an approximate answer was all that was required then the compensation step can be left out.

Example $368 + 93$



368 + 93

Easier:
 $368 + 100 = 468$
 Compensation:
 $468 - 7 = \mathbf{461}$

Using a number board

A similar use can be made of the number board. A multiple of ten can be added easily and the structure of the number board assists with this and then another jump can be made to compensate.

Example: $56 + 19$:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

56 + 19

Instead of +19
 Easier to +20

Compensate by -1

$$56 + 20 = 76$$

$$76 - 1 = \mathbf{75}$$

Compensation strategy for subtraction

The compensation strategy works for subtraction for similar reasons as it does for addition. It is easier to subtract multiples of ten than to subtract other numbers. Students need to realise that it is easier to subtract multiples of ten from another number than it is to subtract from a multiple of ten. So when looking to use the compensation strategy for subtraction, it is the subtrahend (the number being subtracted) that should be adjusted. Then, as with addition, a compensation action can be done to ensure the answer to the computation is exact.

Using a number board

In the example below ($65 - 19$) two jumps back of ten are made (-20) and the compensation this time is to $+1$. With the number board the students can do a comparison by jumping back 1 ten and then 9 ones and they will end up on the same square. They can also conceptualise the compensation needed in terms of “by taking off 20 instead of 19 I took away one too many, so I need to put that one back on”.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

65 - 19

Instead of -19
Easier to -20

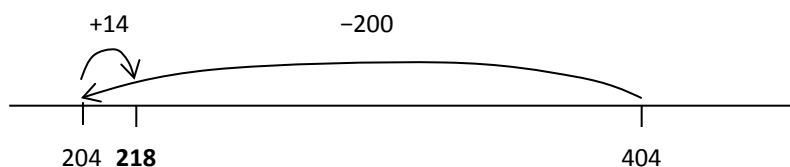
Compensate by $+1$

$$65 - 20 = 45$$

$$45 + 1 = 46$$

Using a number line

The use of over jumps as an adjustment to a subtraction computation followed by a compensation action is another way to model the compensation strategy on a number line. The example below shows $404 - 186$:



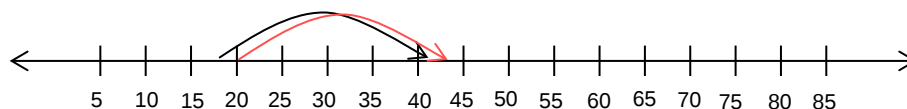
404 - 186

Easy:
 $404 - 200 = 204$
Compensation:
 $204 + 14 = 218$

Using knowledge of adjusting for addition

The compensation strategy for addition was shown to enable a change to be made to both numbers and as long as the total remained the same the easier computation could be completed instead of the original. The same concept can be applied to subtraction computations but instead of the total needing to remain the same it is the difference that needs to remain the same. This can be shown using a number line.

The number line below shows the difference between 18 and 41 using an arrow (black). As a subtraction this would be represented as $41 - 18$. To make this computation easier it is best to adjust the subtrahend (the number being subtracted) to be a multiple of ten. If this was changed to 20 (by $+2$), the difference needs to be kept the same which means that the first number also needs to be changed by $+2$. The second (red) arrow on this number line shows the new computation and that the difference is still the same. The new computation is $43 - 20$. This compensation to maintain equivalence can be generalised to state that to change a subtraction computation you need to do the same to each number.



This strategy can then be applied to larger numbers. As with addition, the strategy can be applied several times until the computation is quite easy. Students need to remember to focus on changing the subtrahend to make the computation easier.

1878 - 674

Change to $1874 - 670$ (-4 and -4)
Change to $1904 - 700$ ($+30$ and $+30$)
 $= 1204$

2.5 Estimation strategies

Estimation is a method of finding approximate answers on large-number additions and subtractions. With a calculator for accurate answers where required, it is adequate for students' computation in terms of YDM. Estimation is very useful in being able to guide whether a calculated solution is realistic.

2.5.1 Strategies

There are four strategies for teaching estimation. They are described below. Like basic facts and algorithms, strategies are the most effective way to teach anything because, in most cases, the strategies, once learnt, are useful in other areas.

Front end

This strategy requires that only the highest place-value (PV) position(s) are considered in addition and subtraction to give a simple estimate that, sometimes, is not that accurate. However, it can be a good start. It is called front end because one way to teach it is to put a card over all the sum but the "front end" of the highest PV position(s). The strategy uses multiple-of-ten basic facts. Examples are:

$4\,678 + 3\,402$ is found by $4\,000 + 3\,000 = 7\,000$

$52\,901 - 34\,287$ is found by $50\,000 - 30\,000 = 20\,000$

Rounding

This strategy rounds the highest PV position(s) to the nearest of these PV positions and then adds and subtracts these numbers. It is a more accurate method than front end in most cases (but not all). The strategy uses multiple-of-ten basic facts. Examples are:

$4\,678 + 3\,402$ is found by rounding to $5\,000 + 3\,000 = 8\,000$

$52\,901 - 34\,287$ is found by rounding to $50\,000 - 30\,000 = 20\,000$

Straddling

This strategy determines, by rounding up and down, approximate answers that are less and more than the example. The strategy uses multiple-of-ten basic facts and the number-size principles (particularly the inverse relation principle). Examples are:

$4\,678 + 3\,402$ is straddled by rounding up, $5\,000 + 4\,000 = 9\,000$, and rounding down, $4\,000 + 3\,000 = 7\,000$. So the addition is between $9\,000$ and $7\,000$. Looking at the numbers, both are about halfway in between, so estimate is $8\,000$.

$52\,901 - 34\,287$ is straddled by rounding up the first number and down the second, $60\,000 - 30\,000 = 30\,000$, and down the first and up the second, $50\,000 - 40\,000 = 10\,000$ (this is because of the inverse relation for subtraction, i.e. bigger second number is a smaller answer). So the answer is between $30\,000$ and $10\,000$. Looking at the numbers, they are both nearer the $10\,000$ below, so estimate is $20\,000$.

Getting closer

This strategy follows on from the above and uses the number-size principles and multiple-of-ten basic facts to get a more accurate estimate. Examples are:

The front-end strategy's estimate for $4\,678 + 3\,402$ is $7\,000$ and is obviously too low as both numbers reduced, so a better estimate is about $1\,100$ more, or $8\,100$.

The rounding estimate for $52\,901 - 34\,287$ is 20 000 and is slightly high as the second number is rounded down an extra 1 500 approximately and so a more accurate estimate is 1 500 less, or 18 500.

2.5.2 Teaching and practising

For expertise in estimation, it is necessary to practise because accurate estimation combines basic facts, multiple-of-ten facts, number-size principles, place value and rounding in situations where decisions have to be made on which of these prerequisites to use – a lot to bring together at the same time as making decisions. Thus familiarity through practice is needed to reduce cognitive load. There are a lot of games available to practise estimation – many use calculators. Here are two of these games.

Gestimate!

This game practises the three strategies – it is suitable for one or many players/teams. It requires the teacher to make up a set of nine estimation examples – three for front end, three for rounding and three for straddling – finally, all nine are made more accurate through Getting Closer.

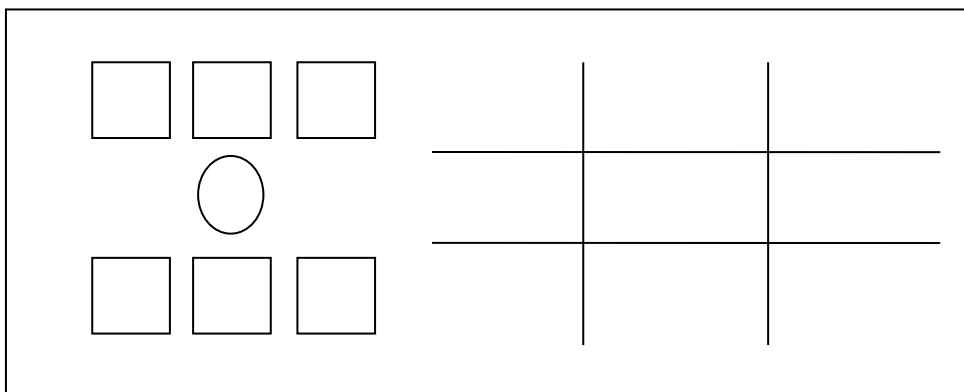
Each player/team receives a game sheet. Players/teams complete A, B and C using strategies then move to Getting Closer for a better solution. Players/teams score 1 point if their answer has the same number of digits as the correct answer (use a calculator for this); 1 point if it has the same first digit; and 1 point if it is within 10% (use a calculator for this also).

FRONT END 1. _____ 2. _____ 3. _____ ROUNDING 4. _____ 5. _____ 6. _____ STRADDLING 7. _____ 8. _____ 9. _____	GETTING CLOSER 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ 7. _____ 8. _____ 9. _____
---	--

Estimation Tic-Tac-Toe

This game requires the teacher to make up six numbers to be added or subtracted. Then + or – are put in the circle and the six numbers randomly in the boxes. This gives nine additions or subtractions; they are calculated and put in the tic-tac-toe figure. This game is suitable for two players/teams. The two players/teams can play on one sheet with a calculator.

Each player/team chooses a colour for their counters. Players/teams take turns in selecting two numbers – one from the top and one from the bottom. Players/teams use a calculator to operate on the two numbers. Players/teams cover the answer on tic-tac-toe board with their colour counter. First player/team with three in a row, wins.



2.6 Word problem solving

This section looks at one of the end points of addition and subtraction and that is problem solving. It discusses how to teach students how to interpret and construct word problems, and how to be a better problem solver.

In section 2.1, we looked at the basic concepts of addition and subtraction as follows:

ADDITION	SUBTRACTION
Action (joining)	Action (separation or taking away)
Part-part → total (know parts and not total)	Total → part-part (know total and one part and not other part)
Inaction – want superset	Inaction – want subset

Now we look at further concepts so that we have a complete set to enable us to solve all types of addition and subtraction word problems.

ADDITION	SUBTRACTION
Inverse action (backward separation)	Inverse action (backward joining)
Change-comparison (end wanted)	Change-comparison (start or change wanted)

Thus, this section first looks at these inverse action and change-comparison concepts before looking at how to interpret, solve and construct addition and subtraction word problems.

The relationship between computation/calculation and problem solving also needs further discussion. The need for computation, particularly in out-of-school settings, arises from problem situations. However, the ability to solve problems comes from the meanings/operating side of operations not the computation/calculating side – mainly from understanding of concepts. Before any calculation can be undertaken, students need to: (a) recognise that calculation will be needed; (b) determine the numbers and operations that will have to be used; and (c) determine if an exact or approximate answer is required. Only then can a method of calculation be determined and the calculation be carried out. The student should also interpret the solution in terms of its reasonableness to the characteristics of the situation (NCTM, 1989).

However, in schools, students often complete many calculations that do not relate to problem situations. Effectively, they are practising the “carry out the calculation” part of this sequence only. The whole sequence is important because problems in the real world are not presented as computations to be completed. Students need practice with the whole sequence for problem solving, from identifying that a computation is required to checking the answer they have found is a reasonable solution to the problem. This section describes the

interpretation and solution of problems using problem-solving strategies as well as the construction of problems which is a necessary skill which practises the reversing idea advocated by YDM.

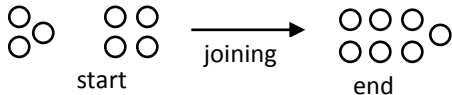
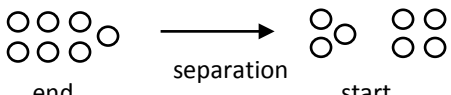
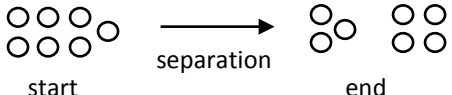
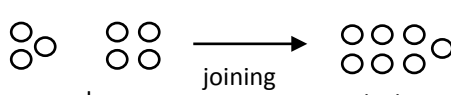
2.6.1 New addition and subtraction concepts

There are two new concepts or clusters of concepts: the inverse or backward concepts, and the change-comparison concepts.

Inverse concepts – operating forwards and backwards

As an action, each operation, addition and subtraction, **takes place over time**. Thus, we can know the start of the problem and want the end (forward problems), or we can know the end and want the start (backward problems).

Interestingly, if we run a joining (addition) activity backwards we get a separating (subtraction) activity and vice versa. This means that backward problems reverse the operation (see table below). In other words, backward joining is subtraction, and backward separating is addition. Examples are provided below to further highlight these varying actions. The intention, like above, is to provide teachers with a rich source of different problem types to work with their students.

ACTION	FORWARD	BACKWARD
Joining 3 and 4 → 7	 start → joining → end Forward joining is starting from parts and joining to get total or sum – as a join, it is the normal/traditional form for addition.	 end → separation → start Backward joining is reversing the join and so starting from the total – it becomes a separation and gives subtraction.
Separating 7 → 3 and 4	 start → separation → end Forward separation is starting from whole and separating into parts – as a separation it is the normal/traditional form for subtraction.	 end → joining → start Backward separation is reversing separation and so ending with the total – it becomes a join and gives addition.

This means that backward joining is subtraction and backward separation is addition, giving the following analysis of addition and subtraction in the real world (see table below).

ADDITION	FORWARD JOINING	BACKWARD SEPARATION
3 + 5 = 8	Joe caught 3 fish in the morning and 5 fish in the afternoon. How many fish did he catch?	Joe caught some fish. He gave away 5 fish and this left 3 fish. How many fish did he catch in total?
SUBTRACTION	FORWARD SEPARATION	BACKWARD JOINING
7 – 4 = 3	Joe caught 7 fish and gave away 4. How many fish did he keep?	Joe caught 4 fish in the morning and some more in the afternoon. He caught 7 fish overall. How many fish did he catch in the afternoon?

With regard to **teaching**, these concepts should be introduced similarly to the action concepts from 2.1 – using RAMR structure in the same manner, that is:

- (a) covering all representations – telling stories $\leftrightarrow$ acting out situations and modelling them with materials $\leftrightarrow$ drawing the situations $\leftrightarrow$ language $\leftrightarrow$ symbols; and
- (b) using both set and number-line models and ensuring stories are in a variety of situations for both these models.

Change and comparison concepts

There are two ways of looking at operations – **relationship and transformation (change)**.

Relationship is the traditional way of looking at operations where three numbers are related, e.g. $7 - 3 = 4$ or $5 + 4 = 9$. However, transformation thinks of operations as movements, e.g. 7 goes to 4 by -3 and 5 goes to 9 by $+4$. They are best represented by Arrowmath as on right. Thus we have the change concept of addition and subtraction.

$$7 \xrightarrow{-3} 4$$

$$5 \xrightarrow{+4} 9$$

Change can also be thought of as additive comparison – instead of 5 going to 9 by the change $+4$, we can think of comparing two things. For example, John has \$4 more in his pocket than Jack. If Jack has \$5, John has \$9.

Thus, it is important to think of operations as change and comparison as well as joining and separation. It is also important to make students aware of, and use, the Arrowmath notation as an alternative to equations. This gives rise to two ways of seeing concepts of operations as follows.

Change – addition is in terms of increase: John's pay was \$5, it increased by \$3 to \$8 ($5 + 3 = 8$); subtraction meaning is in terms of decrease: the price had decreased by \$5 from \$11 and was now \$6 ($11 - 5 = 6$).

Comparison – addition is a comparison where there is more than the starting number: John caught 6 fish, Jack caught 3 more fish than John, Jack caught 9 fish; subtraction is difference: Jack caught 9 fish, John caught 6 fish, the difference was 3 fish.

2.6.2 Solving addition and subtraction problems

This section looks at interpreting problems so as to be able to determine which operation to use, and the role of strategies and plans of attack.

Determining whether multiplication/division or addition/subtraction

The first step in working out the operation for a word problem is to determine whether it is multiplication/division or addition/subtraction. *There is no need to give answers at this point.*

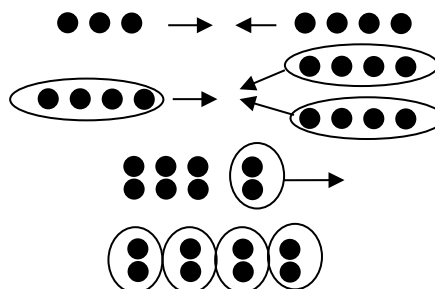
Look at various forms of addition, subtraction, multiplication and division stories. Particularly, compare addition and multiplication, and subtraction and division, using examples as follows.

Addition $3 + 4 = 7$

Multiplication $3 \times 4 = 12$

Subtraction $8 - 2 = 6$

Division $8 \div 2 = 4$



As can be seen:

- (a) addition and multiplication are both joining/combining, while subtraction and division are both separating/partitioning;
- (b) addition and subtraction can have different-sized groups but multiplication and division have same-sized groups; and
- (c) in addition and subtraction, all numbers represent the same thing (here, black discs), while in multiplication and division, two numbers represent the same thing (black discs) but one number represents something different (not discs) that is equivalent to “lots of” or “groups of”.

Thus, we can now differentiate between multiplication/division or addition/subtraction by looking for group size, and whether one number represents groups.

A worksheet (below) or oral questions will allow students to practise. In this practice, use the same numbers for addition and multiplication (e.g. $3 + 4$ and 3×4) and the same numbers for subtraction and division (e.g. $8 - 2$ and $8 \div 2$). Ask students to circle which of $+$, $-$ or $\times \div$ applies. Addition and subtraction is when the two answers are *No-No* and multiplication and division is when the answers are *Yes-Yes*. For the first problem, the dollars (groups), 4 and 5, are different and everything is in dollars; for the second, there are four \$5 (the \$ or groups are all the same) and the 4 represents drinks not dollars. So the first problem is $+/-$ and the second $\times/\div$.

PROBLEM	Multiplication/Division or Addition/Subtraction?
I bought a pie for \$4 and a drink for \$5. How much did I spend? Are groups the same size: Yes / No Does one number represent something different: Yes / No	$+$ $-$ $\times \div$
I went into a shop and bought 4 drinks for \$5 each. How much did I spend? Are groups the same size: Yes / No Does one number represent something different: Yes / No	$+$ $-$ $\times \div$

Interpreting to determine whether addition and subtraction

Once you have decided that a problem is $+/-$, the second step is to use Part-Part-Total (P-P-T) to determine whether it is addition or subtraction. If two parts are known and the total is not, the problem is addition; if the total and one part is known and the other part is not known, the problem is subtraction. All types of addition and subtraction problems (including forward, backward, change and comparison) can be decided in terms of P-P-T.

One way to do this is to set examples (below) and ask students to put numbers and question mark (if unknown) beside the headings Part, Part and Total. Then use these numbers to circle which operation (addition or subtraction) should be used (*the answer is not required*). [In the first problem, the \$47 and \$121 are both parts, so circle addition; in the second problem, the \$45 is a part and the \$87 is the total, so circle subtraction.]

PROBLEM	Addition or Subtraction?
I took \$47 out of my bank account, this left \$121 in the account, how much was in the account to start with? Part: _____ Part: _____ Total: _____	$+$ $-$
Joan has \$45 more than I have, Joan has \$87, how much do I have? Part: _____ Part: _____ Total: _____	$+$ $-$

You can also use the part-part-total figure from subsection 2.1.4 (see on right) instead of listing Part, Part and Total. This can be used to interpret the problem the same way – by using it without the words and writing in the numbers – thus determining if total known (subtraction) or total unknown (addition).

TOTAL	
PART	PART

?	
?	?

2.6.3 Constructing problems

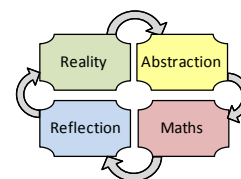
The best way to become an expert at interpreting word problems is to learn how to construct them.

Forwards–backwards stories

Give students an equation such as $5 + 8 = 13$ and ask them to write forward joining, backward separating (and also change-comparison and inaction problems if appropriate) for set and number-line models and for different day-to-day contexts (e.g. shopping, driving, walking, playing sport, and so on). The RAMR cycle can still be useful. Some hints are contained within the following RAMR lesson.

RAMR activity: Constructing problems for addition and subtraction

Constructing problems will give greater insights into interpretation of problems.



Reality

Give students an equation such as $5 + 8 = 13$ and ask them to come up with different problem stories (forward joining, backward separating, change-comparison, and inaction) for set and number-line models and for different everyday contexts (e.g. shopping, driving, walking, playing sport, ...).

Social interaction roles. Set students in groups of three to make up and act out stories by assigning roles of director (leader – makes decisions when there is an impasse), continuity (continuously checks for any errors), and script writer (records and reports on the story and how it will be acted).

Abstraction

Materials. Give students materials to work with (e.g. toy models of people, each other, set up a shop) and ask them to make up and act out their story. (One of the best teaching methods for this approach was by a teacher who organised the students to do a claymation of their story.)

Mathematics

Triad approach. If given $5 + 8 = 13$, write a straightforward joining story with all numbers known, then rewrite with one unknown. This will give three stories, one for $5 + 8 = ?$, one for $? + 8 = 13$, and one for $5 + ? = 13$; the first will be forward and the last two will be backward. After this, write the associated subtraction stories for $13 - 5 = 8$ and $13 - 8 = 5$ and then rewrite each of these with one unknown. The students can be encouraged to see that (a) each equation will give three problems, one for each unknown; (b) 13 unknown means addition is the operation to get the answer; and (c) right-hand side unknown means problems forward, otherwise problems backward.

Use part-part-total. If given $5 + 8 = 13$ and asked for a comparison, the 5 and 8 are parts so they have to be the initial number and the increase while the 13 is the large number and is unknown. So have a start of 5 and have an increase of 8 and then find the end. Then write this into context as a story.

Reflection

Extend an existing problem. (1) Give students a problem, then ask the students to add further words to the problem and change the context of the problem to make it harder/easier. (2) Generalise the understandings from mathematics above by examining how changing the component (part, part or total) that is unknown changes the

type of problem. For each word problem, act out the normal meaning with two knowns and one unknown, then give the unknown a number, and act out the problem with one of the other numbers unknown – does this change addition to subtraction or vice versa? When is the problem addition/subtraction?

Triadic relationship

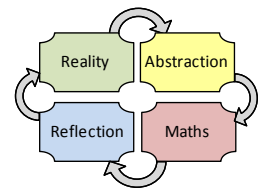
Another way is shown in the table below. As all operations have three components, these components form a triadic relationship. If a problem is written with all three numbers showing, it is possible to transform it into three problems by making each of the three numbers the unknown in turn.

PROBLEM (ALL NUMBERS)	THREE PROBLEMS
Sue ran 3 km, then she ran another 4 km, altogether she ran 7 km.	Sue ran some km, then she ran another 4 km, altogether she ran 7 km, how many km in the first run?
	Sue ran 3 km, then she ran some more km, altogether she ran 7 km, how many extra km did she run?
	Sue ran 3 km, then she ran another 4 km, altogether how many km did she run?
There were 8 boys playing, 3 ran away, there are now 5 boys playing.	There were some boys playing, 3 ran away, there are now 5 boys playing, how many at the start?
	There were 8 boys playing, some ran away, there are now 5 boys playing, how many ran away?
	There were 8 boys playing, 3 ran away, how many are left playing?

RAMR activity: Teaching inverse operation

Reality

Make a list of as many words as the students can think of for addition and subtraction that they use every day. Ensure that this list is flexible – sometimes, the same word may be used in different ways for addition and subtraction.



Abstraction

(1) Take each of these words for addition (e.g. “more”) and act out their normal meaning with two knowns and one unknown. (2) Now give the unknown a number, and act out the problem with one of the other numbers unknown – students will find that it changes addition to subtraction or vice versa (as in RAMR lesson above). For the example, “more means addition”, the following could eventuate:

Original problem: I had 4 books, I got 3 more, how many books do I have?

This is **addition** because part = 4, part = 3, total = unknown

New problem: I had 4 books, I got some more, now I have seven books, how many more did I get?

This is **subtraction** because part = 4, total = 7, part = unknown

(3) Draw diagrams of the two problems. (4) Repeat the above for a subtraction word like “take-away”. Once again students will see that it is possible to have take-away addition, e.g. “I took \$7, this left \$21, how much to start with?” (5) Draw diagrams. (6) Relook at the situation in terms of forward and backward – students should notice that the problems that follow the “normal” use of the words are forward while the problems that give the opposite operation are backward. (7) Look at the drawings – act out the problems – students should see that running addition backward is subtraction and vice versa. (8) Reverse the situation – we start with addition such as $2 + 3 = ?$ This is problem “2 boys join 3 boys, how many boys playing?” Now think – this is also $? - 2 = 3$ backwards, that is “2 boys left the group, 3 boys remaining, how many to start with?”.

Mathematics

Practise backwards problems – give a sum like $6 + 9$. Write it as joining forward and take-away backward. Do similar for a subtraction sum. See how this relates to parts and total and which is unknown, and to forward and backward. Write the sums vertically as well as horizontally. Practise $\text{sum} \rightarrow \text{problem}$ and $\text{problem} \rightarrow \text{sum}$ for inverse meaning.

Reflection

Try to generalise the process. For example, if addition, how do we change that to subtraction? Is there a pattern? Also this change from addition to subtraction can be done on symbols – see on right.

$\begin{array}{r} 47 \\ + 34 \\ \hline \end{array}$	$\rightarrow$	$\begin{array}{r} ?7 \\ + 3? \\ \hline 81 \end{array}$
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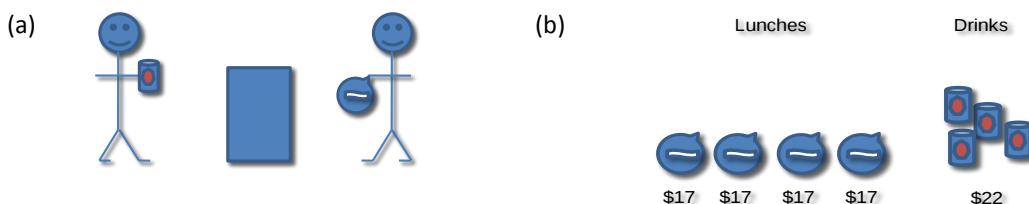
2.6.4 Multi-step problems

It is common for problems to have more than one step. Consider an example, “I bought pants for \$75 and a shirt for \$49, how much change from \$200?”. This example requires the addition of \$75 and \$49 and the subtraction of the total from \$200. Thus this is a scenario where there are two steps required, one is addition and the other is subtraction in this example ($\$200 - (\$75 + \$49) = \$200 - \$124 = \76). In multi-step problems such as these there are two components: (a) determining the steps, and (b) determining the operations in each step. Component (a) requires students to have a good repertoire of strategies and a general plan of attack. Component (b) uses the part-part-total approach to identify where addition and subtraction have to be used.

Strategies and plan of attack

For simpler two- and three-step problems, there are six particularly useful **strategies**. These are listed below. They are described using the problem: “I bought 4 lunches for \$17 each and drinks for \$22. How much did I pay?”

1. **Break problem into parts.** Do not try to do it all at once; look at the problem as a series of steps – for this problem there would be two steps, the 4 lunches and the drinks.
2. **Be systematic/exhaust all possibilities.** This means use tables or charts to check you have covered everything – list the purchases, 4 lunches and drinks and ensure everything is accounted for (e.g. were the drinks each or a total?) – not complicated in this example but helps in larger problems such as planning a vacation.
3. **Make a drawing/act it out.** Get students to make drawings (act it out if this helps) and discuss with them which is the most useful picture and why? For example, (b) is the most useful drawing below to solve the problem.



4. **Given, needed and wanted.** Get students to identify these three things and record:
 - (a) Given: 4 lunches, \$17 for each lunch, \$22 for drinks
 - (b) Needed: $4 \times \$17$ to work out lunches
 - (c) Wanted: the total amount (food and drinks)
5. **Restate the problem/Make it simpler.** Think of it in an easier way and write it down – make the numbers smaller if this helps, work out what to do with these smaller numbers, then replicate this with the larger problem. Think: How could I make the problem easier for a student without giving them the answer? For example, “Work out what 4 lunches at \$17 each is and add that answer to \$22 for drinks to get how much you spent.”

6. **Check/learn.** Check the answer, reflect on solution and learn from it for later problems.

The best **plan of attack** – a metacognitive process for attacking the problem – is Polya’s four stages:

See – spend time simply working out what the problem is – understand the problem.

Plan – make up a plan based on known strategies and knowledge to tackle the problem.

Do – implement the plan, see if it works and calculate an answer.

Check – check the answer for reasonableness and see what you can learn from what you have done.

Framework

It is useful to have a framework within which to work. The following have been found useful.

1. **Extension of part-part-total.** The diagram used for one-step problems can be extended – for the example above, it becomes as on right.

What spent				
4 lunches				\$22
\$17	\$17	\$17	\$17	

2. **Working page.** The following is a working page that YDC staff have found useful. Use the page to solve the following problems:

- (a) A koala was climbing a 10 m tree. He climbed up 3 m and back 2 m each hour. How many hours to the top of the tree?
- (b) Fred set out on a 90 km walk. On the first day he walked 16 km, the second 32 km and the third 23 km. On the fourth day he finished the 90 km walk and then walked another 11 km to the motel. How far did he walk altogether on the fourth day?

Problem:	
Drawing:	Given: Needed: Wanted:
Restating:	
Working:	

2.7 Directed number addition and subtraction

As we saw in the YDM Number book, the application of numbers to continuous entities (such as length or temperature) changed the nature of number and allowed 0 to become other than nothing (e.g. the first point on a ruler). It also allowed number lines to be bi-directional and to go into the negative (e.g. a lift going up and down through floors and basements; a thermometer showing temperatures above and below zero). Now we look at how to do operations with such numbers.

2.7.1 Prerequisites

Ensure that the students have the following understandings for this topic, that is, directed number and the operations of addition and subtraction.

Directed number

1. Start with developing ideas of opposites (up, down; hot, cold; large, small; happy, sad; multiplication, division; add, subtract; plus, minus; positive, negative). Discuss -1 is opposite $+1$; -4 is opposite $+4$, etc.
2. Teach the language: -1 is negative 1, not minus 1; $+1$ is positive 1, not plus 1; -4 is negative 4, not minus 4; $+4$ is positive 4, not plus 4.
3. Relate language to notation: -1 is also (-1) ; -3 is also (-3) (the brackets allow the sign to be similar to a subtraction minus sign).

4. Ensure that students know that if there is no sign the number is positive; for example, 6 on its own represents +6 or (+6).
5. Relate to the real world using examples relevant to students; for example, temperature on thermometers; lifts in buildings with underground car parks; football penalties, etc. (*Note:* Not all students will be familiar with such examples depending on where they live and their experiences.)
6. Relate to vertical number line as done in the YDM Number book. Also ensure students know that line in many instances is horizontal – ensure students do not confuse place value with directed number when using horizontal line.

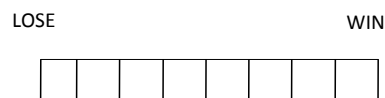
Operations

1. Recap what addition and subtraction actually mean.
2. Act addition and subtraction out with counting numbers (e.g. $6 - 2$ is up 6 and back 2).
3. Repeat this for a horizontal number line (e.g. $6 - 2$ is 6 to right and 2 to left).
4. It is important to do these prerequisites – in earlier years to prepare for directed number and to rebuild ideas at the time of doing directed number.

2.7.2 Addition and subtraction with directed number

Introductory activity

Game: *Materials* – number line like on right; two ends – one good and the other not good (we'll call ours WIN/LOSE); dice with F1, F2, F3, B1, B2, B3 (F – forward, B – backward); dice with W, W, W, L, L, L (W – facing towards WIN, L – facing towards LOSE).



Set up – set up a story like someone seeing if they can win or lose in some way which involves walking along a number track – this person will win or lose depending on throws of dice.

Throw the two dice – the W or L die gives direction facing, the other die gives the number of steps forward or back. The player starts in the middle of the track.

As game is played (player can change), start to record the moves informally (e.g. L-F2, L-B3, W-B1 and so on); discuss movements (e.g. L-B3 actually moves towards WIN); change recording to formal (e.g. W is +, L is –, F1 is +1 and B2 is -2, and so on; and call the middle 0 with to the right (to WIN) positive and to the left (to LOSE) negative.

Record positions as well as movements; e.g. starting at -3 and getting a L and B2 ends at -1 and is recorded as:

$$-3 - 2 = -1 \quad \text{or} \quad -3 - (-2) = -1$$

Acting out operations

Number line

Movement of units to either the left or right (or up and down – depending on form of line), the number gives the number of units moved, + before number means facing to right, and – before number means facing to left. Operation of addition is moving forward and operation of subtraction is moving backwards. For example:

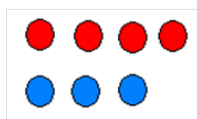
Sentence $+7 + (-6)$ [or $7 + -6$] means you start at 7, 6 means 6 units of movement, + means you move forward, and the – before the 6 means you face to left. This gives starting at 7 facing left and moving 6 units forward to +1.

Sentence $-1 - (-3)$ means that you start at -1 , $-$ means move back, $-$ before 3 means facing left, and 3 means 3 units. This gives starting at -1 , facing left and moving 3 units backward to $+2$.

This can be done using students in class by drawing a line on floor, or using tape, with clearly marked numbers and divisions. Student walks to starting number, faces left or right and moves forward or back depending on operation signs and numbers.

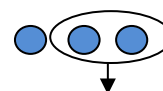
Cancelling

Use different coloured counters to represent positive and negative numbers. Make sure counters are of similar size and easily align under each other. Red is positive, blue is negative.

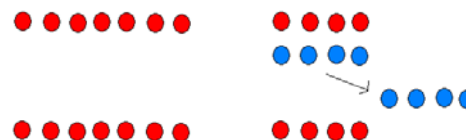


The figure on the left represents $4 + (-3)$; each positive negative pair cancel; so answer is positive 1. (Note: You can also do cancelling using male and female students. Girls can be positive, boys are negative or vice versa.)

Cancelling is harder to use with subtraction. Consider $(-3) - (-2)$, you start out with 3 negative counters, and you take away 2 negative counters (as on right). So you are left with 1 negative counter.



In $7 - (-4)$, you start out with 7 positive counters, but you are supposed to take away 4 negative counters. However, you have 7 positive counters but no negative counters to subtract. So you add 4 pairs of positive and negative counters so they cancel to zero. You can then take away (subtract) 4 negative counters leaving 11 positive counters, so $7 - (-4) = 11$.



Game: "Hit Me" card game

Take a deck of playing cards and remove jokers and picture cards (leave only Aces and 2–10). Make red positive, black negative. The idea is to get to zero in shortest number of deals (7 is maximum). Deal each player a single card face up. Each player is then dealt another card face down which only the player can look at. If player wants another card he/she says "hit me" (maximum of 5 hit me requests). When everyone either rests or reaches maximum "hit me's", all cards are turned over. Whoever reaches ZERO is winner and dealer for next game OR whoever has value closest to zero is winner.

Patterns

Patterns can be used to justify rules for subtracting integers. Begin with known simple subtractions and discuss patterns when students have done answers – students to observe and identify patterns. You can also use the number line. For example $7 - 9$, place your finger at 7, and draw an arrow that is 9 units long towards the left. Arrow should stop at (-2) . Do the same using a negative number as a starting point, like $(-6) - 3$. Start at (-6) and move 3 units to the left. Again the pattern works – have student observe the answers, and then continue the pattern. Finally, justify subtraction of a negative number to give a positive number; that is two negatives turn into a positive.

$6 - 3 = 3$	$(-3) + 2 = -1$	$5 - 3 = 2$	$(-4) - 2 = -6$
$6 - 4 = 2$	$(-3) + 1 = -2$	$5 - 2 = 3$	$(-4) - 1 = -5$
$6 - 5 = 1$	$(-3) + 0 = -3$	$5 - 1 = 4$	$(-4) - 0 = -4$
$6 - 6 = 0$	$(-3) - 1 = -4$	$5 - 0 = 5$	$(-4) - (-1) = -3$
$6 - 7 = -1$	$(-3) - 2 = -5$	$5 - (-1) = 6$	$(-4) - (-2) = -2$
$6 - 8 = -2$	... and so on	$5 - (-2) = 7$	$(-4) - (-3) = -1$
... and so on		$5 - (-3) = 8$	$(-4) - (-4) = 0$
		... and so on	$(-4) - (-5) = 1$
			$(-4) - (-6) = 2$
			... and so on

2.7.3 Using mathematical structure

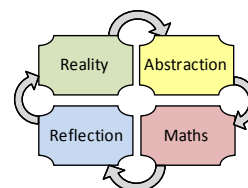
If during the school years before the advent of directed number, students can gain an understanding of the structure of number and operations (the structure called “Field” by mathematicians), they can do directed number **as a consequence** of their understanding of the structure. For example, the Field structure means there is no $-$ operation, $7 - 3$ is $7 + (-3)$ where -3 is the inverse of $+3$ and undoes $+3$. Thus $7 + (-3)$ is just $7 - 3 = 4$. This also means that $7 - (-3)$ is adding the inverse of the inverse of 3 which is adding the inverse of -3 which is $7 + 3 = 10$.

2.8 Extension to algebra

The final step of operations in this section is extending the meaning of addition and subtraction to variables, that is, algebra.

2.8.1 Introducing variable

An effective method for introducing understanding of addition and subtraction to variables is to give problems with more than one unknown as the following RAMR model shows.



Reality

Revise turning symbols into stories: $4 + 7 = ?$ could be “I caught 4 fish and my friend caught 7 fish. How many fish did we catch altogether?”

Abstraction

Discuss with students scenarios that don’t have all the information – “I bought a box of chocolates and then a meal that cost \$9. How much did I spend?” Discuss the following:

- Why can’t this problem be answered? [Don’t know the cost of the chocolates]
- What information would I need to be able to calculate something? [Could calculate spending if given the cost of the chocolates]
- What else is possible? [Calculate the cost of the chocolates if given the total amount of spending]

Note: The YDM Algebra book has other ways of introducing this – patterns, function machines, mass balance and length models. Also can use something for a variable – cups with counters for numbers, envelopes, strips of paper, jumps along a line, and so on.

Mathematics

Let students give amounts and calculate some answers (e.g. “what if...”) and write the equations.

- What if the chocolate cost \$3 – the equation is $3 + 9 = ?$
- What if the chocolate cost \$1.50 – the equation is $1.50 + 9 = ?$

Discuss if it is possible to write the equation if the spending is known and the cost of the chocolate is not known. Allow students to devise their own ways of representing this.

What if the total spending is \$11 – the equation could be: **chocolate cost** + 9 = 11

or $C + 9 = 11$

or  + 9 = 11

If students are struggling to write an equation, have the students write the problem verbally on the thinkboard. Allow student to construct problems using their own method before introducing letters as the Western mathematics symbol for unknown and variable.

Reflection

Use these symbols to do two types of activities:

- From symbols with unknowns/variable, write the story without all the numbers.
- From the story without all the numbers, write the symbols with unknowns/variable.

Have students give the three stories for $5 + 9 = 14$; each of the stories has a different part of the equation “unknown”.

I scored 5 goals in the first half of the match and then 9 goals in the second. How many goals did I score altogether?

I scored 5 goals in the first half and I scored 14 goals in total. How many goals in the second half?

I scored some goals in the first and 9 in the second half. I scored 14 goals altogether. How many did I score in the first half?

Do this for more than one unknown – I bought a pie and a can of coke. How much did I spend?

P = cost of pie

C = cost of coke

T = total cost

Equation $P + C = T$

2.8.2 Interpreting and constructing

This can proceed in the same way word problems are interpreted in section 2.6 but now unknowns are given letters. For example, *There were cows, 5 sheep were put in the same pen. How many animals?* This problem can be stated now as $a + 5 = c$ or as $c - 5 = a$, depending on whether we interpret it as total unknown or part unknown.

We can even have all numbers unknown. For example, *There were cows, sheep were put in the same pen. How many animals?* This problem can be stated now as $a + b = c$ or as $c - a = b$ or $c - b = a$, depending on whether we interpret it as total unknown or part unknown. In fact, algebra allows much more flexibility on whether it is addition or subtraction.

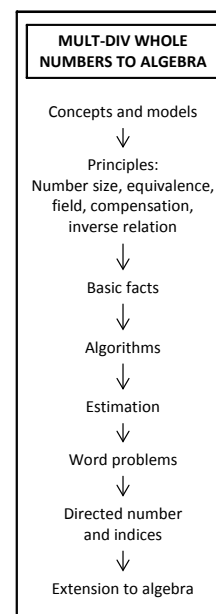
Finally, constructing problems is very flexible. For example, “Write a story for $x + y = z$ ” can be anything: *I bought a pie and a cake and paid money for them.*

3 Multiplication and Division

This is the second of the operations chapters. It covers multiplication and division for whole numbers, moving from meanings and models for one-digit multiplication and division through to algebraic representations. Similar to Chapter 2, this chapter is built around the **two components** for operations:

1. **Meaning/operating.** What does multiplication and division mean (what are their concepts), where are they used in their world, and what are their properties?
2. **Computation/calculating.** What is the answer to the multiplication and division operation, what ways can we work it out, and how accurate does this have to be? Like number, multiplication and division operate on the world in two ways – on individual objects, and on measures.

The **sequence** for the chapter is based on the figure on right: concepts, principles, basic facts, algorithms, estimation, word problems, directed number/indices, and extension to algebra. Problem solving and understanding of algebraic applications of multiplication and division are based on meaning (i.e. concepts and principles), not computation. Obviously, algorithms are computation along with basic facts and estimation.



3.1 Concepts and models for multiplication and division

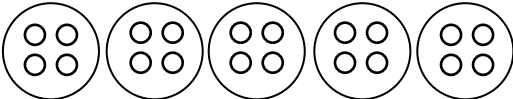
There are five concepts for multiplication and division: **combining/partitioning** (equal groups), **inverse, change/comparison, combinations**, and **factor-factor-product**. There are four models for multiplication and division: **set, number line, array/area**, and **tree diagrams** (tree diagram model for later year levels only). These concepts and models reflect everyday meanings such as “lots of” and sharing. There are a number of notational/symbol forms for multiplication (horizontal and vertical with symbol $\times$) and division (symbols $\div$ and $\overline{)}$). We believe in giving students the full set of concepts, models and notations but spread out across the relevant year levels.

Operations such as multiplication and division are understood in terms of connections between different concepts, models and representations (e.g. real-world situations, models, language and symbols). Thus, the complexity of multiplication and division does not lie in what they do but in the many ways they can be understood. For example, because in multiplication the two numbers being multiplied represent different things, division has two options: (a) when the unknown is the number of groups (this is called grouping or repeated subtraction); and (b) when the unknown is the number in each group (called sharing). This often means that division seems to have 10 concepts.

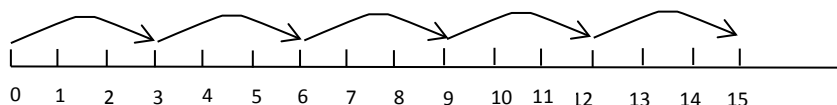
This section looks at the early concepts of combining and partitioning, and factor-factor-product for the set, number-line and array models, and the role of equals. The other concepts and models are in section 3.6.

3.1.1 Combining and partitioning

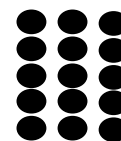
Multiplication as **combining** involves taking equal groups (factors) and combining them to make a total (product). This is common in our society. For example, buying 6 bottles of water for \$4 a bottle, giving 6 people 4 throws each at a “knock ‘em down”, or packing 6 bags of 4 apples, are all examples of $6 \times 4 = 24$. Often this is seen initially as repeated addition where each group added on is the same thing or unit (as for addition). However, for multiplication one factor is applied to the number in the group while the other factor is the number of groups. For example, the 4 is apples while the 6 is bags of apples.

ACTION	NUMBER SENTENCE
	5 groups of 4 are combined $5 \times 4 = 20$

Similarly, combining can be shown on number-line models where jumps may be combined. The most common real-world example of this is fencing where fence panels are a consistent length. For example, 12 fence panels of 3 m each would give the expression 12×3 . The total length of fence from combining these panels would be $12 \times 3 = 36$ m. This can be shown on a number line although this does emphasise the repeated addition meaning of multiplication more than sets/groups of (see number line below for $5 \times 3 = 15$).

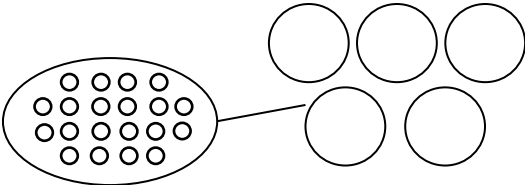
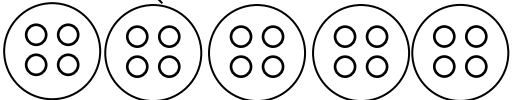


The array model for multiplication is most important for its application to area. In the array model groups are combined in rows and link most easily to the formula for area.



Division as **partitioning** involves taking a quantity (product) and partitioning this product into equal parts (factors). Again this is a common action in our world. For example, sharing \$24 among 4 children or putting 24 apples into packs of 4 are both examples of $24 \div 4$.

Interestingly, the fact that numbers in multiplication can refer to two different things causes there to be two types of division – the first where the number of equal groups is known (sharing), and the second where the number in each group is known (grouping). Thus \$24 shared among 6 people and \$24 given to people \$6 at a time (or put in groups of \$6) are both examples of $24 \div 6$. See figures below for examples of sharing and grouping. It is important for real-life applications to understand both of these.

ACTION	NUMBER SENTENCE
	$20 \div 5 = 4$ (sharing)
	$20 \div 4 = 5$ (grouping)

Students need to know the difference between grouping and sharing using set models, array models and number-line models.

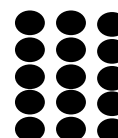
1. The set model drawing on the right is:

- $15 \div 5 = 3$ (5 groups – how many in each group?) in terms of sharing; and
- $15 \div 3 = 5$ in terms of grouping (3 in each group – how many groups?).



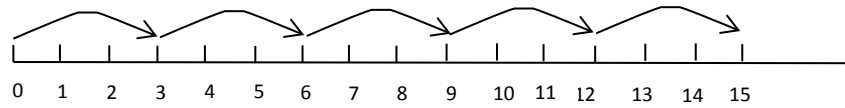
2. The array drawing on right is:

- $15 \div 5 = 3$ in terms of sharing since there are 5 rows; and
- $15 \div 3 = 5$ in terms of grouping since there are 3 objects in each row.



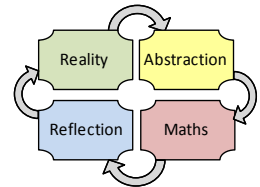
3. The number-line drawing below is:

- $15 \div 5 = 3$ in terms of sharing since there are 5 jumps; and
- $15 \div 3 = 5$ in terms of grouping since there are 3 spaces in each jump.



This means that sharing for $15 \div 3$ is the same as grouping for $15 \div 5$ and this means 3 bags of 5 for set model, 3 rows of 5 for array, and 3 jumps of 5 for number line.

3.1.2 Teaching division as partitioning



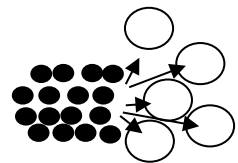
Reality

Act out problems using the students: for example, multiplication and set model, “There were 3 bags of lollies, each bag had 5 lollies, how many lollies overall?”; and division and set model, “15 apples were shared equally among 5 bags, how many apples in each bag?” and “15 apples were put into bags of five, how many bags?”

Note: It is important to differentiate multiplication and division from addition and subtraction at the start. Compare the operations with those previously introduced as you start on an operation. For example, when introducing multiplication for the first time, compare it with addition. Use examples, e.g. “How do we act out $3 + 4$ and 3×4 . What is the same about them, what is different?”

Abstraction

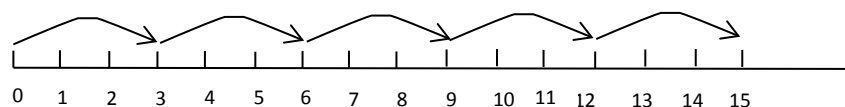
Set model ($15 \div 5 = 3$ – sharing). Start with acting out a set problem (and finding the answer, if important) and getting students to model the acting out with counters; we shall look at a sharing example: “15 apples were shared equally among 5 bags, how many apples in each bag?”



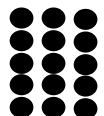
Then get students to draw a picture; “15 apples were shared equally among 5 bags, how many apples in each bag?”



Number-line model ($15 \div 5 = 3$ – sharing). Start with acting out a number-line problem (and finding the answer, if important) and getting students to model the acting out with a number line: “The relay race was 15 km, 5 runners to a relay team, how far does each runner have to run?”



Array model ($15 \div 5 = 3$ – sharing). Start with acting out an array problem (and finding the answer, if important) and getting students to model the acting out with counters in rows and columns: “15 students lined up in 5 rows, how many students in each row?”



Mathematics

Introduce language: “Fifteen divided by five is three.”

$$15 \div 5 = 3$$

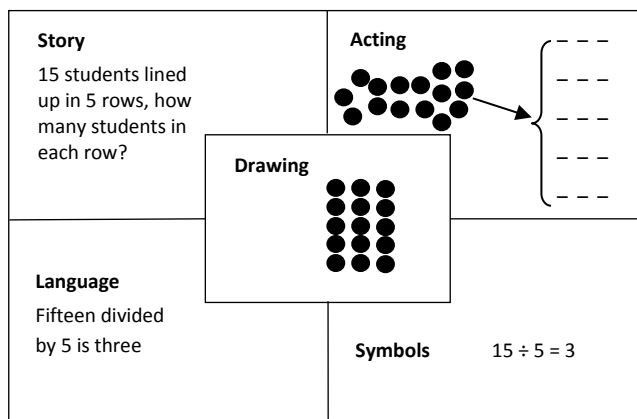
Finally introduce symbols: “How many apples were there?” [15] “How many bags?” [5] and “How many apples in each bag?” [3]

$$\begin{array}{r} 3 \\ 5 \overline{)15} \end{array}$$

Reflection

Begin with any of the representations and complete the others, that is, relate all five representations: stories $\leftrightarrow$ act out $\leftrightarrow$ pictures $\leftrightarrow$ language $\leftrightarrow$ symbols as below. Use the diagram/mat on the right, fill in one area (any of the areas) and ask for the other areas to be completed. It is important to ensure that:

- **all meanings and models are covered**, that is, multiplication, division (group and sharing) and set, number-line and array models;
- **all connections are both ways**, that is, that students can write a story for language or symbols, and that a drawing can be interpreted in a story or symbols;
- stories can be for a **variety of situations** – that is, they can write shopping, sporting, fishing, driving, TV stories, and so on; and
- **operations are generic** – that $5 \times 3 = 15$ means that 5 bags of 3 fish is 15 fish, 5 bottles at \$3 each is \$15, 5 men run 3 km each to run the 15 km, and so on. Thus, $5 \times 3 = 15$ holds for every set of objects and every measure in the world.



3.1.3 Factor-factor-product

The four operations involve situations with parts that are known and parts that need to be found. Addition and subtraction situations involve two parts and a total. Multiplication and division situations involve two parts (factors) and a product.

Multiplication is a situation when parts (factors) are known and the product is wanted.

In multiplication the situation and representation have an important difference. Where in addition and subtraction the numbers represent a quantity of things, in multiplication and division one of the numbers refers to a quantity of things and the other number is the number of groups of those things– that is, not the number of things. Multiplication involves the combination of equal groups, which is common in our society. For example, buying 6 bottles of water for \$4 a bottle, giving 6 people 4 throws each at a “knock ‘em down”, or packing 6 bags of 4 apples, are all examples of $6 \times 4 = 24$. Interestingly, this means that the 6 and the 4 do not refer to the same things. For example, the 4 is apples while the 6 is bags of apples.

Division is a situation when product and one part (factor) is known and the other part (factor) is wanted. For division, the unknown may be the number of groups (called grouping or repeated subtraction) or the number in each group (called sharing).

Similar to subtraction, division is not an operation in a strictly mathematical sense as it is neither commutative (e.g. $24 \div 3 \neq 3 \div 24$) or associative (e.g. $(24 \div 4) \div 2 \neq 24 \div (4 \div 2)$). Division is the opposite action (or inverse) of multiplication. Thus division is breaking the total or product into equal groups. Again this is a common action in our world. For example, sharing \$24 among 4 children or putting 24 apples into packs of 4 are both examples of $24 \div 4$. As discussed earlier in section 3.1.1, the fact that numbers in multiplication can refer to two different things causes there to be two types of division – the first where the number of equal groups is known (sharing), and the second where the number in each group is known (grouping). Thus \$24 shared among 6 people and \$24 given to people \$6 at a time (or put in groups of \$6) are both examples of $24 \div 6$.

The concepts of multiplication and division can be combined under *factor-factor-product* (F-F-P). This allows all concepts to be integrated and a single method to be used to determine whether a problem is multiplication or division. (Note: There are difficulties in integrating comparison.)

OPERATION	MEANING	PROBLEM	THINKING
Multiplication	Know parts – want product	<i>Money was divided among the employees, each received \$436, there were 57 employees, how much money was divided?</i>	“The \$436 is a part. The 57 is a part. The wanted amount is the product. So, the operation is multiplication.”
Division	Know product – want a part	<i>The number of apples is 8 times the number of oranges, there are 56 apples, how many oranges?</i>	“The 8 is a part. The 56 is the product. The wanted amount is also a part. So, the operation is division.”

F-F-P and inaction. F-F-P is not the only way of seeing multiplication and division that does not involve actions or changes; there is another – inaction. This inaction meaning looks at different sets and then the number in the superset that contains both as below.

- Multiplication – there were 5 Holdens, 5 Fords, and 5 Toyotas; this made 15 cars.
- Division – there were 15 cars made up of equal numbers of Fords, Holdens, Toyotas, Mazdas and Hyundais; there were 3 Holdens.

3.1.4 Teaching factor-factor-product

To teach factor-factor-product, do the following:

- act out multiplication and division situations, like $4 \times 5 = 20$, with materials identifying the group of 20 by the name “product”, the groups of 5 by the name “factor” and the number of groups of 5 also by the name “factor”;
- show how the stories of multiplication and division are the reverse of each other (multiplication as $F \times F \rightarrow P$; and division as $P \rightarrow F \times F$; and
- generally do the same activities as for teaching combining and partitioning but continuously use the terms factor and product instead of the language or symbols.

To use factor-factor-product, use a storyboard or thinkboard to represent different problems that have an element missing, as in the summary below, and to identify factor, factor and product and which is unknown. Use the type of thinking that is given below.

OPERATION	MEANING	PROBLEM	THINKING
Multiplication	Know factors – want product	Straightforward problem: <i>I bought 15 radios at \$127 per radio, how much did I pay?</i>	“The 15 and \$127 are factors. The wanted amount is the product. So, the operation is multiplication.”
		Complex problem: <i>I divided the winnings among the 15 in the group, each member got \$127, how much did we win?</i>	“The 15 and \$127 are factors. The wanted amount is the product. So, the operation is multiplication.”
Division	Know product – want a factor	Straightforward problem: <i>I had \$3427. I shared it among 23 people. What did each person get?</i>	“The \$3427 is the product. The 23 is a factor. The wanted amount is a factor. So, the operation is division.”
		Complex problem: <i>Each of the 23 people received the same amount of money. The total given out was \$3427. How much did each person get?</i>	“The \$3427 is the product. The 23 is a factor. The wanted amount is a factor. So, the operation is division.”

3.1.5 Role of equals

For many students the equals sign has become a symbol for “put the answer here” or “do something” when its real meaning is “same value as”. Thus although 7 multiply 4 is 28 is represented with symbols as $7 \times 4 = 28$, it must be seen as **7×4 is the same value as 28**. This means that it is possible and equally correct to show $7 \times 4 = 28$ as $28 = 7 \times 4$. Thus many forms of equations are possible and all relate to stories, as the following shows.

STORY	SYMBOLS
There were 7 bags of 4 lollies.	7×4
There were 7 bags of 4 lollies, how many lollies altogether?	$7 \times 4 =$
There were 7 bags of 4 lollies, this made 28 lollies altogether; OR There were 7 bags of 4 lollies on the table, this was the same number of lollies that were on the bench which was 28.	$7 \times 4 = 28$
There were 28 lollies on the plate, this was the same number of lollies as on the bench where there were 7 bags of 4 lollies; OR 28 lollies is the same number of lollies as in 7 bags of 4 lollies.	$28 = 7 \times 4$
Jack had 7 bags of 4 lollies, Frank had 30 lollies but he ate 2, both Jack and Frank had the same number of lollies.	$7 \times 4 = 30 - 2$, or $30 - 2 = 7 \times 4$

The important point here is that the teaching must focus on

- (a) the line in $\begin{array}{r} 7 \\ \times 4 \\ \hline 28 \end{array}$ and the equals sign as meaning “same as value as”;
- (b) the sequence, stories $\rightarrow$ acting out/modelling $\rightarrow$ pictures $\rightarrow$ language $\rightarrow$ symbols;
- (c) the reverse of this sequence; and
- (d) the RAMR cycle, reality $\rightarrow$ abstraction $\rightarrow$ mathematics $\rightarrow$ reflection.

The wide variety of problems given to students should give **precedence to equals as “the same value as”** because it is the long-term meaning used in algebra.

3.2 Early number-sense principles

This section looks at the properties of multiplication and division that hold regardless of the numbers or variables being multiplied and divided. It begins by looking at the number-size principles, then looks at the relationship principles (called Field principles), and finally the equals principles.

3.2.1 Number-size principles

The number-size principles are properties of multiplication and division that hold for all numbers (whole numbers, decimal numbers, common fractions, measures, and so on).

Nature of principles

We can work the principles out by looking at the two simple examples in the figure on the next page: $6 \times 3 = 18$ and $8 \div 4 = 2$ and the activities (a) to (p).

For these examples, a number and an upward arrow means that the number (in the algorithm) is increasing; a number and a downward arrow means that the number is decreasing; and a number and an equals sign means that the number stays the same.

The line next to a number means that the activity requires determining whether that number moves $\uparrow$, $\downarrow$ or $=$ as a result of the changes given. For example (a), “6 $\uparrow$ ” means that 6 in $6 \times 3 = 18$ is increasing, while “3 $=$ ” means that the 3 stays the same. The “18 $______$ ” means that we have to find what will happen to 18 in this situation. Thus, the activity requires the student to determine whether 18 is $\uparrow$, $\downarrow$ or $=$ when 6 is $\uparrow$ and 3 is $=$. The way to do this is to think of different scenarios and check what happens. If the 6 increases to 9, for example, and $9 \times 3 = 27$, this means that the 18 increases, that is 18 $\uparrow$.

For example (i), “6 $\uparrow$ ” and “18 $=$ ” means that the 6 increases and the 18 stays the same in $6 \times 3 = 18$, while “3 $______$ ” means we have to find whether 3 increases, decreases or stays the same. Once again, this can be done by increasing the 6 (say, to 9), then we have $9 \times 2 = 18$ and the 3 has been decreased, that is “3 $\downarrow$ ”.

If all examples (a) to (p) are completed, students can discover differences between multiplication and division that are important – these are described below.

Important principles

The principles are as follows (* shows differences between multiplication and division):

Exercises:			
		$6 \times 3 = 18$	
		$8 \div 4 = 2$	
(a)	6 $\uparrow$	3 $=$	18 $______$
(b)	6 $\downarrow$	3 $=$	18 $______$
(c)	8 $\uparrow$	4 $=$	2 $______$
(d)	8 $\downarrow$	4 $=$	2 $______$
(e)	6 $=$	3 $\uparrow$	18 $______$
(f)	6 $=$	3 $\downarrow$	18 $______$
(g)	8 $=$	4 $\uparrow$	2 $______$
(h)	8 $=$	4 $\downarrow$	2 $______$
(i)	6 $\uparrow$	18 $=$	3 $______$
(j)	6 $\downarrow$	18 $=$	3 $______$
(k)	8 $\uparrow$	2 $=$	4 $______$
(l)	8 $\downarrow$	2 $=$	4 $______$
(m)	3 $\uparrow$	18 $=$	6 $______$
(n)	3 $\downarrow$	18 $=$	6 $______$
(o)	4 $\uparrow$	2 $=$	8 $______$
(p)	4 $\downarrow$	2 $=$	8 $______$

What relationships and differences can you see between multiplication and division?

CHANGE	MULTIPLICATION	DIVISION
First number increases/decreases	Answer increases/decreases	Answer increases/decreases
Second number increases/decreases	Answer increases/decreases	Answer decreases/increases *
First increases/decreases, answer equals	Second decreases/increases	Second increases/decreases *
Second increases/decreases, answer equals	First decreases/increases	First increases/decreases *

The results above mean that:

1. Multiplication and division **act differently** for change in second number (the division change is called the **inverse relation principle**).
2. Multiplication and division **act differently** to keep answer the same (called the **compensation principle**).

Are there more principles? [Hint – What if the answer increases? What could happen to the other numbers?]

Teaching the principles

The main teaching approach is to keep following the RAMR framework.

- Try to think of reality/kinaesthetic activities from world of students – a good one is a relay race – if there are 3 runners and each runs 4 km, then total is 12 km; if there are only 2 runners and each still runs 4 km, then total is 8 km. Get the students outside in pairs with batons – set up walking short-course relay races – change the number of students in each group or change how far they each walk.
- Act out/model what is happening with counters and on number lines.
- Record all findings.
- Reflect back into the world and look for extensions.

One powerful way to teach the principles is to run this as an investigation – let students simply investigate multiplication and division for properties and record all they find. The activities below are also a way to do this.

3.2.2 Relationship or Field principles

The relationship principles or the Field principles are properties of numbers and operations. Although we are looking at multiplication and division, only multiplication is a real operation in mathematics because division does not obey all these principles.

Nature of principles

The relationship principles are as follows. (***) denotes principles for which division does not hold.)

1. **Closure.** Numbers and a multiplication operation always give another number (e.g. $2.17 \times 4.3 = 9.331$; for any numbers a and b , $a \times b = c$ which is another number).
2. **Identity.** There exists a number (1) that does not change the number when multiplied. That is, anything multiplied by 1 = itself (i.e. for any number a , $a \times 1 = 1 \times a = a$). Anything multiplied by 0 = 0.
3. **Inverse.** For any number, there exists a number which when multiplied gives the identity (e.g. the inverse of 7 is $^{-1}/7$ because $7 \times ^{1}/7 = ^{1}/7 \times 7 = 1$; for any number a , there exists a^{-1} so that $a \times a^{-1} = a^{-1} \times a = 1$).
4. **Associative.** Numbers can be associated/multiplied in any way and give the same answer (e.g. $5 \times 3 \times 6 = 15 \times 6 = 90$ and $5 \times 3 \times 6 = 5 \times 18 = 90$; for any numbers a , b and c , $(a \times b) \times c = a \times (b \times c)$). ***
5. **Commutative.** Numbers can change order without changing the answer (e.g. $7 \times 4 = 4 \times 7 = 28$; for any numbers a and b , $a \times b = b \times a$). ***
6. **Distributive.** Multiplication distributes across all additions (e.g. $7 \times (4 + 9) = 28 + 63$; for any numbers a , b and c , $a \times (b + c) = (a \times b) + (a \times c)$).

Note: This is the full list of Field principles – they hold for addition and multiplication and do not all hold for subtraction and division. Distributive laws do hold for all operations (e.g. $7 \times (8 - 3) = (7 \times 8) - (7 \times 3)$, $(56 + 21) \div 7 = (56 \div 7) + (21 \div 7)$, and $(56 - 21) \div 7 = (56 \div 7) - (21 \div 7)$).

Teaching the principles

There are two ways of teaching these principles. The first is to use the RAMR cycle (reality → abstraction → mathematics → reflection) with attention to set and number-line models, and kinaesthetic activities. The second is to use patterns and calculators, for example:

1. *Do these with calculators:*

(a) $2.5 \times 3.8 =$ _____; $3.8 \times 2.5 =$ _____; (b) $34.9 \times 7.8 =$ _____; $7.8 \times 34.9 =$ _____; and so on.

2. *Do these without calculators:*

(a) $34 \times 1.6 = 54.4$; $1.6 \times 34 =$ _____; (b) $9 \times 3.5 = 31.5$; $3.5 \times 9 =$ _____; and so on.

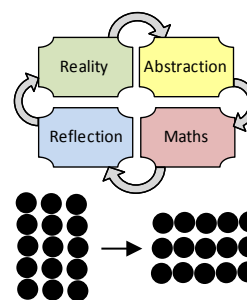
The students are asked to see the patterns in the calculator activities that enable them to do the non-calculator activities without a calculator.

Examples of RAMR with the principles

Commutative principle

Reality. Find something in the world where $a \times b = b \times a$.

Abstraction. (a) Show with objects that, e.g. 3 groups of 5 is the same as 5 groups of 3. (b) Show with number line that, e.g. 5 hops of 3 is the same as 3 hops of 5. (c) Show by rotating array 90 degrees that, e.g. 5 rows of 3 is 3 rows of 5 (as on right).



Mathematics. Record above activities with symbols. Validate this with calculator for any two numbers, e.g. $456 \times 327 = 327 \times 456$.

Reflection. Explore whether this is true for division.

Associative principle

Reality. Find something in the world where you can multiply numbers in any order and it does not matter.

Abstraction. Show with objects and number lines that, for any three numbers, e.g. 3, 5 and 6, it does not matter whether 3 and 5 or 5 and 6 are multiplied first, $(3 \times 5) \times 6$ always equals $3 \times (5 \times 6)$.

Mathematics. Record above activities with symbols as you go. Validate with calculator for large numbers.

Reflection. Explore whether this is true for division.

Identity principle

Reality. Discuss actions that leave things unchanged; for example, do nothing, rotate 360 degrees, drive in a circle.

Abstraction. Show through sets and number lines that $\times 1$ leaves numbers unchanged. This is difficult for some students to understand – you may have to use patterns, e.g. $5 \times 1 = 5$, $6 \times 1 = 6$, $7 \times 1 = 7$, $8 \times 1 = ??$.

Mathematics. Do these patterns with materials and recordings.

Reflection. Discuss that the following equations are allowed and what they mean, and discuss how to relate equations to stories and vice versa: $2 \times 3 = 6$, $6 = 2 \times 3$, $8 \times 5 = 40$, $40 = 8 \times 5$, $4 \times 2 = 8 \times 1$ and $8 \times 1 = 4 \times 2$. Stress the need to have understandings that cover stories with two equal components as well as reaching-an-answer stories (e.g. “Cats were sitting in pairs on 3 fences; this made the same number of cats that were on another fence where there was one group of 6 cats” as well as “Cats were sitting in pairs on 3 fences; there were 6 cats on the fence”). Write equations such as $3 \times 4 = 12$ and $4 \times 3 = 12$ where students write their own meaning for the rule, and so on.

3.2.3 Equals principles

The equals principles are the properties that always hold for equals – there are three of them:

1. **Reflexive.** Any number a is equal to itself, that is, $a = a$.
2. **Symmetric.** For any numbers a and b , if $a = b$ then $b = a$ (order of an equation can be reversed or “turned around”); this is important for equations as it means that $3 \times 4 = 12$ and $12 = 3 \times 4$, $3 \times 4 = 24 \div 2$ and $24 \div 2 = 3 \times 4$ are all allowed and are all correct and true.
3. **Transitive.** For any numbers a , b and c , $a = b$ and $b = c$ means $a = c$; this is also important for equations because it means we can say: $4 \times 8 \div 2 = 32 \div 2 = 16$ and so $4 \times 8 \div 2 = 16$.

Once again, the way to teach them is to use the RAMR cycle and the set and number-line models. However, the best model for these principles is the mass model – to think of equals as two sides being balanced.

3.3 Basic fact strategies

Once the concepts of the operations are introduced, it is time to teach ways to calculate the answers more quickly than representing the operation with counters and counting to get the answer. The first of the calculations to teach are those that form the basis of the later algorithms and estimation – the basic facts. While it is widely accepted that these facts have to be learnt off by heart, that is, automated by practice (drill), it is not something that should have an inordinate amount of time spent on memorising to the detriment of time for other concepts that must be learnt. The reason for still automating facts is that automated facts are available in task situations without taking any thinking away from the task – automated facts have no cognitive load.

The basic facts are all the calculations with numbers less than 10 for multiplication and the inverse operations for division, that is:

$0 \times 0, 0 \times 1, 0 \times 2, \dots, 0 \times 9; 1 \times 0, 1 \times 1, 1 \times 2, \dots, 1 \times 9; 2 \times 0, 2 \times 1, \dots, 2 \times 9; \dots; 9 \times 0, 9 \times 1, 9 \times 2, \dots, 9 \times 9.$

$1 \div 1, 2 \div 1, \dots, 10 \div 1; 2 \div 2, 4 \div 2, \dots, 18 \div 2; 3 \div 3, 6 \div 3, \dots, 27 \div 3; \dots; 9 \div 9, 18 \div 9, \dots, 81 \div 9.$

Basic fact strategies are usually quicker than completing the calculation on a calculator and are valuable as a means of teaching basic facts to students. Students can learn the basic fact strategies as a way of working out basic facts, and have these as a fall-back method if they can't remember the fact "off by heart". The use of strategies as a teaching tool also helps students to understand the operations they are working with. This is especially important with multiplication and division if students are to develop an ability to identify and use these operations to solve problems. The goal of basic fact learning is automaticity.

There are four strategies for multiplication and division basic facts:

- use a rule ($0 \times$ and $1 \times$);
- use ten ($5 \times$, $9 \times$ and $10 \times$ even though this is not a basic fact);
- use doubles ($2 \times$, $4 \times$ and $8 \times$); and
- use connections ($3 \times$, $6 \times$, $7 \times$ and $9 \times$).

The big ideas relevant for the facts are *identity* and *inverse* and the *commutative*, *associative* and *distributive* laws.

3.3.1 Multiplication strategies

For students to be able to work out multiplication (and then division) basic facts they need to understand the operation concept of multiplication. To understand the concept of multiplication, students need to see the two numbers in the computation differently than they do for addition and subtraction. In addition (e.g. $5 + 4$), the two numbers both refer to a quantity of items (e.g. 5 birds and 4 more birds); $9 - 5$ can be 9 birds and 5 birds flew away. In both these situations the numbers represent birds. However, in multiplication the numbers are not both quantities of items. One number is a quantity of items but the other is the number of sets or groups of these items. So 2×5 is 2 groups of 5 birds.

The multiplication basic facts are the $0 \times$, $1 \times$, $2 \times$, $3 \times$, $4 \times$, $5 \times$, $6 \times$, $7 \times$, $8 \times$ and $9 \times$. Using the $4 \times$ as an example, students can think of these as 4 ones, 4 twos, 4 threes, 4 fours and so on, which look like:

4 ones	● ● ● ●
4 twos	●● ●● ●● ●●
4 threes	●● ●● ●● ●● ● ● ● ●

The multiplication basic fact strategies link to the representation and presenting the basic facts in the way described below initially will help students to understand the connection between the fact and the strategy. Once this is understood the commutativity principle will of course apply. The value in understanding how these

strategies relate to the particular multiplication will be helpful as students can apply these strategies to numbers beyond basic facts. In the example above this representation for the $4\times$ basic facts highlights the connection to doubling. It is clear that $2\times$ of a quantity is double and $4\times$ is double double. This strategy will work for $2\times$ and $4\times$ any number.

Use a rule

There are two rules which apply to multiplication basic facts. The first is the **concept of zero** in multiplication meaning there are no groups; for example, 0×5 is no groups of 5 so the result is zero. The same works as a turnaround, e.g. 5×0 is 5 groups of zero which is clearly zero. This can be generalised so that anything multiplied by zero will result in zero. Students need to conceptualise this as a special property of zero and know that this rule is always true.

The second rule which applies to multiplication links to the **identity principle**. This principle is about leaving things unchanged. The identity that leaves things unchanged when multiplied is 1. So the $1\times$ basic facts can be taught using this principle: $1\times$ any number will be that number, e.g. $1 \times 7 = 7$; $1 \times 9 = 9$.

Use ten

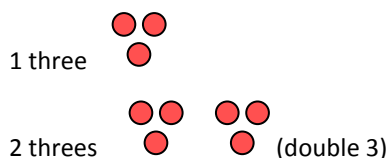
The Use Ten multiplication basic fact strategy helps students understand the $5\times$ basic facts and it can be used for the $9\times$ facts as well through connections (see later). By thinking of $10\times$ instead of $5\times$ the students can determine an answer using their knowledge of place value and then halve the result to work out the $5\times$ facts. For example, 5 sixes: think 10 sixes (60) and then halve the result (30). This strategy can help students who are still working on automaticity but it will also provide them with a strategy for multiplying any number by 5; for example, 5×42 : think 10×42 (420) and halve it (210).

This strategy is also effective for $\div 5$. Instead of dividing by 5 a number can be divided by 10. This means you end up with twice the number of divisions so to adjust this you need to double the answer. For example, $40 \div 5$: think $40 \div 10$ (4), then double 4 is 8.

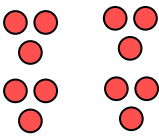
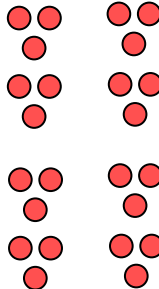
Another multiplication basic fact series that can be worked out with a Use Ten strategy is the $9\times$ facts. Instead of working out 9 ones, 9 twos, 9 threes, this strategy uses the $10\times$ instead and then takes away the one group extra. For example, 9 sixes: think 10 sixes (60) then take away 1 six ($60 - 6 = 54$). This strategy is also described as a Use Connections as it utilises the connection between $9\times$ and $10\times$.

Use doubles

The Use Doubles multiplication basic fact strategy builds on the concept that double something means you have twice the quantity. This holds for the doubles strategy in addition ($3 + 3$ is 2 threes or double 3) and this provides a link between the addition basic fact strategy and the multiplication basic fact strategy. When this basic fact is modelled students can clearly see that they have a double.



So $2\times$ any number is a double. Two threes is double 3. The relationship continues: when you double the double three there are now 4 threes (double double three or 4 threes) and when this is doubled there are 8 threes (double double double three or 8 threes).

1 three		
2 threes		is double three
4 threes		is double the double three
8 threes		is double the double double three

The $2\times$, $4\times$ and $8\times$ multiplication basic facts can be found using doubling. This strategy also works beyond the basic facts, e.g. 8×75 is double double double 75. Double 75 is 150, double 150 is 300, double 300 is 600.

The Use Doubles strategy also applies to the inverse operation of division through halving. To $\div 2$ is to halve a number. To $\div 4$ is half and half again. To $\div 8$ is to use half, half, half. A basic fact example is $24 \div 4$: half of 24 is 12; half of 12 is 6. Again, this strategy also applies beyond basic facts, e.g. $128 \div 4$: half of 128 is 64, half of 64 is 32. It also can work with decimals, e.g. $156 \div 8$: half of 156 is 78, half of 78 is 39, half of 39 is 19.5.

Use patterns

An alternative strategy to those above is patterns. This strategy applies to any of the basic facts for which there is a pattern that could help students remember the facts. The following basic facts have patterns – note that most have been covered above:

BASIC FACT	PATTERN
$0\times$	Gives zero for all multiplication, e.g. $3 \times 0 = 0 \times 3 = 0$; $8 \times 0 = 0 \times 8 = 0$
$1\times$	Gives the number (identity), e.g. $3 \times 1 = 1 \times 3 = 3$; $8 \times 1 = 1 \times 8 = 8$
$2\times$	Doubles: 2, 4, 6, 8, 0, ... and so on
$5\times$	Fives: 5, 0, 5, 0, 5, ... and so on; 5, 10, 15, ...; half the $10\times$ tables; hands; clockface (minutes in one hour)
$9\times$	Nines: tens are one less than number to be multiplied by 9, ones are such that tens and ones digits add to 9; 9, 18, 27, ... and so on

These patterns can be most easily seen with a calculator, see-through counters, and large and small 99 boards. The table for the pattern is chosen (e.g. $4\times$). The number of the table is entered on the calculator and [=] [=] pressed (e.g. [4] [=] [=]). The result (4) is covered on the 99 board with a Unifix. From there, [=] is continually pressed (adding 4) and the number shown is covered. Once sufficient numbers are covered to see the visual pattern on the 99 board, this pattern is transferred to the small 99 board by colouring squares.

The numbers coloured are discussed to arrive at the pattern. If more reinforcement is needed, [number] [=] [=] [=] [=] [=] ... is pressed on the calculator and the ones or tens called out at each [=] press (see below). This enables students to verbally hear patterns. The numbers could also be written down for inspection for pattern.

Press [5] [=] [=] [=] ... stating the ones position

Press [9] [=] [=] [=] [=] ... stating the ones position, then repeat, stating the tens position

Press [4] [=] [=] [=] [=] ... stating the ones position

Some teachers also do $4\times$ and $3\times$ with patterns as below.

$4\times$ Fours – 4, 8, 2, 6, 0, 4, and so on; odd tens is 2 and 6 for ones and even tens is 0, 4 and 8 for ones, that is:

0	4	8
12	16	
20	24	28
32	36	

$3\times$ Threes – the “one back” pattern, that is:

0	3	6	9
12	15	18	
21	24	27	

Use connections

This strategy works for the remaining multiplication basic facts ($3\times$, $6\times$, $7\times$). Here, the unknown fact is connected to known facts using the distributive principle. We can use counters, Unifix, dot paper and graph paper for the models. The idea is that the answers to the unknown facts are found from the known facts.

UNKNOWN FACT	KNOWN FACT(S)	CONNECTION	EXAMPLE	DIAGRAM
$3\times$	$2\times$	$3\times$ is $2\times + 1\times$	3×7 is the same as $2\times 7 + 1\times 7 = 14 + 7 = 21$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
$6\times$	$2\times$, $3\times$	$6\times$ is double $3\times$ or $6\times$ is $3\times + 3\times$	6×7 is the same as $3\times 7 + 3\times 7$, i.e. double 3×7 is double $21 = 42$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
$6\times$	$5\times$	$6\times$ is $5\times + 1\times$	6×7 is the same as $5\times 7 + 1\times 7 = 35 + 7 = 42$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
$7\times$	$2\times$, $5\times$	$7\times$ is $5\times + 2\times$	7×7 is the same as $5\times 7 + 2\times 7 = 35 + 14 = 49$	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

In terms of a teaching sequence, taking into account patterns and connections, a suggestion is $2\times$, $5\times$, $9\times$, $4\times$, $8\times$, $3\times$, $6\times$, and $7\times$. This, of course, is not the only correct or appropriate sequence. For example, $2\times$, $4\times$, $8\times$, $5\times$, $3\times$, $6\times$, $9\times$, $7\times$ is also efficient.

3.3.2 Division facts – think multiplication

The think multiplication strategy is used for division facts. The idea is not to do division but to think of the facts in multiplication terms. For example, $36 \div 9$ is rethought as “what times 9 equals 36”. To use this strategy, students need to understand that division and multiplication are inverses of each other.

This strategy works for all division facts. It can be taught by looking at combining and partitioning: Take 3 groups of 5 counters and combine. Partition 15 into groups of 5. Repeat. State “15 divided by 5 is the same as 5 multiplied by what is 15”. Do the same for $4 \times 7 = 28$.

For each multiplication/division fact, there are four members of the fact family, for example:

$$3 \times 5 = 15, 5 \times 3 = 15, 15 \div 5 = 3, \text{ and } 15 \div 3 = 5.$$

Families for 4×7 and $36 \div 9$ are:

$$4 \times 7 = 28, 7 \times 4 = 28, 28 \div 7 = 4, 28 \div 4 = 7$$

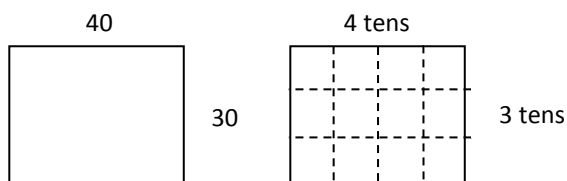
$$4 \times 9 = 36, 9 \times 4 = 36, 36 \div 9 = 4, 36 \div 4 = 9$$

3.3.3 Multiplication and division extended facts

Like with addition and subtraction extended facts, the multiplication and division extended facts are facts where the strategies used to develop and complete basic facts are used for other computations beyond the basic facts. Generally, extended facts relate to **multiples of ten** so the basic fact strategy is used for tens or hundreds etc., rather than for ones. For example, $6 \times 4 = 24$ means $60 \times 4 = 240$ and $4 \times 4 = 16$ means $40 \times 40 = 1600$. We will now look at how to build multiple-of-ten facts from ordinary facts. We will look at two methods.

Method 1: Arrays/area model

To find 40×30 , consider it as 4 tens $\times$ 3 tens. This can be shown as on right. It shows that 40×30 is 3 rows of 4 ten $\times$ ten squares, which is equal to 12 ten $\times$ ten squares. As ten $\times$ ten = hundred, (i.e. $10 \times 10 = 100$), then 12 ten $\times$ ten squares = 1200, so $30 \times 40 = 1200$.



Note: These facts are not as simple as addition and subtraction. They are based on the tens facts such as:

$$10 \times 10 = 100, 10 \times 100 = 1\,000, \text{ and } 100 \times 100 = 10\,000$$

Method 2: Patterns

Another way to do multiple-of-ten facts is to look for patterns using calculators as below.

- Step 1.** Complete with a calculator. (The students are asked to complete the examples.)

$2 \times 4 =$ _____	$5 \times 3 =$ _____	$8 \times 7 =$ _____
$20 \times 4 =$ _____	$50 \times 3 =$ _____	$80 \times 7 =$ _____
$2 \times 40 =$ _____	$5 \times 30 =$ _____	$8 \times 70 =$ _____
$20 \times 40 =$ _____	$50 \times 30 =$ _____	$80 \times 70 =$ _____
$200 \times 40 =$ _____	$500 \times 30 =$ _____	$800 \times 70 =$ _____
and so on ...		

- Step 2.** Look for a pattern. (The students are asked to look for any patterns that emerge that enable the multiple-of-ten facts to be calculated from the basic facts.) The pattern that should emerge is that the “factors of ten” are combined, that is, $4 \times 6 = 24$ means that 40×60 is 24 with three “factors of ten”, i.e. 24 000.

3. **Step 3.** Complete without a calculator. (To check that the students understand the patterns, they are asked to complete new exercises by using patterns and without a calculator.)

$$7 \times 6 = \underline{\hspace{2cm}} \quad 70 \times 6 = \underline{\hspace{2cm}} \quad 70 \times 600 = \underline{\hspace{2cm}} \quad 7000 \times 60 = \underline{\hspace{2cm}}$$

Note: Must be careful with examples like 40×50 because $4 \times 5 = 20$, so $40 \times 50 = 2000$ (three “factors of ten” but one is from the 4×5).

In a similar manner, division multiple-of-ten facts could be done with examples such as those below and the “complete with calculator – look for a pattern – complete without calculator” method.

$12 \div 6 = \underline{\hspace{2cm}}$	$21 \div 7 = \underline{\hspace{2cm}}$	$56 \div 8 = \underline{\hspace{2cm}}$
$120 \div 6 = \underline{\hspace{2cm}}$	$210 \div 7 = \underline{\hspace{2cm}}$	$560 \div 8 = \underline{\hspace{2cm}}$
$120 \div 60 = \underline{\hspace{2cm}}$	$210 \div 70 = \underline{\hspace{2cm}}$	$560 \div 80 = \underline{\hspace{2cm}}$
$1200 \div 60 = \underline{\hspace{2cm}}$	$2100 \div 70 = \underline{\hspace{2cm}}$	$5600 \div 80 = \underline{\hspace{2cm}}$
and so on ...		

Studying these examples can lead to the pattern that, when numbers are divided, the “factors of ten” are subtracted, that is, $56\,000 \div 80$ is 700 because $56 \div 8 = 7$ and three “factors of ten” subtract one “factor of ten” = two “factors of ten”.

Once again, students’ patterns can be checked by asking them to complete the following without a calculator:

$$45 \div 9 = \underline{\hspace{2cm}} \quad 450 \div 9 = \underline{\hspace{2cm}} \quad 45\,000 \div 90 = \underline{\hspace{2cm}} \quad 450\,000 \div 9\,000 = \underline{\hspace{2cm}}$$

Similarly to multiplication, the $3000 \div 60$ examples are difficult as one of the “factors of ten” is from the 6×5 – so the answer is 50 not 500. Give special attention to these and always get students to check by multiplying so they are sure that they are not using a “factor of ten” twice. For example, $40\,000 \div 80$ has a 5 in it. Because $8 \times 5 = 40$, there are only three “factors of ten” in the 40 000 to be considered; thus, $40\,000 \div 80$ is 500 not 5000.

3.3.4 Automaticity

As for addition and subtraction, the goal of learning basic facts is for students to develop automaticity to reduce cognitive load. We believe that developing basic fact automaticity should be the focus after learning of the strategies and investigation of the patterns and connections within the basic facts has occurred. The added benefit to learning the strategies is their applicability to computations beyond basic facts.

All the multiplication and division basic facts can be completed using one or more of the strategies described in this section. Some basic facts can be completed using more than one basic fact strategy. As multiplication is **commutative**, the **turnarounds** can lead to the use of different strategies. For example 5×6 can be worked out with a Use Ten strategy (5×6 : think 10×6 then halve it; half of 60 is 30) but it can also be thought of as 6×5 which enables a connection strategy to be used: 6×5 is $5 \times 5 + 1 \times 5$. Depending on the student’s number sense they may prefer one of these strategies over the other. Once students are confident with the commutativity principle this can be an advantage for working out basic facts but also for multiplication computations with numbers beyond basic facts.

Give students a list of random multiplication basic facts to complete. Keep all students together on the facts by reading each fact with a short time to write the answer. Mark and record the results on a multiplication grid.

The multiplication grid can be used to determine both the facts with which students make errors and the strategies needed to help students with their errors. If a student’s errors are placed on the grid, the position of the errors will determine which strategy or strategies are needed. For example, if students know one fact but not its turnaround, they need to be taught “turnarounds”.

×	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Strategy

	Use a rule
	Use doubles
	Use ten
	Use connections

Practising basic facts

Once students have been taught the various basic fact strategies, practising facts that use these strategies will assist in strengthening their understanding and application of the strategies as well as boosting their automaticity and recall of the basic facts. Activities that focus on speed can be effective for students with knowledge of the strategies. Ideas for practice mirror those listed in Addition and Subtraction including speed and overall automaticity with a mix of basic facts, student tracking worksheets, daily practice program and recording of successes and errors to guide future practice.

3.4 Strategies for computation

As stated in section 2.4, computation is the calculation of answers to the application of operations and the strategies used to complete these calculations. Calculation has always been a significant component of school mathematics programs. The National Research Council back in 1989 noted that “the teaching of mathematics is shifting from a preoccupation with inculcating routine skills to developing broad-based mathematical power” (p. 82). A key element of this mathematical power is the ability to calculate exact answers efficiently and with understanding. Australian curriculum documents in recent years have shown a shift in emphasis from written algorithms for computation to mental computation, the use of computation strategies and electronic tools like calculators and computers (Australian Curriculum, Assessment and Reporting Authority, 2011; Australian Education Council, 1991; Queensland Studies Authority, 2004). These statements hold for multiplication and division as well as for addition and subtraction. As for addition and subtraction, we advocate that mental computation strategies be used for numbers up to three digits and operations on larger numbers be carried out using calculators.

Three computation strategies are described in this section. Each of these strategies provides a means for students to transform calculations to make them more manageable. Each of the strategies applies to each of the operations. The three strategies are: **separation**, **sequencing** and **compensation**. Each section describes these strategies and provides sample activities for using the strategy for multiplication and division. However, it should be noted that, when one number is a single digit, or the operation requires one number not to be separated, it is difficult to classify the strategy as separation or sequencing. Thus, choices have had to be made about where methods will be placed and sometimes they are placed in both strategies.

Important note: We advocate always doing calculation in **problem situations** – do not give students worksheets with exercises in numeral form – **always give worksheets with written problems** where the student has to both create the setting out and undertake the calculation.

3.4.1 Separation strategy

The separation strategy works on transforming the numbers in a calculation by partitioning them or breaking them into parts. The parts are then worked on and the result is found by recombining the parts. There are three

ways that numbers in a calculation can be separated: (a) into place-value parts; (b) into compatible parts; or (c) into a combination of place-value and compatible parts.

As for addition, in the separation strategy for multiplication both (or all) of the numbers in the calculation are separated. Students need to understand the structure of our number system and how place values are multiplicatively related so that when numbers are separated they are considered in terms of their value not as digits. The separation strategy is the basis of the traditional written algorithm. The traditional algorithm has a tendency for students to work with digits in columns rather than numbers that have value. Because of this it is valuable to teach and practise separation strategies using PVCs and size materials such as bundling sticks, MAB and money placed on top of these PVCs. MAB and money can be used similarly to the bundling sticks. It is crucial that students experience multiplication with real bundling sticks, MAB and money before moving on to the virtual bundling stick, MAB and money activities. The separation strategy is used in a variety of contexts – it is useful for whole numbers (e.g. 346×8), decimal numbers (e.g. 4.65×0.8), measures (e.g. 3 m 342 mm $\times$ 5), mixed numbers (e.g. $3\frac{1}{6} \times 4$), and algebra (e.g. $2a \times (3a + 2b)$).

Separation strategy for multiplication

The nature of the operation of multiplication is that when numbers are partitioned the operation of multiplication needs to apply to all the parts. Students who do not understand this or have not had experiences to see why this works will make errors, particularly when multiplying two-digit numbers by two-digit numbers. While we advocate the use of calculators for larger numbers it is worth exploring this strategy with whole numbers as the application to understanding the distributive law and especially the application of this process to algebra are well grounded by starting here with whole and decimal numbers.

The error that students often make with multiplication of two-digit numbers using the separation strategy occurs when they multiply the tens and then multiply the ones and add the resulting numbers. This strategy is not surprising as it works like this for addition and subtraction. A student example of this error is presented on the right.

$$15 \times 16$$

$$10 \times 10 = 100$$

$$5 \times 6 = 30$$

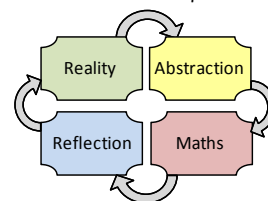
$$100 + 30 = 130$$

Using set materials

The following strategy for multiplication can be modelled using PVCs and set material, e.g. bundling sticks, MAB, money, etc. It can be considered **both** as a separation and a sequencing strategy. The following example shows the use of MAB to model a situation that was described to the students as the purchase of 4 meals at \$37 as a real-life context.

RAMR activity: Separation/sequencing strategy: Multiplication

Using the set model (MAB) for a multiplication (e.g. $37 \times 4 = 148$).



Reality

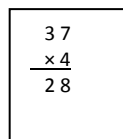
If it costs \$37 for a meal, how much do we pay for 4 meals? Act this out with money.

Abstraction

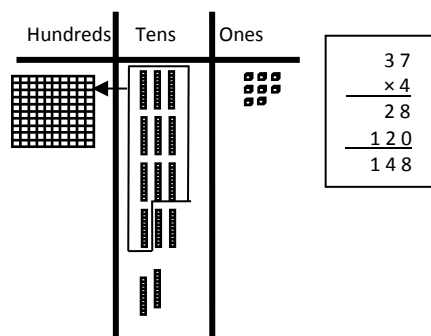
Students have to recognise that 4 meals at \$37 is reality for 37×4 and that this means 4 lots of 37.

Step 1. Put out 4 lots of 37 with tens and ones (here MAB but could be money) in Tens and Ones columns. Record as you go. See on right.

Hundreds	Tens	Ones	
			$\begin{array}{r} 37 \\ \times 4 \\ \hline \end{array}$



Step 3. Combine the tens and trade to make hundreds, moving the hundred to the Hundreds PV position. This means that 4 lots of 3 tens



Mathematics

$$\begin{array}{r} 37 \\ \times 4 \\ \hline 120 \quad (4 \times 30) \\ \underline{28} \quad (4 \times 7) \\ 148 \quad (Total) \end{array}$$

works and why.

area model

an area problem

side and 15 tiles

into sub-areas

16

15

10 6

10 5

10 6

10 5

10×10

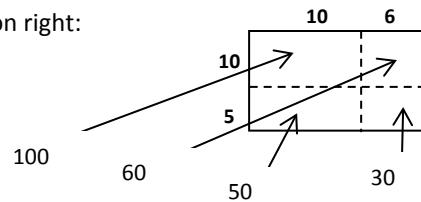
10×6

5×10

5×6

Step 4. Calculate each sub-area and then combine as on right:

$$100 + 60 + 50 + 30 = 240$$



This process can be recorded using the traditional written algorithm.

Note: This algorithm can be done both ways: starting with the smaller

PVs, i.e. $5 \times 6 \rightarrow 10 \times 10$, as well as starting with the larger

PVs, i.e. $10 \times 10 \rightarrow 5 \times 6$

$$\begin{array}{r} 16 \\ \times 15 \\ \hline 100 \quad 10 \times 10 \\ 60 \quad 10 \times 6 \\ 50 \quad 5 \times 10 \\ \hline 30 \quad 5 \times 6 \\ 240 \quad \text{Product} \end{array}$$

Separation strategy for division

The separation strategy where both numbers in a computation are broken into parts does not suit the operation of division.

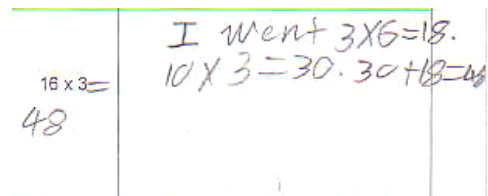
3.4.2 Sequencing strategy

The sequencing strategy is different to the separation strategy as follows: separation breaks both components into parts (usually on place value), while the sequencing strategy breaks one number up (can be place value but need not be) and keeps the other number whole. The parts are done with the whole number in sequence. In these examples, we will just show the steps through abstraction to mathematics. So the sequencing strategy works on the partitioning of numbers like the separation strategy. The operation being applied to the parts will vary depending on the operation of the overall computation. The sequencing strategy is often more efficient than the separation strategy as there are fewer parts when only one number is partitioned and so the entire computation will involve fewer steps.

As with separation strategies there are three ways that the number in a calculation being partitioned can be separated: (a) into place-value parts; (b) into compatible parts; or (c) into a combination of place-value and compatible parts. The sequencing strategy works for each operation. This strategy is best taught from an area understanding of multiplication.

Sequencing strategy for multiplication

Multiplication, being commutative, means that with the sequencing strategy either number can be left untouched, and then the other number can be partitioned and the multiplication done as a number of smaller computations. A student work sample on right shows this strategy being used for multiplication.



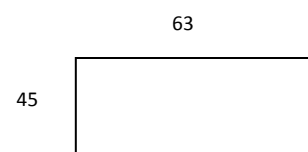
Using the array/area model

The sequencing strategy for multiplication also works well using the array/area model. The example below shows abstraction to mathematics. It should be preceded by a problem from the reality of the students.

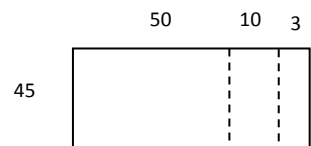
Example $45 \times 63 = 2835$ (e.g. The cost of the trip was \$45 per students.

How much had to be raised for 63 students to go on the trip?)

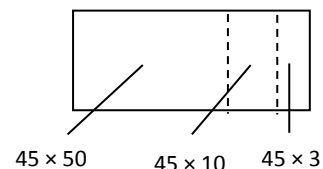
Step 1. Represent 45×63 with an area as shown on right.



Step 2. Keep 45 as is and break 63 into parts with which 45 can multiply easily. These would be 10s and 1s, and also 5s ($\frac{1}{2}$ a ten), 25s ($\frac{1}{4}$ a 100), and 50s ($\frac{1}{2}$ a 100). That is, the 63 would break into 50, 10 and 3.



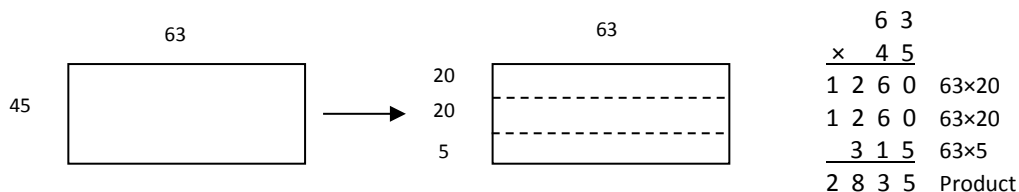
Step 3. Calculate the components – 45×50 is $\frac{1}{2}$ of 45×100 is $\frac{1}{2}$ of 4500 = 2250; $45 \times 10 = 450$; and 45×3 is $45 + 45 + 45 = 135$. Altogether is $2250 + 450 + 135$ and this equals 2835.



Step 4. This is imitated with an algorithm as on right:

$$\begin{array}{r}
 45 \\
 \times 63 \\
 \hline
 2250 \quad 45 \times 50 \\
 450 \quad 45 \times 10 \\
 135 \quad 45 \times 3 \\
 \hline
 2835 \quad \text{Product}
 \end{array}$$

We could also have kept the 63 unseparated and separated the 45 as below, and imitate the sequencing as in the algorithm – 63×20 is double 63 tens = 1260 and 63×5 is $\frac{1}{2}$ of $63 \times 10 = 315$.



Sequencing strategy for division

The sequencing strategy suits the operation of division as the concept of division is to partition into equal parts. Therefore the strategy works very well for division.

Using set materials

The sequencing strategy can be modelled using set materials, e.g. MAB, bundling sticks, money, etc. The following example shows how MAB can be used to model the sequencing strategy for division.

Students need to recognise that \$92 shared among 4 people is $92 \div 4$ and that this means starting with 9 tens and 2 ones and having 4 groups to share among. Below is a RAMR sequence for this example using sharing and the set model.

RAMR activity: Sequencing for division using set model

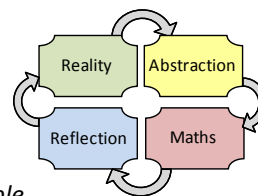
Reality

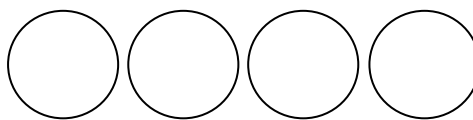
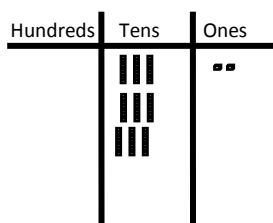
Look at a real-world situation for $92 \div 4$; say, where we share \$92 equally among 4 people.

Abstraction

Students need to recognise that \$92 shared among 4 people is $92 \div 4$ and that this means starting with 9 tens and 2 ones and having 4 groups to share among.

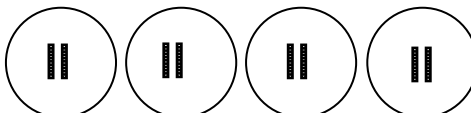
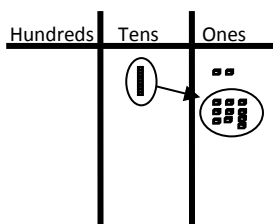
Step 1. Put the 92 with MAB on the PV chart and put out the 4 groups. Record as you go.





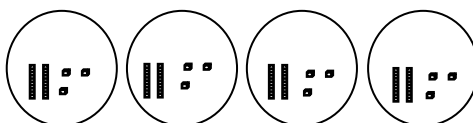
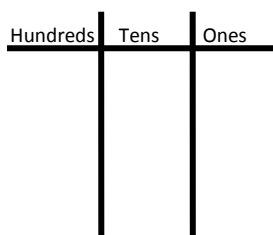
$$4 \overline{) 92}$$

Step 2. Share the tens, trade left-over tens for ones – record as you go. Ask: “How many tens can each person get? Can we give one ten to each person, can we give two tens?”, and so on. “How many tens did each person get? – put this number above.” “How many tens were used? – put this number below.” “How many tens left? – do the subtraction. Trade left-over tens for ones – how many ones left?”



$$\begin{array}{r} 2 \\ 4 \overline{) 92} \\ \underline{8} \text{ share tens} \\ 12 \end{array}$$

Step 3. Share the ones – record as you go. Ask: “How many ones can each person get? Can we give one one to each person, more than one one?” “How many ones did each person get? – put this number above.” “How many ones were used? – put this number below.” “How many ones are left? – do the subtraction.” [If ones left over, these become remainder].



$$\begin{array}{r} 23 \\ 4 \overline{) 92} \\ \underline{8} \text{ share tens} \\ 12 \\ \underline{12} \text{ share ones} \\ 0 \end{array}$$

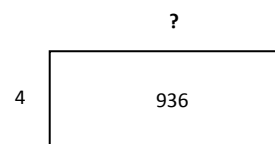
Mathematics

Students imagine materials in their mind and then complete algorithm without material – either pen and paper or in the mind.

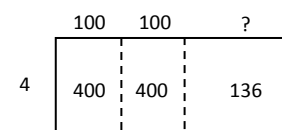
Using the array/area model

The array/area model that was used for multiplication is also effective for modelling the sequencing strategy for division. The initial area is partitioned into sections based on knowledge of multiples of the divisor. In the example below the divisor is 4 so the partitions need to be multiples of 4. Using known multiples to start with will reduce the remaining partition allowing other factors to be identified. For example, $936 \div 4 = 234$.

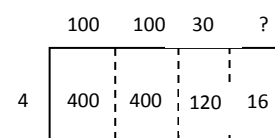
Step 1. Represent $936 \div 4$ with an area as on the right. It has to be thought of as a grouping – how many lots of 4 in 936? It is ? we have to find.



Step 2. Ask “Are there 100 fours?” – answer yes, there are 100 fours in 936, this is 400 and reduces 936 to 536, so adjust the area model. Then again ask “Is there a second 100 fours?” Yes, there is a second 100 fours in 536 and this reduces the amount to 136 (as on right).



Step 3. Ask “Are there some fours in the remaining 136?” – well 10 fours is 40 and there are three 40s or 120 in 136. So let’s remove them and adjust our diagram as on right.



Step 4. Look at what is left – the 16 is 4 fours, so this completes the top – and the diagram. It shows that 100 fours, 100 fours, 30 fours and 4 fours gives 936, so $100 + 100 + 30 + 4 = 234$ is the number of fours in 936.

	100	100	30	4
4	400	400	120	16

Step 5. Imitate this with the grouping algorithm on right.

$$\begin{array}{r}
 4 \overline{) 936} \\
 \underline{- 400} \\
 536 \\
 \underline{- 400} \\
 136 \\
 \underline{- 120} \\
 16 \\
 \underline{- 16} \\
 0
 \end{array}
 \begin{array}{l}
 100 \text{ lots of } 4 \\
 100 \text{ lots of } 4 \\
 30 \text{ lots of } 4 \\
 4 \text{ lots of } 4 \\
 \hline
 234 \text{ lots of } 4
 \end{array}$$

3.4.3 Compensation strategy

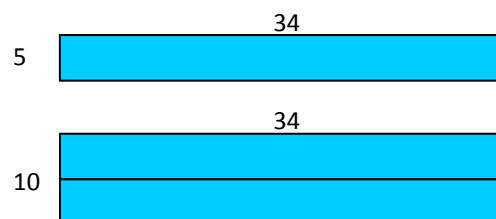
The compensation strategy does not partition the numbers involved in a computation but adjusts or changes the computation to use numbers that are more manageable. Sometimes there is a compensation for the change as an extra step after the change and sometimes the compensation is managed as part of the change. The focus is on making the computation more manageable. Often the changes that are made involve multiples of ten as it is easier to calculate with these numbers than others. Many of the compensation strategies use number-sense and basic fact understandings that students develop in early years of schooling.

Compensation strategy for multiplication

The compensation strategy for multiplication works by changing the numbers in the computation to make them more manageable. The difference with this strategy for the operation of multiplication is that the changes need to be multiplicative. For the example 64×89 , think “this is easier to do if it was $64 \times 100 = 6400$ and then, to compensate, I will subtract 64×11 ”, as on right.

6400	64×100
$- 640$	subtract 64×10
$- 64$	subtract 64×1
5696	which is 64×89

The compensation strategy for multiplication works in a similar way to the compensation strategy for addition in that when a change is made to both numbers it is by doing the opposite multiplicative change to each number that the product will remain the same. The example on right shows this relationship using 34×5 . The area model shows the product as a rectangle. It would be easier to multiply by 10



(5×2) so it is $34 \times 10 = 340$. The product is easier to work out but the area is clearly double what is needed. To return to the original computation the 340 needs to be halved which can be done mentally (170).

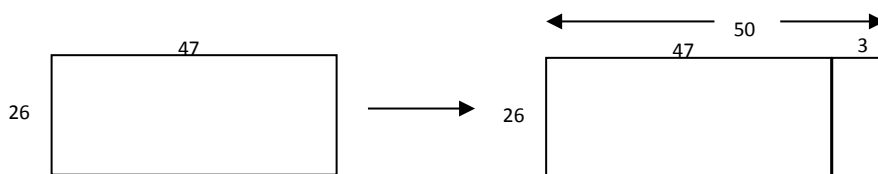
This aspect of the strategy works best for computations involving the multiplication of multiples of 5 (e.g. 5, 25, 50, 75, 125, etc.) as the adjustment can be to use double/half as the compensation. This strategy also works more easily if the other number is even but if not there will be a decimal answer.

28×25 Change to 14×50 ($\div 2$ and $\times 2$) Change to 7×100 ($\div 4$ and $\times 4$) $= 700$

Using the array/area model

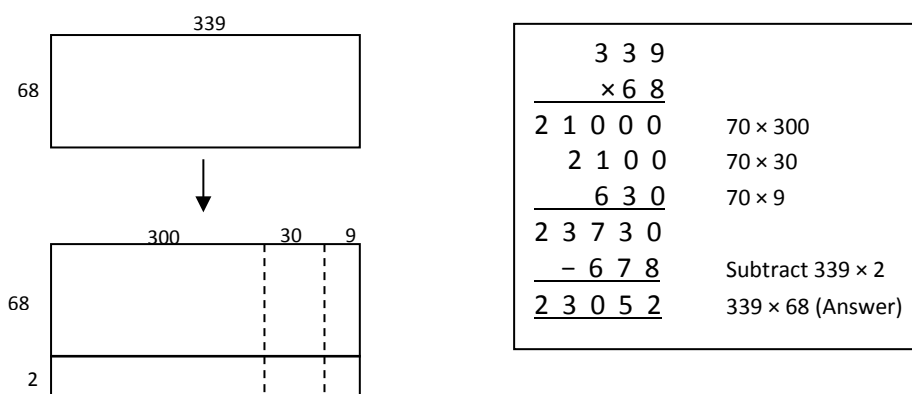
Often it is useful to use the sequencing strategy with the compensation strategy. The area model can help in some instances. The array/area model helps to make it clear what the compensation needs to be.

In the case of 26×47 the model shows the change to 26×50 instead of 26×47 and the extra 26×3 that will need to be removed to compensate for the change.



26×50 can be done by $\times 100$ and halve the result (as described above), $2600 \div 2 = 1300$. From this we need to remove 26×3 (78); $1300 - 78 = 1222$.

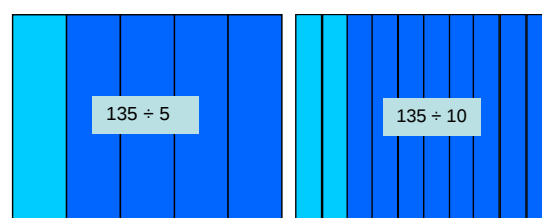
Example 339×68 combines the compensation strategy and the sequencing strategy. Thinking compensation, it is easier to do 339×70 and then subtract $339 \times 2 = 678$. To do 339×70 , think of 339 as 300 and 30 and 9 (sequencing). Thus, the area diagrams (if used) and the algorithm are as follows:



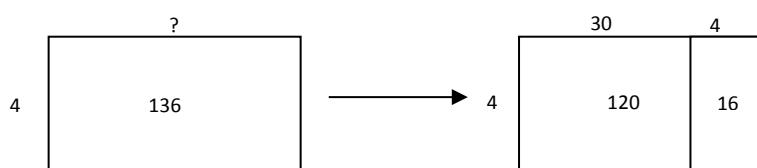
Compensation strategy for division

The compensation strategy for division works like the same strategy for multiplication. The changes to the numbers that make the computation easier need to be multiplicative. As with the compensation method for changing both the numbers to maintain the same difference with subtraction, the changes when made to both numbers for division need to be the same to keep the quotient the same.

The diagram on right illustrates $135 \div 5$ (135 is shown as the area of the rectangle and $\div 5$ is shown by divisions into 5 equal parts). The answer would be the area of one of those parts. It would be easier to divide by 10 (5×2) than by 5. If this is done there are more divisions for the same area. It is clear that to compensate for the change the result needs to be doubled. As with the compensation strategy for multiplication, this strategy works best with multiples of 5 that will make a compensation easier when doubled. Another example is provided below.



To calculate $136 \div 4$, change 136 to something that is easily divisible by 4, say 120. Then we have to compensate by adding the difference between 120 and 136 divided by 4. Thus $136 \div 4$ is $(120 \div 4) + (16 \div 4) = 30 + 4 = 34$.



The use of compensation for calculation $1652 \div 28$ requires thinking about how to make the computation easier. One way would be to change 1652 to something that 28 divides easily into. Two options are 2800 ($2800 \div 28 = 100$) and 1400 ($1400 \div 28 = 50$). Let us choose 1400 because it is closer to the original number. Now

1652 is $1400 + 252$, so to compensate for reducing 1652 to 1400, we have to add $252 \div 28$. Since 252 is $28 \times 10 - 28$, $252 \div 28 = 9$, so the answer is 59.

$ \begin{array}{r} 28 \overline{) 1652} \\ \underline{1400} \\ 252 \\ \underline{252} \\ 0 \end{array} $	50 lots of 28 <u>9 lots of 28</u> 59 lots of 28
--	---

3.5 Estimation strategies

Estimation is a method of finding approximate answers on large-number multiplications and divisions. With a calculator for accurate answers where required, it is adequate for students' computation in terms of YDM. Estimation is very useful in being able to guide whether a calculated solution is realistic.

There are four strategies for teaching estimation. They are described below. Like basic facts and algorithms, strategies are the most effective ways to teach estimation because, in most cases, once the strategies are learnt, they are useful in other areas.

3.5.1 Strategies

Front end

This strategy requires that only the highest place-value (PV) position(s) are considered in multiplication and division to give a simple estimate that, at times, is not that accurate, however, it can be a good start. It is called front end because one way to teach it is to put a card over all the sum but the "front end" of the highest PV position(s). The strategy uses multiple-of-ten basic facts. Examples are:

$4\,678 \times 34$ is found by $4000 \times 30 = 120\,000$

$42\,901 \div 362$ is found by $42\,000 \div 300 = 140$

Rounding

This strategy rounds the highest PV position(s) to the nearest of these PV positions and then multiplies and divides these numbers. For division, the rounding is not necessarily to the nearest 10, 100, 1000, and so on, but is to something which will divide. It is a more accurate method than front end in most cases (but not all).

The strategy uses multiple-of-ten basic facts. For example:

$4\,678 \times 34$ is found by rounding to $5\,000 \times 30 = 150\,000$

$42\,901 \div 362$ is found by rounding to $44\,000 \div 400 = 110$

Straddling

This strategy determines, by rounding up and down, approximate answers that are less and more than the example. The strategy uses multiple-of-ten basic facts and the number-size principles (particularly the **inverse relation principle**). For example:

$4\,678 \times 34$ is straddled by rounding up, $5\,000 \times 40 = 200\,000$, and rounding down, $4\,000 \times 30 = 120\,000$. So the multiplication is between 200 000 and 120 000. Looking at the numbers, they seem to be about halfway in between, so estimate is 160 000.

$42\,901 \div 362$ is straddled by rounding up the first number and down the second (but making the round up divisible by 3), $45\,000 \div 300 = 150$, and down the first and up the second (making the round divisible by 4). $40\,000 \div 400 = 100$. This is because of inverse relation for division; that is, a bigger second number means a smaller answer. So the estimation is between 150 and 100. Looking at the numbers, 42 901 changes by a similar amount in the round up and round down, but 362 changes more (proportionally) in the round down, so the higher straddle is less correct, so the estimate should be nearer to the lower straddle, say 120.

Getting closer

This strategy follows one of the above and uses the number-size principles and multiple-of-ten basic facts to get a more accurate estimate. Examples are:

The front-end strategy's estimate for $4\,678 \times 34$ is 120 000 and is obviously too low as both numbers are reduced, so a better estimate is about 160 000.

The rounding estimate for $42\,901 \div 362$ is 110. The rounding is $44\,000 \div 400$. Both numbers have increased, but the 362 has increased more (proportionately). So the answer is too low. A better estimate is 120.

3.5.2 Teaching and practising

As for addition and subtraction, for expertise in estimation, it is necessary to practise because accurate estimation combines basic facts, multiple-of-ten facts, number-size principles, place value and rounding in situations where decisions have to be made on which of these prerequisites to use – a lot to bring together at the same time as making decisions. Thus familiarity through practice is needed to reduce cognitive load.

There are many games/activities available to practise estimation. Two were given in section 2.5.2 and these can be adapted to multiplication and division. Three more games are given below.

Target

This game is suitable for one to many players/teams. Materials – calculators, table worksheet.

Players/teams enter the starting number in calculator and press $\times$. Players/teams guess what number multiplied by the starting number gives the target and type their guess then $=$ into calculator.

Players/teams use whether their guess was too high or too low to make a better guess. Players record guesses in the "too high" or "too low" columns on chart below. Score is the number of guesses to get to the answer.

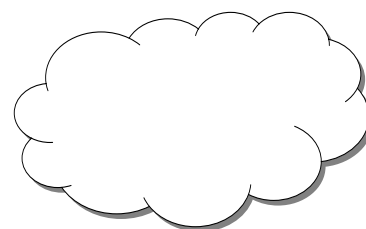
Player/team with least total guesses wins. Make up examples to put into the table.

Start	Target	Too high	Too low	Correct guess	No. of guesses
37	1702				
41	14 227				

Estimation Clouds

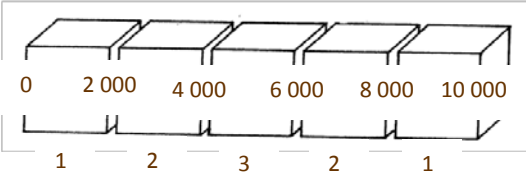
This game is suitable for two players/teams. Materials – calculators, cloud area, game board.

Numbers to be multiplied are chosen (such as 45×126) and placed in the cloud – numbers need to be chosen such that the biggest multiplication is just under a multiple of 10, here 10 000.



Players/teams take turns in selecting any two numbers from cloud. Circle them. (They can be used only once.)

Using a calculator, players/teams multiply the two numbers, find the box for their answer (between 0 and 2 000, between 2 000 and 4 000, and so on), and keep track of their points. For this example, the boxes are divided between 0 and 10 000 and given a point value of 1, 2 or 3.



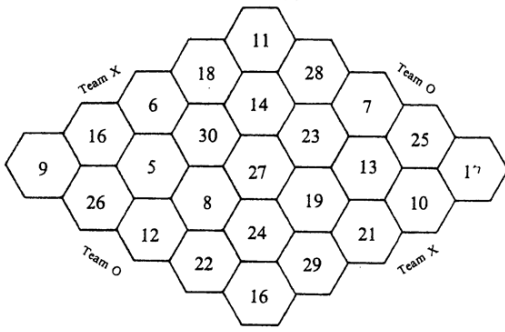
The winner is the player/team with the greater number of points after all numbers have been used.

Estimation Hex

This game is suitable for two players/teams. Materials – calculators, game board (hexagons with answers in each space), a collection of numbers or exercises in a box.

Each player/team chooses a colour for their counters. Players/teams take turns in selecting any two numbers or exercises from the box.

Using a calculator, players/teams multiply the numbers selected (or calculate the exercises – can have division examples). Players/teams look for the answer on the game board on right, and cover it with their colour counter if it is there.



The first player/team to get a path across the game board wins.

3.6 Word problem solving

This section looks at one of the end points of multiplication and division, namely problem solving. It discusses how to teach students how to interpret and construct word problems, and how to be better problem solvers.

Previously we looked at the basic concepts of multiplication and division as follows:

MULIPLICATION	DIVISION
Combining Factor-factor→Product (know factors and not product)/Inaction (know subsets)	Partitioning Product→factor-factor (know product and one factor, and not other factor)/Inaction (know superset)

Now we look at further concepts so that we have a complete set to enable us to solve all types of multiplication and division word problems.

MULTIPLICATION	DIVISION
Inverse action (backward combining)	Inverse action (backward partitioning)
Change-comparison (end wanted)	Change-comparison (start or change wanted)
Combinations	Combinations

This section first looks at these inverse action, change-comparison and combinations concepts before looking at how to interpret and solve and construct multiplication and division word problems.

The relationship between computation/calculation and problem solving also needs further discussion. The need for computation, particularly in out-of-school settings, arises from problem situations. However, the ability to solve problems comes from the meanings/operating side of operations not the computation/calculating side – mainly from understanding of concepts. Before any calculation can be undertaken, students need to: (a) recognise that

calculation will be needed; (b) determine the numbers and operations that will have to be used; and (c) determine if an exact or approximate answer is required. Only then can a method of calculation be determined and the calculation be carried out. The student should also interpret the solution in terms of its reasonableness to the characteristics of the situation (NCTM, 1989).

However, in schools, students often complete many calculations that do not relate to problem situations. Effectively, they are practising the “carry out the calculation” part of this sequence only. The whole sequence is important because problems in the real world are not presented as computations to be completed. Students need practise with the whole sequence for problem solving from identifying that a computation is required to checking the answer they found is a reasonable solution to the problem. This section describes the interpretation and solution of problems using problem-solving strategies as well as the construction of problems, which is a necessary skill that practises the reversing idea advocated by YDM.

3.6.1 New concepts

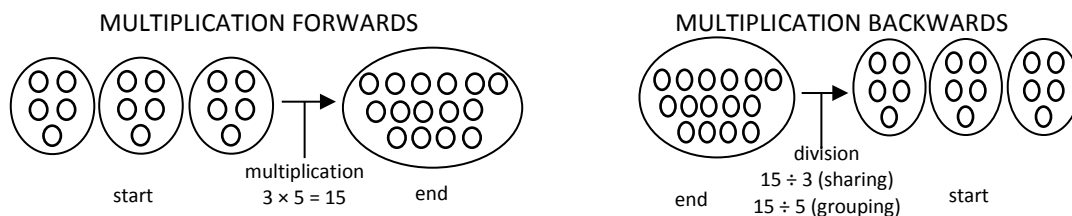
There are three new concepts or clusters of concepts: the inverse or backward concepts, the change-comparison concepts, and the combinations concept. The latter is very important because as array models move to area models, the concept changes from combining/partitioning to combinations.

Inverse concepts – operating forwards and backwards

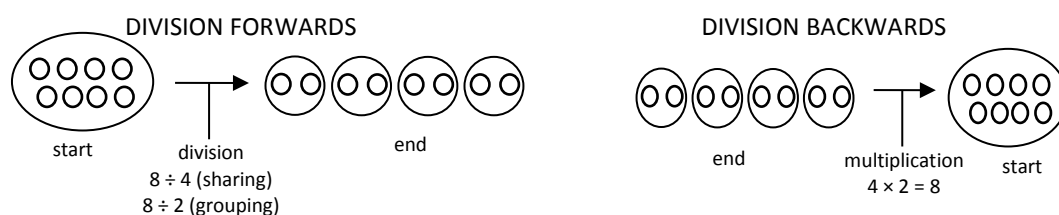
As an action, each operation, multiplication and division, **takes place over time**. Thus, we can know the start of the problem and want the end (forward problems), or we can know the end and want the start (backward problems). Similar to addition and subtraction, if we run a combining (multiplication) activity backwards, we get a partitioning (division) and vice versa. This means that backward problems reverse the operation. In other words, backward combining is division and backward partitioning is multiplication. Examples are provided below to further highlight these varying actions. The intention, like above, is to provide teachers with a rich source of different problem types to work with their students. The table below provides a summary.

OPERATION	FORWARD	BACKWARD
Multiplication	Combining equal groups to get a product.	Reversing partitioning situations and looking for the number of groups or the number in each group.
Division	Starting from the product and partitioning by sharing or grouping into factors.	Reversing combining situations and looking for the size of the whole group (the product).

Thus, **multiplication forwards** is starting from factors and combining into a product, while **multiplication backwards** is reversing partitioning situations and looking for the product, as below.



Division forwards is starting from product and partitioning, by sharing or grouping, into factors, while **division backwards** is reversing combining situations and looking for the size of the whole group, as below.



This means that backward multiplication is division and backward division is multiplication, giving the following analysis of multiplication and division in the real world.

MULTIPLICATION	FORWARD	BACKWARD
$3 \times 7 = 21$	"3 men went out fishing. Each man caught 7 fish. How many fish were caught in total?"	"7 men went out fishing and all caught the same number of fish. In total, they caught 21 fish. How many fish did each man catch?"
DIVISION	FORWARD	BACKWARD
$21 \div 7 = 3$	"Joe caught 21 fish and shared them equally among his 7 friends. How many did each friend get?"	"Joe caught some fish. He divided them equally among the 3 families. Each family got 7 fish. How many fish did Joe catch?"

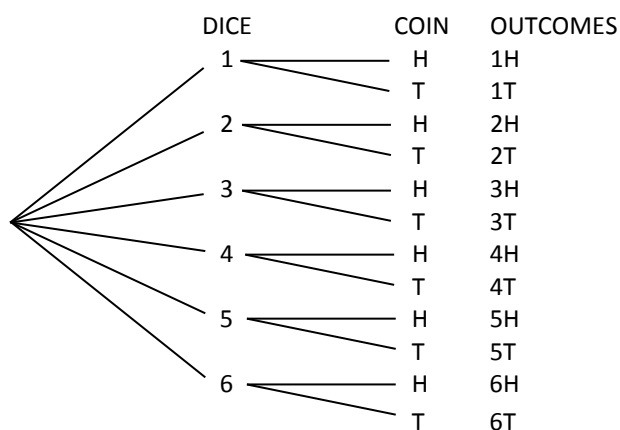
These concepts/meanings should be taught using RAMR structure in the same manner, that is: (a) covering all representations – telling stories $\leftrightarrow$ acting out situations and modelling them with materials $\leftrightarrow$ drawing the situations $\leftrightarrow$ language $\leftrightarrow$ symbols; and (b) using both set and number-line models and ensuring stories are in a variety of situations for both these models.

Changing-comparing

Multiplication is a change-comparison where there is a "number" times more than the starting number. For example, *John caught 3 fish, Jack caught 5 times as many fish as John, Jack caught 15 fish*. Division is the opposite: *Jack caught 5 times as many fish as John, Jack caught 15 fish, John caught 3 fish*. Change and combinations are the basis of powerful methods to do percent, rate and ratio (see Chapter 5) and to do algebra (see YDM Algebra book).

Combinations

In this meaning, two things are put forward to make a composite. For example, 3 shirts can be mixed and matched with 5 pants to make 15 outfits; or length of 3 m can put with a width of 5 m to make a rectangle whose area is 15 m^2 . This type of multiplication is common in probability where tree diagrams are used. Suppose that a die is thrown and then a coin is tossed. This gives the following diagram:



The number of outcomes is the number of outcomes from a die (6) $\times$ the number of outcomes from the coin (2) and this is 12.

Examples of this meaning are: (a) multiplication – *5 men and 7 women meet; how many possible couples?* ($5 \times 7 = 35$); and (b) division – *A die is tossed and a spinner is spun; there are 18 outcomes from these two actions; how many options in the spinner?* ($18 \div 6 = 3$).

Arrays to area: Two bases for multiplication

A $3\text{ m} \times 4\text{ m}$ area can be thought of as 3 rows of 4 square metre areas. This is the array model and is part of the combining-equal-groups concept. However, the area can also be thought of as 3 m by 4 m giving 12 m^2 and this is part of the combinations concept. This is because 3 rows of 4 m^2 is the same as 3 bags of 4 marbles, while $3\text{ m} \times 4\text{ m} = 12\text{ m}^2$ is the same as 3 pants $\times$ 4 coats = 12 outfits.

This means that the change from array model to area model **must be treated with care** and **time spent on students seeing that area can be seen as an array, but is also a model on its own leading straightforwardly to calculation**. However, the most important outcome is that it enables multiplication (and, by inverse, division) to be seen as having only two really different forms: number $\times$ rate, and number $\times$ number.

Number by rate. In combining, we have a number of sets, bags, hops, groups, and so on and a number of objects per these sets, bags, and so on. For example, 3 bags of lollies with 4 lollies in each bag is 3 bags $\times$ 4 lollies/bag, that is, number by rate. Similarly, 5 hops of 2 m is 5 hops $\times$ 2 m/hop, again number by rate.

For change/comparison, we have John with 6 times the money Fred has and Fred having \$5; this is 5 Fred-\$ $\times$ 6 John-\$ per 1 Fred-\$ which is number by rate again. If we think of multiplication being number of groups $\times$ number in each group, we see that this is number $\times$ rate.

Interestingly, number $\times$ rate is additive in nature in that the 3 bags with 4 lollies in each bag can be thought of as $4 + 4 + 4$. Also, the answer is the same as the rate factor, that is, the subject of the rate becomes the subject of the answer:

$$3 \text{ bags} \times 4 \text{ lollies/bag} = 12 \text{ lollies}$$

Number by number. In combinations, we have something different. Here 3 coats and 4 pants form 12 outfits, and we have number $\times$ number. Similarly, 3 options in the spinner (1,2,3) and 6 options in the die (1,2,3,4,5,6) becomes 18 outcomes (1-1, 1-2, ..., 1-6; 2-1, 2-2, ..., 2-6; 3-1, 3-2, ... 3-6).

This also holds for the area model in its abstract form – $3\text{ m} \times 4\text{ m} = 12\text{ m}^2$ – this is number $\times$ number.

Interestingly, number $\times$ number is not structurally additive in that it is not 6 lots of \$7 or something like that – it is 6 shirts $\times$ 7 pants – it goes from 6×7 direct to 42 (actually a 6×7 matrix) in one step. Also, the attribute of the answer is different to the attributes of the numbers, although related – in the examples above, clothes goes to outfits, options to outcomes, and m to m^2 .

3.6.2 Solving problems

This section looks at interpreting problems, determining which operation to use, and the role of strategies and plans of attack. Information on all of these has been given in section 2.6 (for addition and subtraction).

The steps in solving were as follows.

1. **Determining whether multiplication/division or addition/subtraction.** This is fully explained in subsection 2.6.2. Multiplication and division is when groups are the same size and one number is not the objects but the number of groups/sets; addition and subtraction is when groups can be different size and all numbers refer to the same objects.
2. **Interpreting to determine whether multiplication and division.** To interpret problems, look at all types of problems (including forward and backward) in terms of factor-factor-product. Then get students to recognise what is a factor and what is a product, and that if the product is unknown it is multiplication and if a factor is unknown it is division.

One way to do this is to set examples and ask students to put numbers and question mark (if unknown) beside the headings Factor and Product. Then use these numbers to circle which operation (multiplication or division) should be used (*the answer is not required*). An example of this is given in the table on the next page.

Another suggestion is to use a factor-factor-product (F-F-P) diagram as on right to write the numbers into and determine if product is unknown (multiplication) or known (division).

PRODUCT?	
FACTOR?	FACTOR?

PROBLEM	Multiplication or division?
I took money out of bank, and shared it among my 9 nephews, each got \$50, how much did I take out of the bank? Factor: _____ Factor: _____ Product: _____	$\times$ $\div$
Joan went shopping, she bought 7 dresses, each cost the same, she spent \$420, how much for each dress? Factor: _____ Factor: _____ Product: _____	$\times$ $\div$
Sue and Jane went out together, Sue had 3 times the money Jane had, Sue had \$39, how much did Jane have? Factor: _____ Factor: _____ Product: _____	$\times$ $\div$

Note: Some educators use part-part-total for all operations instead of part-part-total for addition and subtraction and factor-factor-product for multiplication and division. It does not matter which you use if students understand how it works and can use it practically – but we are advocating factor-factor-product as more correct because when you have $3 \times 4 = 12$, it is difficult to think of the 3, which is the number of groups, as a part of the 12 objects – it is really correctly a factor.

3.6.3 Constructing problems

The best way to become an expert at interpreting word problems is to learn how to construct them.

Forwards–backwards stories

Give students an equation such as $5 \times 3 = 15$ and ask them to write forward combining, backward partitioning, separating, change-comparison, combinations and inaction/F-F-P problems for set, array and number-line models and for different day-to-day contexts (e.g. shopping, driving, walking, playing sport, and so on). The RAMR cycle can still be useful.

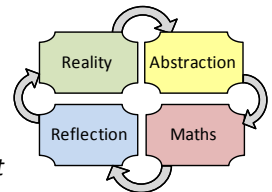
Some hints are as follows.

- Materials.** Give students materials to work with (e.g. the students in the class, toy models of people, cars, animals and so on, materials to set up a shop) and ask them to make up and act out a story for $5 \times 3 = 15$. (*Note:* One of the best teaching methods seen for this approach was by a teacher who organised the students to do a claymation of their story.)
- Social interaction roles.** Set up students in groups of three to make up and act out stories by assigning roles of director (leader – makes decisions when there is an impasse), continuity (continuously checks that group is not making any errors), and script writer (records and reports on the story and how will be acted).
- Triad approach.** If given $5 \times 3 = 15$, write a straightforward combining story with all numbers known, then rewrite with each of the numbers as the unknown – this will give three stories, one where the answer is found by multiplication and two where the answer is found by division. After this, write the associated division stories for $15 \div 3 = 5$ and $15 \div 5 = 3$, and then rewrite these with numbers unknown. Once again, there are three stories for each sum. And in the three cases, one of the stories is multiplication (the one with 15 unknown) and two are division. Furthermore, students can be encouraged to see that 15 unknown for $5 \times 3 = 15$ is forward and 15 unknown for $15 \div 3 = 5$ and $15 \div 5 = 3$ is backward.
- Use factor-factor-product.** If given $5 \times 3 = 15$ and asked for a multiplicative comparison story, the 5 and 3 are factors so they have to be two factors in the story (e.g. the number of groups and the number in each

group, or the start number and the change multiple, the things being combined, or the length and the width), while the 15 is the product (e.g. the total number, the end number, or the number of combinations or area) and is unknown. So have a start of 5, multiply by 3 and then find the end number. Then write this into context as a story.

5. **Extend an existing problem.** Give students a problem, then ask the students to add further words to the problem and change the context of the problem to make it harder/easier.
6. **Opposite operation.** Another interesting way to construct stories is to think of words that most people associate with an operation, then try to write problems using those words (and actions) that have the opposite operations.

RAMR activity: Constructing problems for multiplication and division



Reality

Make a list of as many words the students can think of for multiplication and division that they use every day (e.g. “times”, “share”, “lots of”). Ensure that this list is flexible – sometimes, the same word may be used in different ways for multiplication and division.

Abstraction

Take each of these words and act out their normal meaning with two knowns and one unknown. Now give the unknown a number, and act out the problem with one of the other numbers unknown – does this change multiplication to division or vice versa? Draw diagrams of the two problems.

- I was given 4 lots of books where each lot had 3 books, how many books do I have?
First factor is 4, second factor is 3, product is unknown – so it is multiplication.
- I was given 4 lots of books, each lot had the same number of books, I now had 12 books, how many books in each lot?
First factor is 4, second factor is unknown, product is 12 – so it is division.

Mathematics

Write the two problems and determine the factors and the product and which is unknown. Relate the problems to the factor-factor-product approach, write the example down as a “sum”, and calculate the answers.

Reflection

Try to generalise the process. For example, if multiplication, how do we change that to division? Is there a pattern? Also this change from multiplication to division can be done on symbols – see on right.

$$\begin{array}{r} 47 \\ \times 4 \\ \hline ? ? ? \end{array} \longrightarrow 1 ? 8 \div 4 = 4 ?$$

Using triadic relationship

The following tables of examples show how the triadic relationship can be used to construct a variety of problems. Mix up these problems and give them to students, this will really determine whether they understand multiplication and division.

PROBLEM (ALL NUMBERS)	THREE PROBLEMS (EACH WITH ONE UNKNOWN)
4 women each ran 3 km of the relay race. The race was 12 km long.	Women ran a relay race where each ran 3 km. The length of the race was 12 km. How many women? 4 women ran a relay race. Each ran the same distance. The length of the race was 12 km. How far did each woman run? 4 women each ran 3 km of the relay race. How long was the race?
Jess shared the \$21 equally among 7 people. Each person received \$3.	Jess shared the money equally among 7 people. Each person received \$3. How much was shared? Jess shared the \$21 equally among a group of people. Each person got \$3. How many people received a share? Jess shared the \$21 equally among 7 people. How much did each person get?

3.6.4 Multi-step problems

Multi-step problems were looked at in detail in subsection 2.6.4. The inclusion of multiplication and division along with addition and subtraction allows more variety. Part and factor can be considered as similar and total and product also similar, allowing the F-F-P and the P-P-T figure to be extended (using P-P-T to cover for all operations) to cover complex problem solving using all the operations. For example:

- I bought a coat for \$113 and shared the change from \$200 with my three nieces, how much did each niece get?* This can be recorded on the P-P-T figure as on right.

\$200			
Coat \$113	\$29	\$29	\$29

- John bought four radios for \$105 each. He spent \$25 more than Jack who bought a suit for \$360 and a box of chocolates. How much did the chocolates cost?* This can also be put on the P-P-T figure as on right. The first step is to get the total – the largest number – this is the four radios.

\$105	\$105	\$105	\$105
Suit \$360		Chocolates ?	\$25

3 nieces \$87
\$35

Six strategies or strategy clusters can be used, as described in subsection 2.6.4: break into parts, systematic/exhaustion, make a drawing/act it out, given-needed-wanted, restate/simpler, and check/learn. Polya's four stages can also be used – See, Plan, Do, Check. Finally, a framework can be used as above and in 2.6.4.

3.7 Directed number, indices, and primes and composites

In this subsection, we look at multiplication and division of directed numbers, indices, and prime and composite numbers.

3.7.1 Multiplication and division of directed numbers

This is an extension of the work done in subsections 2.7.1 and 2.7.2 on addition and subtraction of directed numbers. Since directed numbers are associated with number lines, multiplication will be repeated jumps forward along the line and division repeated jumps backward.

Prerequisites

Ensure that prerequisites are known as in subsection 2.7.1, that is, that students understand:

- the idea of opposites;
- modelling directed numbers on number lines;

- use of language, symbols and conventions; and
- reality problems and situations for directed number.

Also ensure that students understand the operations:

- what multiplication and division mean; and
- how multiplication and division operate on number line (in particular).

Multiplying and dividing teaching methods

1. **Multiplication on number line.** Set up $+2$ as being 2 forward or to right and -3 as 3 backward or to left, and $4 \times$ as 4 jumps facing forward/right and $-5 \times$ as 5 jumps facing backward/left, and so on. Thus:
 - $3 \times (-2)$ is 3 jumps facing forward with each jump 2 backwards; this gives 6 backward, that is, $3 \times (-2) = -6$; and
 - $(-3) \times (-2)$ or $-3 (-2)$ is 3 jumps of 2 backward facing backward; this give 6 forwards, that is, $(-3) \times (-2) = +6$.
2. **Division on number line.** Set up $\div 2$ as number of 2 jumps forward, but facing either forward or backward, to give initial jump and $\div -2$ as number of 2 jumps backward, but facing either forward or backward, to give initial jump. Facing forward means the number of jumps is positive, facing backwards means the number of jumps is negative. Thus:
 - $6 \div -2$ is how many 2 jumps forward will give 6 backward, this can only be done if facing backward, so answer is 3 jumps forward but facing backwards, so $6 \div -2 = -3$; and
 - $-6 \div -2$ is how many 2 jumps backward will give 6 backward, which is 3 facing forward, so $-6 \div -2 = 3$.
3. **Acting out.** Use the setting up above to act out with students on large number line or counters on small number line the various multiplication and division options.
4. **Patterns.** The complexity of multiplication and division on a number line means that patterns can be the best way to teach the multiplication and division rules for directed numbers, as follows.

$4 \times 2 = 8$	$-4 \times 2 = -8$	$2 \div 1 = 2$	$2 \div -1 = -2$
$4 \times 1 = 4$	$-4 \times 1 = -4$	$1 \div 1 = 1$	$1 \div -1 = -1$
$4 \times 0 = 0$	$-4 \times 0 = 0$	$0 \div 1 = 0$	$0 \div -1 = 0$
$4 \times -1 = -4$	$-4 \times -1 = 4$	$-1 \div 1 = -1$	$-1 \div -1 = 1$
$4 \times -2 = -8$	$-4 \times -2 = 8$	$-2 \div 1 = -2$	$-2 \div -1 = 2$
... and so on	... and so on	... and so on	... and so on

Using structure

If students have acquired knowledge of operations as a structure, then $3 \times (-4)$ would be 3 lots of inverse of 4 which is inverse of 12, so $3 \times (-4) = -12$. However for $(-3) \times (-4)$ it is necessary to work off a pattern. We can show $(-1) \times (-1)$ is $+1$ and that $(-n) = (-1) \times n$. Thus $(-3) \times (-4)$ is $(-1) \times 3 \times (-1) \times 4 = 12$.

3.7.2 Indices

This involves representing numbers by using indices for repeatedly multiplying the same number, for example:

- $7 \times 7 = 7^2$, $7 \times 7 \times 7 = 7^3$, $7 \times 7 \times 7 \times 7 = 7^4$;
- by patterns, $7 = 7^1$, $1 = 7^0$, $\frac{1}{7} = 7^{-1}$, $\frac{1}{7} \times \frac{1}{7} = 7^{-2}$, $\frac{1}{7} \times \frac{1}{7} \times \frac{1}{7} = 7^{-3}$, and so on; and
- $72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2 = 2^3 3^2$.

To introduce these ideas, we can use patterns, investigations and structure.

1. **Patterns for indice sequence.** These show that the notation is reasonable:

	Four 7s multiplied together	$= 7^4$
Divide by 7	Three 7s multiplied together	$= 7^3$
Divide by 7	Two 7s multiplied together	$= 7^2$
Divide by 7	One 7 multiplied on own	$= 7^1$
Divide by 7	Identity 1 is	$= 7^0$
Divide by 7	One divided by one 7	$= 7^{-1}$
Divide by 7	One divided by two 7s	$= 7^{-2}$
	... and so on	

2. **Introduction by investigation.** Set up an investigation that gives indices. For example – if we can go up stairs 1, 2, 3 or any number of steps at a time, how many different ways can we go up 1 step, 2 steps, 3 steps, ... any number of steps (pattern?):

STEPS	WAYS	STEPS	WAYS
1	1	2	2
3	4 or 2×2	4	8 or $2 \times 2 \times 2$
5	16 or $2 \times 2 \times 2 \times 2$	6	32 or $2 \times 2 \times 2 \times 2 \times 2$
...	and so on		

The result of this investigation is that n steps means $2 \times 2 \dots \times 2$ $n-1$ times – we can use this to introduce the notation 2^n , 2^{n-1} and so on. It is a similar result in the old story of placing a grain of rice on the first square on a chess board, 2 grains on the second square, 4 grains on the third square, and so on, continuing to double until you get to the 64th square – how many grains on the 64th square?

This approach makes the introduction of the indices more palatable.

3. **Patterns for multiplication rules.** Use pattern to get across rules for multiplying indices. For example:

$$\begin{aligned}
 4^1 \times 4^1 &= 4 \times 4 = 4^2 & 4^3 \times 4^1 &= 4 \times 4 \times 4 \times 4 = 4^4 \\
 4^2 \times 4^1 &= 4 \times 4 \times 4 = 4^3 & 4^3 \times 4^2 &= 4 \times 4 \times 4 \times 4 \times 4 = 4^5 \\
 4^3 \times 4^1 &= 4 \times 4 \times 4 \times 4 = 4^4 & \dots \text{and so on} & \\
 \dots \text{and so on} & & \text{generalisation } 4^p \times 4^q &= 4^{p+q}
 \end{aligned}$$

$$\begin{aligned}
 4^3 \div 4^1 &= 4 \times 4 \times 4 \div 4 = 4 \times 4 = 4^2 \\
 4^3 \div 4^2 &= 4 \times 4 \times 4 \div (4 \times 4) = 4 = 4^1 \\
 4^3 \div 4^3 &= 4 \times 4 \times 4 \div (4 \times 4 \times 4) = 1 = 4^0 \\
 4^3 \div 4^4 &= 4 \times 4 \times 4 \div (4 \times 4 \times 4 \times 4) = \frac{1}{4} = 4^{-1} \\
 \dots \text{and so on} & & \text{generalisation } 4^p \div 4^q &= 4^{p-q}
 \end{aligned}$$

It is also possible by patterns as above to show that, for example, $(6^p)^q = 6^{p \times q}$ and $(6^p)^{-q} = 6^{p \times -q}$, and more generally that $(n^p)^q = n^{p \times q}$ and $(n^p)^{-q} = n^{p \times -q}$.

Using structure

Structurally, $6 \div 2$ is not an operation; it represents $6 \times (\frac{1}{2})$ where $\frac{1}{2}$ is the multiplicative inverse of 2. In terms of inverses $\frac{1}{2}$ is 2^{-1} , thus $6 \div 2 = 6 \times 2^{-1}$. This means that $\frac{1}{\frac{1}{2}}$ is inverse of $\frac{1}{2}$ which is the inverse of the inverse of 2 which is 2, that is, that $(2^{-1})^{-1} = 2$. This can also be seen as 2 by considering how many $\frac{1}{2}$ s in 1. Patterns can show that $(n^{-p})^{-q} = n^{pq}$.

3.7.3 Primes, composites and other number types

An interesting study is the factors of a number: 7 has only two (1 and 7); 9 has three (1, 3 and 9), 6 has four (1, 2, 3 and 6), and 12 has six (1, 2, 3, 4, 6 and 12). (Note: A factor is something which divides into a number or which equals the number when multiplied by another factor.)

Number types. Factors have been used to classify numbers as follows (look up the internet for more names):

- if only two factors, numbers are **prime**;
- if more than two factors, numbers are **composite**;
- if factors other than itself add to the number (e.g. factors of 6 other than 6, $1 + 2 + 3 = 6$), the number is **perfect**; and
- if factors other than itself add to more than the number (e.g. factors of 12 other than 12, $1 + 2 + 3 + 4 + 6 = 16$), the number is **abundant**.

A number is the multiple of its factors (e.g. $6 = 1 \times 6$ and $= 2 \times 3$), but factors can appear more than once and can use indices (e.g. $72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$).

Sieve of Eratosthenes. One way to find the primes is to **remove the composites**. Do this by removing all multiples of smaller primes, e.g. 2, 3, 5 and so on (4s are removed with the 2s). Use a 100 board – go through and remove all multiples of 2 by crossing them out, then remove in order, multiples of 3, 5, and 7. (Note: Why do we stop at 7?) What is left is the primes.

Prime factors and factors of factors. If a factor of a number is a composite (e.g. 6, which is a composite, is a factor of 24), then the prime factors of that factor are factors of the original number. In fact, any factor of a factor is a factor of the original number (e.g. $6 = 2 \times 3$ so 2 and 3 are factors of 24 and this is true as $2 \times 12 = 24$ and $3 \times 8 = 24$). This can be shown by examples, but how can it be shown to be always true?

Get students to **explore factors** – for example, $28 = 7 \times 4 = 14 \times 2 = 2 \times 2 \times 7 = 1 \times 28$; so 1, 2, 4, 7 and 28 are factors. Let's look at factor 4 – it is composite (1, 2 and 4 are factors of 4) and it is a factor of 28 because $4 \times 7 = 28$. This means that 2 is a factor of 28. Is that true? Get students to construct their own rules.

Look at it in terms of a factor-factor-product diagram as on right. Factor 3 is a factor of Factor 1 (Factor 3 $\times$ Factor 4 = Factor 1) and Factor 1 is a factor of the number (Factor 1 $\times$ Factor 2 = Number). Obviously Factor 3 is also a factor of the number.

Number	
Factor 1	Factor 2
Factor 3	Factor 4

3.8 Extension to algebra

In this section, extending the meaning of multiplication and division to variables, that is, algebra, is done for one multiplication or division problem first and then for problems with two operations from multiplication, division, addition and subtraction. Two-operation problems often involve more than one step. Make these problems straightforward at first.

3.8.1 Introducing variable

An effective method for extending understanding of multiplication and division to variables is to give problems with more than one unknown. This can be reinforced with physical materials such as cups and counters (and other materials which have ones and something that can be said to hold or represent any number).

One operation – cups and counters. It is possible to support students' understanding of algebraic terms like $4x$ by using cups and counters. In this, the counters represent numbers and the cups represent an unknown or variable amount of counters depicted by letters. Thus, $4x$ is 4 unknowns or 4 cups.



Two operations. This is an extension to problems with two operations and is introduced the same way – by leaving one number out. Thus, we have the following:

I bought 3 pies for \$5 and a chocolate for \$6	$3 \times 5 + 6$
I bought 3 pies, and a chocolate for \$6	$3p + 6$

I received a third of the \$18 and an extra \$5	$18 \div 3 + 5$
I received a third of the money and an extra \$5	$m \div 3 + 5$

This can be reinforced for multiplication with cups and counters. This allows different meanings to be explored as follows:

$4y + 1$ is 

$4(y + 1)$ is 

3.8.2 Interpreting and constructing

Interpreting problems in algebra can proceed in the same way word problems were interpreted in section 3.6 but now unknowns are given letters. For example, *The farmer put the same number of cows in each of 5 pens. How many cows?* This problem can be stated now as $a \times 5 = c$, or as $c \div 5 = a$, depending on whether we interpret it as product unknown or factor unknown. We can even have all numbers unknown. For example, *The cows were put in some pens. How many cows?* This problem can be stated now as $a \times b = c$, or as $c \div b = a$, or $c \div a = b$, depending on whether we interpret it as product unknown or factor unknown. In fact, algebra allows much more flexibility on whether it is multiplication or division.

Constructing problems in algebra is very flexible. For example, “Write a story for $x \times y = z$ ” can be anything: *I bought two chocolates for each of my four friends and paid money for them.*

RAMR activity: Introducing variable for one operation

Reality

Revise turning symbols into stories, e.g. 4×7 could be “I have 4 bags each with 7 chocolates”.

Abstraction

Discuss with students scenarios that don’t have all the information, e.g. “I bought 4 boxes of chocolates. How much did I spend?”. Discuss the following:

- Why can’t this problem be answered? [Don’t know the cost of the chocolates]
- What information would I need to be able to calculate something? [Could calculate spending if given the cost of the chocolates]
- What else is possible? [Calculate the cost of the chocolates if given the total spent]

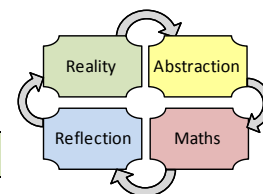
Mathematics

Let students give amounts and calculate some answers (e.g. “what if.....”) and write the equations.

- What if the chocolate cost \$3? – the equation is $4 \times 3 = ?$ (answer is \$12)
- What if the chocolate cost \$1.50? – the equation is $4 \times 1.50 = ?$ (answer is \$6)

Discuss if it is possible to write the equation if the spending is known and the cost of the chocolate is not known. Allow students to devise their own ways of representing this before introducing letters.

Consider example: “I bought 4 boxes of chocolates; I spent \$24”. Move through the following sequence of equations. If students are struggling to write an equation, have the students write the problem verbally on the thinkboard.



$$4 \times \text{chocolate cost} = 24$$

$$4 \times \text{chocolate cost} = 24$$

$$4 \times C = 24 \text{ or } 4C = 24 \text{ (introduce the notation where the multiplication sign is not used)}$$

Allow students to construct problems using their own method before introducing letters as the Western mathematics symbol for unknown and variable. Ensure students experience simple expressions such as $3 \times a$ (3 bags with the same number of apples in each bag) and equations like $3a = 15$.

Note: The YDM Algebra book will have other ways of introducing this – patterns, function machines, mass balance and length models. It is also possible to use something to represent the variable – cups with counters for numbers, envelopes, strips of paper, jumps along a line, and so on.

Reflection

Use these symbols to do two types of activities:

- from symbols with unknowns/variable, write the story without all the numbers; and
- from the story without all the numbers, write the symbols with unknowns/variable.

Have students give the three stories for $5 \times 9 = 45$ where each of the stories has a different part of the equation “unknown”.

- There were 5 teams of 9 player, how many players?
- There were 5 teams all with the same number of players; altogether, there were 45 players, how many players on each team?
- There were teams of 9 players, there were 45 players altogether, how many teams?

Do this for more than two unknowns – I bought a number of pies, how much did I spend?

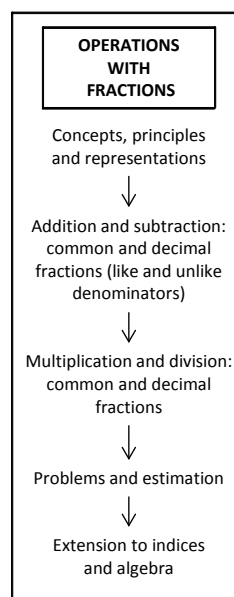
$$c = \text{cost of one pie}; n = \text{number of pies}; T = \text{total cost} \rightarrow \text{Equation } n \times c = T$$

The above can also be done for division. For instance: “I share \$24 equally among 6 friends” is $24 \div 6$ (which has an answer of 4). Thus, “I share \$24 equally among my friends”, where I do not give the number of friends, is $24 \div n$ or $24 \div x$ (depending on what I choose for the letter). Likewise, “I share the money equally among 6 friends”, is $m \div 6$.

4 Operations with Fractions

This chapter covers addition, subtraction, multiplication and division of fractions, for both decimal and common fractions. The sequence for this chapter is based on the figure on the right.

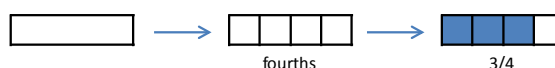
The chapter begins with meaning, revisiting the ideas from Chapters 2 and 3 to set up concepts, principles and representations (section 4.1). The chapter moves on to addition and subtraction, first as an extension of whole numbers for decimal numbers and separated into two stages (like and unlike denominators) for common fractions and mixed numbers (section 4.2). The next section looks at multiplication and division, beginning with the area model and moving on to algorithms (section 4.3). Both sections 4.2 and 4.3 focus on strategies. The chapter is completed with problem solving and estimation (section 4.4) and extensions to indices and algebra (section 4.5). Both sections 4.4 and 4.5 are considered in relation to whole-number operations.



4.1 Concepts, principles and representations

The concepts and principles for the four operations remain as they were developed in Chapters 2 and 3. However, the representation changes when moving from grouping ones to partitioning ones – there are two options:

- common fractions** where the denominator gives the number of parts that the whole has been partitioned into and the numerator the number of these parts that are being considered; and



- decimal fractions** where numbers are presented in terms of fractional place values based around partitions of ten – tenths, hundredths thousandths, and so on.

TH	H	T	O	t	h	th
			0	• 7	5	

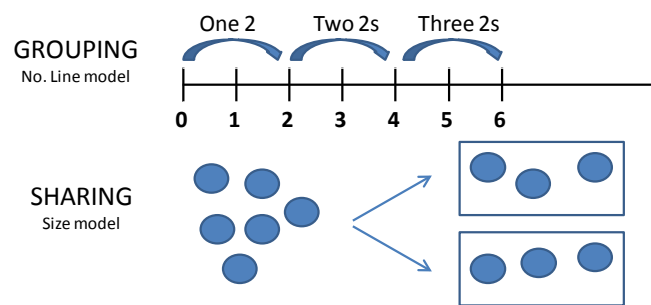
75 hundredths

4.1.1 Concepts and models

Summarising Chapters 2 and 3, the concepts of the operations are as follows:

OPERATION	CONCEPTS	INTEGRATING CONCEPT					
Addition	Joining, inverse of taking away, change/comparison (end unknown), inaction (where have to find superset)	Part-part-total (total is unknown)	<table><tr><td colspan="2">?</td></tr><tr><td>Part</td><td>Part</td></tr></table>	?		Part	Part
?							
Part	Part						
Subtraction	Taking away, inverse of joining, change/comparison (start or change unknown), inaction (where have to find subset)	Part-part-total (part is unknown)	<table><tr><td colspan="2">Total</td></tr><tr><td>Part</td><td>?</td></tr></table>	Total		Part	?
Total							
Part	?						
Multiplication	Combining equal amounts, inverse of partitioning, change/comparison (end unknown), combinations	Factor-factor-product (product is unknown)	<table><tr><td colspan="2">?</td></tr><tr><td>Factor</td><td>Factor</td></tr></table>	?		Factor	Factor
?							
Factor	Factor						
Division	Partitioning into equal amounts (sharing and grouping), inverse of combining, change/comparison (start or multiplier unknown), combinations	Factor-factor-product (a factor is unknown)	<table><tr><td colspan="2">Product</td></tr><tr><td>Factor</td><td>?</td></tr></table>	Product		Factor	?
Product							
Factor	?						

For example $6 \div 2$, grouping is how many twos in six (there are three twos in six), while sharing is how much does each person get if six is shared equally between two (each person gets three) – see below.

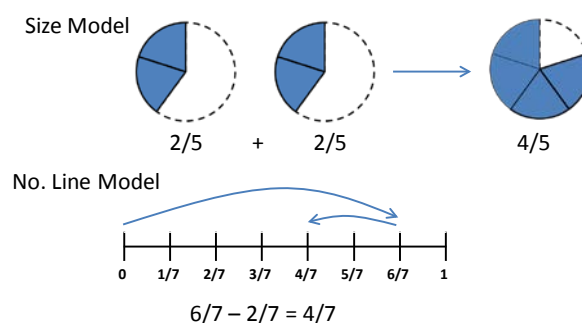


Again from Chapters 2 and 3, the models are as follows. We use the term **size** to replace the set model because with fraction you will not have sets of objects, you will have $\frac{2}{3}$ of an object – the model has to show the difference between $\frac{2}{3}$ and $\frac{3}{4}$ so it does this by showing size.

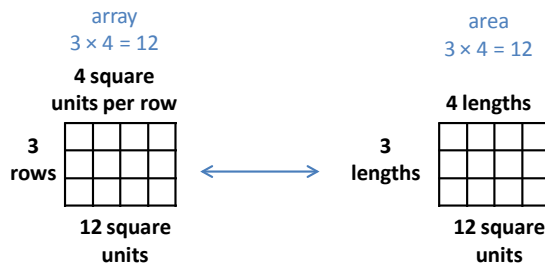
OPERATION	MODELS	INTEGRATING CONCEPT
Addition	Set/size, number line	$2 + 3$
Subtraction	Set/size, number line	$7 - 3$
Multiplication	Set/size, number line, arrays, area	3×4
Division	Set/size, number line, arrays, area	$12 \div 3$

With regard to fractions, some of these meanings and models become difficult to apply. For example:

1. **Addition and subtraction.** The nature of addition and subtraction means that it is necessary to **add or subtract like things** (e.g. ones to ones and tens to tens). This can be extended to fractional place value in decimal numbers (e.g. tenths added or subtracted to tenths) and for like denominators (e.g. 2 fifths + 2 fifths = 4 fifths, 6 sevenths – 4 sevenths = 2 sevenths) – see below. However, if denominators are unlike, then something must be done before addition and subtraction can occur (e.g. 2 fifths + 3 sevenths cannot be done unless changed to the same thing, i.e. thirty-fifths – see subsection 4.2).

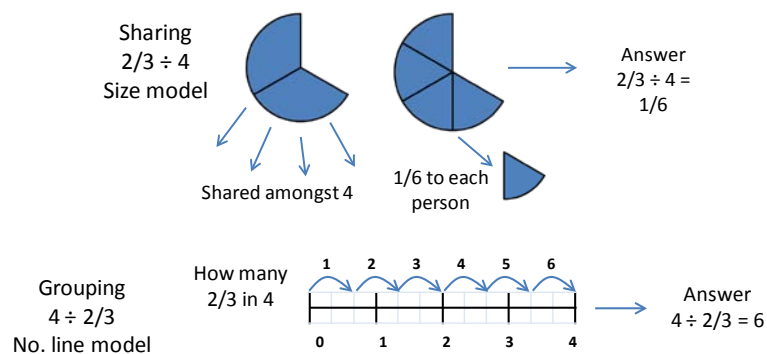


2. **Multiplication.** Multiplication has been built around combining equal amounts; however, this assumes a whole number of these amounts. A fraction multiplied by a whole number can be represented and modelled using the set and number line models, but to represent multiplication of a fraction by a fraction we need to use an extension of the array model – the area model. It is possible for 3 rows of 4 square units to be seen as a 3×4 rectangle. In both examples (see below), the answer is 12 square units. However, in arrays, the 3 is the number of rows, the 4 is the number of square units in each row and the 12 is square units (i.e. number times rate, $3 \text{ rows} \times 4 \text{ square units/row}$); but in area, the 3 is lengths of the side of the square unit, the 4 is the same while the 12 is square units (i.e. number $\times$ number, $3 \text{ lengths} \times 4 \text{ lengths} = 12 \text{ square units}$). The extension of arrays to area must be taken with care as it is a change from the combining concept to the combinations concept.



As well as this, multiplication of fractions makes things smaller (i.e. is very different to multiplication of whole numbers); for example, multiplying by $\frac{1}{3}$ changes 6 to 2.

3. **Division.** Division also is difficult with fractions – sharing requires a whole number of sharers, while grouping requires a whole number of groups (see below). To understand fraction divided by fraction requires using the proper mathematical meaning of division, that is, multiplication by reciprocal (e.g. $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3}$).



As well, division by fractions makes things larger (i.e. is very different to division by whole numbers which makes things smaller); for example, dividing by $\frac{1}{3}$ changes 6 to 18.

4.1.2 Principles, language and symbols

All the number-size principles/big ideas continue to hold for fraction operations. This means the following:

OPERATIONS	MAJOR NUMBER-SIZE PRINCIPLES
Addition and Multiplication	- If either number becomes larger/smaller, the answer goes the same way for either the first or second number: $3 + 4 = 7 \rightarrow 3 + 5 = 8$; $5 > 4$ and $8 > 7$. - There is no situation in which something becoming larger/smaller makes the answer go the opposite way. - If either number becomes larger/smaller, and the answer stays the same, the other number has to go the opposite way: $3 + 4 = 7 \rightarrow 2 + 5 = 7$; $2 < 3$ and $5 > 4$.
Subtraction and Division	- If the first number becomes larger/smaller, the answer goes the same way: $8 - 3 = 5 \rightarrow 10 - 3 = 7$; $10 > 8$ and $7 > 5$. - If the second number becomes larger/smaller, the answer goes the opposite way: $24 \div 3 = 8 \rightarrow 24 \div 4 = 6$; $4 > 3$ and $6 < 8$. - If either number becomes larger/smaller and the answer stays the same, the other number has to go the same way: $11 - 4 = 7 \rightarrow 9 - 2 = 7$; $9 < 11$ and $2 < 4$.

All the Field principles hold for addition and multiplication as the following shows. The equals or equivalence principles also hold (as we have seen, this partitions all the fractions into equivalence classes – sets all equivalent to the same starting fraction).

FIELD PRINCIPLES	APPLICATION/EXAMPLES
Closure	Fraction + fraction = fraction; fraction $\times$ fraction = fraction
Identity	0 for addition and 1 for multiplication: $\frac{2}{3} + 0 = \frac{2}{3}$; $\frac{2}{3} \times 1 = \frac{2}{3}$
Inverse	A fraction has additive inverse of $-\text{fraction}$ (e.g. $-\frac{3}{4}$ is the inverse of $\frac{3}{4}$); a fraction has multiplicative inverse of reciprocal of fraction (e.g. $\frac{4}{3}$ is the inverse of $\frac{3}{4}$)
Associativity	$(\frac{2}{11} + \frac{3}{11}) + \frac{4}{11} = \frac{2}{11} + (\frac{3}{11} + \frac{4}{11})$; $(\frac{2}{3} \times \frac{3}{5}) \times \frac{4}{7} = \frac{2}{3} \times (\frac{3}{5} \times \frac{4}{7})$
Commutativity	$\frac{2}{11} + \frac{5}{11} = \frac{5}{11} + \frac{2}{11}$; $\frac{2}{3} \times \frac{3}{5} = \frac{3}{5} \times \frac{2}{3}$
Distributivity	$\frac{2}{3} \times (\frac{3}{11} + \frac{5}{11}) = (\frac{2}{3} \times \frac{3}{11}) + (\frac{2}{3} \times \frac{5}{11})$

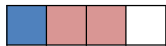
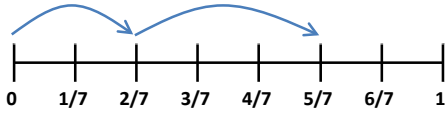
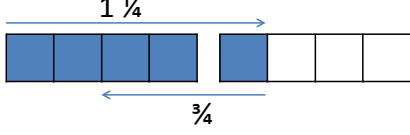
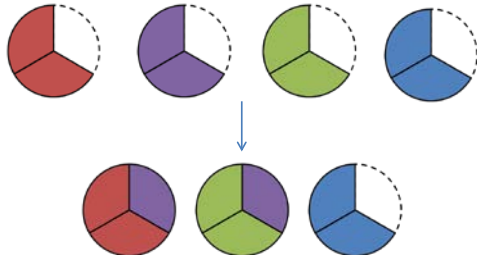
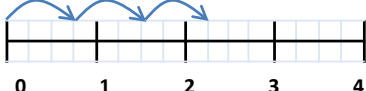
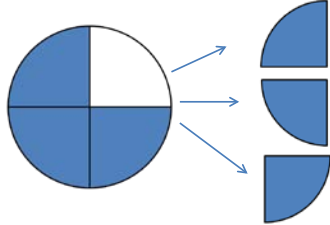
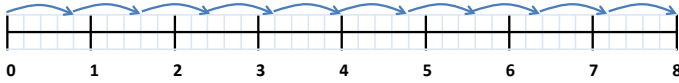
Representations for fractions mean the language, models (physical, virtual and pictorial) and symbols for these fractions. Models have been covered, so this leaves language and symbols.

There are two symbol representations for fractions – common fractions and decimals (e.g. $\frac{3}{8}$ and 0.375 – both of these represent three-eighths). The decimal fraction symbols are an extension of whole-number symbols but common fraction symbols are very different. Overall the decimal symbols predominate in a metric society. However, the language for both decimal and common fractions is the same and is the common fraction language, but related to only some denominators – the place-value positions of tenths, hundredths, thousandths and so on. Thus, $\frac{3}{8}$ is stated as “three-eighths” and 0.375 is stated as “three hundred and seventy-five thousandths”.

4.1.3 Using models

It is important at the start of fraction work to transfer the concepts, principles and representations (including models) from whole numbers to decimal and common fractions. In particular, it is important to be able to use the set/size, number line and array/area models for operating with fractions. However, it is also important to modify the models for the needs of fractions. The modifications will be covered in the next section. The following table comprises examples where the whole-number ideas can be easily transferred.

Use the models beside the operations to work out the answer to the operations for the models given.

OPERATION	MODEL	WORKING	ANSWER
$\frac{1}{4} + \frac{2}{4}$	Set/Size	 $\frac{1}{4} \quad \frac{2}{4}$	$\frac{3}{4}$
$\frac{2}{7} + \frac{3}{7}$	Number line		$\frac{5}{7}$
$1\frac{1}{4} - \frac{3}{4}$	Set/Size		$\frac{2}{4}$
$4 \times \frac{2}{3}$	Set/Size		$\frac{8}{3}$ or $2\frac{2}{3}$
$3 \times \frac{3}{4}$	Number line		$\frac{9}{4}$ or $2\frac{1}{4}$
$3 \div \frac{3}{4}$	Set/Size		4
$\frac{3}{4} \div 3$	Set/Size		$\frac{1}{4}$
$8 \div \frac{4}{5}$	Number line		10

4.2 Addition and subtraction of fractions

Addition and subtraction of fractions requires students to have a good understanding of the many concepts of addition and subtraction and of what a fraction is. Addition and subtraction require like things to be added or subtracted. This will not be a problem for like denominator fractions nor for decimal numbers where we will add tenths to tenths and hundredths to hundredths in an extension of whole-number addition and subtraction. However, for unlike denominators, fractions need to be renamed so that the operation is acting on like things, that is, like denominators. Students need a well-developed understanding of equivalence ideas relating to fractions to be able to complete computations involving fractions. This section includes like denominator fractions, decimal fractions, and unlike denominator fractions. Mixed numbers will be integrated into the common fraction work.

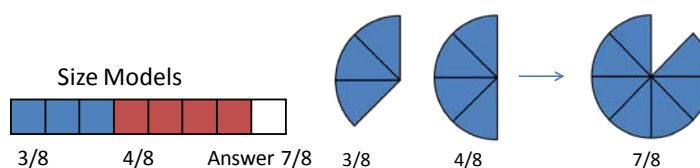
4.2.1 Addition/subtraction for like denominator common fractions

When fractions have the same denominators the numerators can be added using any whole-number computation strategy that would work for the numbers involved. So the separation, sequencing and compensation strategies described for whole numbers would each apply to the addition and subtraction of fractions with the same denominators. Examples are provided below for both addition and subtraction.

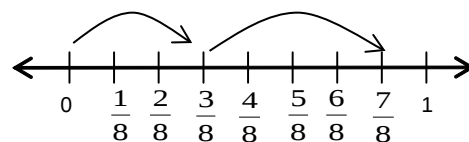
There are two options here – size and number line models. Students need to understand addition as parts combined to make a total. The models represent the parts and the total and allow the computation strategies to be used. The models act similarly to whole numbers. There are three steps.

1. Simple operations

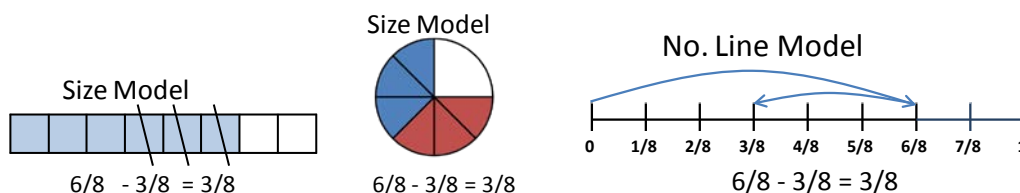
Example $\frac{3}{8} + \frac{4}{8}$. For the **size model**, shading areas will work as will joining eighth pieces (see below).



For the **number-line model**, the shading or the materials are replaced with hops along a line as on right. Eighths are shown on a line 0–1 divided into eight equal parts. This can be done by 3 hops and then 4 hops, or by a hop of $\frac{3}{8}$ and a hop of $\frac{4}{8}$.



Example $\frac{6}{8} - \frac{3}{8}$. Subtraction is done the same way as below.



2. Patterns

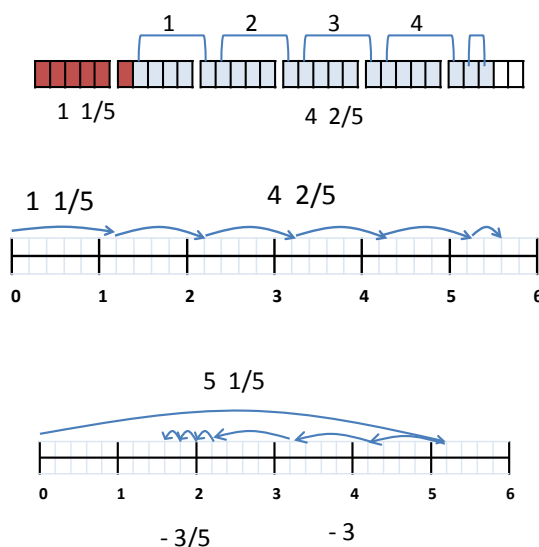
Complete examples as in (1) above, record what is done in symbols, look at symbols and discover pattern:

$$\frac{m}{p} + \frac{n}{p} = \frac{(m+n)}{p} \quad \text{and} \quad \frac{m}{p} - \frac{n}{p} = \frac{(m-n)}{p}$$

3. Mixed numbers

Example $1\frac{1}{5} + 4\frac{2}{5}$ and $5\frac{1}{5} - 3\frac{3}{5}$. Mixed number addition and subtraction can be done in the following three ways.

(a) Use models as in Step 1:



(b) Change to improper fractions and use pattern:

$$1\frac{1}{5} + 4\frac{2}{5} = \frac{6}{5} + \frac{22}{5} = \frac{28}{5} = 5\frac{3}{5}; \quad 5\frac{1}{5} - 3\frac{3}{5} = \frac{26}{5} - \frac{18}{5} = \frac{8}{5} = 1\frac{3}{5}$$

(can also do **additive sequencing**: $3\frac{3}{5} \rightarrow \frac{2}{5}$ to 4 $\rightarrow \frac{1}{5}$ to $5\frac{1}{5}$, so difference is $1\frac{3}{5}$;
and **compensation**: $5\frac{1}{5} - 3\frac{3}{5} = 5\frac{1}{5} - 4$ by adding $\frac{2}{5}$ to each number).

(c) Use a whole-part chart:

Whole	Part	Whole	Part	Whole	Part	Whole	Part
1	$\frac{1}{5}$			4	$\frac{6}{5}$		
+ 4	$+\frac{2}{5}$			- 3	$-\frac{3}{5}$		
5	$\frac{3}{5}$			1	$\frac{3}{5}$		

The above examples use sequencing for (a); sequencing, additive sequencing and compensation for (b); and separation for (c). This also extends work done in the YDM Number book for (a) and (c) – showing that, where possible, operations should be done as an extension of numeration work, not as a new topic.

4.2.2 Addition/subtraction for decimal numbers

Addition and subtraction of decimal numbers can be done as an extension of whole-number operations. In fact, where possible, we recommend doing decimal addition and subtraction **as the reflection stage** of RAMR lessons of whole-number addition and subtraction. We will now look at the three strategies for algorithms and how they are applied to decimals.

1. Separation

The separation strategy for addition works by adding like place values and then combining. Students need to understand decimal place values to partition numbers into parts and manage the value of these parts. An example of this strategy being used with decimals is provided on the right. Using this strategy also has potential to assist students' understanding of place value as they need to consider the value of the digits in the computation not just the digits themselves as can be the case when using the traditional written algorithm.

23.6 + 45.58

$$23 + 45 = 68 \text{ (whole numbers)}$$

$$0.6 + 0.5 = 1.1 \text{ (tenths)}$$

$$0 + 0.08 = 0.08 \text{ (hundredths)}$$

$$68 + 1.1 + 0.08 = 69.18$$

Three examples of the separation strategy for subtraction involving decimals are provided below. The first example manages the decimal component of the number as hundredths. The second example manages the decimal component one place value at a time. The third example shows regrouping – ways of separating numbers to manage subtraction so as to manage computations requiring bridging.

34.78 – 23.55

Separate 34.78 into $34 + 0.78$

Separate 23.55 into $23 + 0.55$

$$34 - 23 = 11 \text{ (whole numbers)}$$

$$0.78 - 0.55 = 0.23 \text{ (hundredths)}$$

$$11 + 0.23 = \mathbf{11.23} \text{ (sum of parts)}$$

34.78 – 23.55

Separate 34.78 into $34 + 0.7 + 0.08$

Separate 23.55 into $23 + 0.5 + 0.05$

$$34 - 23 = 11 \text{ (whole numbers)}$$

$$0.7 - 0.5 = 0.2 \text{ (tenths)}$$

$$0.08 - 0.05 = 0.03 \text{ (hundredths)}$$

$$11 + 0.2 + 0.03 = \mathbf{11.23} \text{ (sum of parts)}$$

29.315 – 18.27

Separate 29.315 into $29 + 0.31 + 0.005$

Separate 18.27 into $18 + 0.27$

$$29 - 18 = 11 \text{ (whole numbers)}$$

$$0.31 - 0.27 = 0.04 \text{ (hundredths)}$$

$$0.005 - 0 = 0.005 \text{ (thousandths)}$$

$$11 + 0.04 + 0.005 = \mathbf{11.045} \text{ (sum of parts)}$$

2. Sequencing

This strategy will also work with decimals. For simpler examples, it is possible to develop a number board that involves decimals and to use it in the same way as a number board with whole numbers (the jumps will change so that to go down a row will not be +10 but will be +0.1 or 0.01 depending on the numbers used). It is also fairly straightforward to use a number line similar to whole numbers to do the operations. An example is given on the right.

26.75 + 43.88

Start with 26.75

Separate 43.88 and add progressively

$$+ 40 = 66.75$$

$$+ 3 = 69.75$$

$$+ 0.25 = 70$$

$$+ 0.25 = 70.25$$

$$+ 0.3 = 70.55$$

$$+ 0.05 = 70.60$$

$$+ \mathbf{0.03} = \mathbf{70.63}$$

$$= 43.88$$

3. Compensation

All of these representations can be used to assist with addition and subtraction computations involving decimals (see two examples below). The adjustments and compensations will require sound understandings of decimal place value. Introducing these strategies and practising them could work to improve student understanding of decimals as well.

23.86 + 18.75

Change to $24 + 18.61$ (+0.14 and -0.14)

Change to $30 + 12.61$ (+6 and -6)

$$= \mathbf{42.61}$$

123.32 – 58.75

Change to $123.57 - 59$ (+0.25 and +0.25)

Change to $124.57 - 60$ (+1 and +1)

$$= \mathbf{64.57}$$

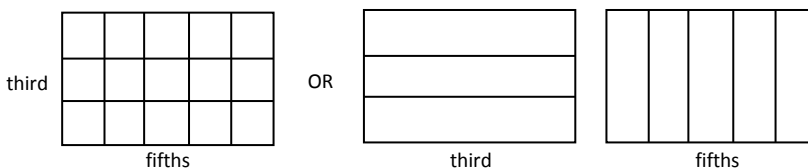
4.2.3 Addition and subtraction for unlike denominator fractions

To complete operations with fractions that have different denominators students need to use understandings of equivalent fractions. There are three models that can be used to do this.

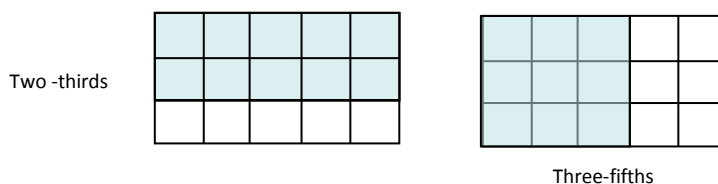
1. Array/Area model

Representing fractions using an area model was advocated as a representation of the one of the concepts of a fraction – a fraction is equal parts of a whole. When fractions with different denominators are to be added there needs to be consideration of equivalent fraction versions of the fractions being added. This can be helped using an area model.

Example $\frac{2}{3} + \frac{3}{5}$. To represent these fractions the whole needs to be the same. Using the one rectangle to show both fractions assists with the maintenance of the whole and in seeing the equivalent fractions. One dimension of the rectangle is used to show each fraction. For example, in the figure on the right, one dimension is divided into thirds and the other into fifths. If this is difficult for students to visualise they can start with two congruent rectangles and divide one into thirds and the other into fifths and overlay them for the same effect.



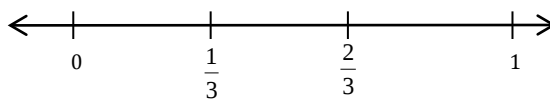
Using the combined diagram on right, the fractions in the computation can be represented. It can now be seen that two-thirds is the same as $\frac{10}{15}$ and that three-fifths is the same as $\frac{9}{15}$. The two fractions now have the same denominator and can be added using a computation strategy that works for the whole numbers that are the numerators, $10 + 9$ in this case. The answer will be fifteenths which gives $\frac{19}{15}$. This can be converted to a mixed number or other equivalent fraction.



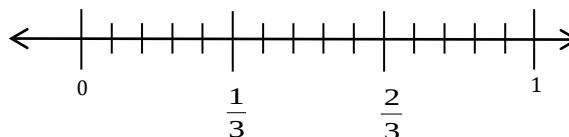
Example $\frac{3}{4} - \frac{1}{3}$. Use this area method to find the answer to this example.

2. Number lines

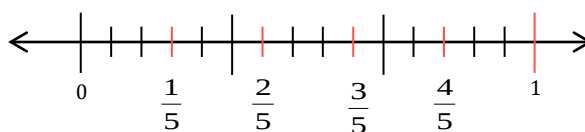
Example $\frac{2}{3} + \frac{3}{5}$. The representation of equivalent fractions can be done on a number line in a similar way to the area model. The same example as above is shown here using a number line. One of the fractions is used to divide a line into equal parts.



Then each segment is divided into the number of equal parts represented by the other fraction (fifths).



Two-thirds is represented as $\frac{10}{15}$ so $\frac{2}{3}$ is equivalent to $\frac{10}{15}$. This number line can now also be shown divided into fifths (which will be every third mark). $\frac{3}{5}$ is seen to be $\frac{9}{15}$. So, as with the area model, the two equivalent fractions with the same denominator can be added by focusing on the numerators and adding these as whole numbers.



Example $\frac{3}{4} - \frac{1}{3}$. Use this number-line method to find the answer to this example.

3. Fraction sticks

The YDM Number book introduced fraction sticks in section 5.4.2 (a full set of these is provided at the end of this subsection). The sticks are named by their left hand digit (e.g. 2 stick, 5 stick and so on). A fraction is made by putting two rows together, that is, $\frac{2}{5}$, is the 2 stick above the 5 stick as below (note that the 2 stick is the multiples of 2 and the 5 stick is the multiples of 5):

2	4	6	8	10	12	14	16	18	20
5	10	15	20	25	30	35	40	45	50

The sticks then give the sequence for equivalent fractions. This can be used to show the pattern for equivalent fractions, and they can be used to **compare, add or subtract** fractions.

The **equivalent fraction pattern** was shown in the YDM Number book and is:

- that the fractions equivalent to $\frac{2}{5}$ are found by multiplying $\frac{2}{5}$ by 1, that is multiplying $\frac{2}{5}$ by $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$ and so on (e.g. $\frac{2}{5} \times \frac{2}{2} = \frac{4}{10}$; $\frac{2}{5} \times \frac{3}{3} = \frac{6}{15}$, and so on), and that, reversing, fractions are equivalent to $\frac{2}{5}$ if they cancel down to $\frac{2}{5}$ (e.g. $\frac{14}{35} = \frac{2 \times 7}{5 \times 7} = \frac{2}{5}$, cancelling the 7);
- that two fractions are equivalent to each other if they cancel down to the same starting fraction; and
- if we have a fraction, then any fraction with a denominator which is a multiple of the first fraction's denominator can be equivalent to it if its numerator is the same multiple of the first fraction's denominator (e.g. $\frac{7}{12} = \frac{21}{36}$ because $36 = 3 \times 12$ and $21 = 3 \times 7$).

Examples $\frac{3}{7} + \frac{2}{5}$ and $\frac{3}{7} - \frac{2}{5}$. There are four steps as follows.

- Step 1: Use sticks.** Put out the 3 stick and the 7 stick to make $\frac{3}{7}$ and the 2 stick and the 5 stick to make $\frac{2}{5}$. Look along the denominators for a common denominator, in this case 35ths. Align the 35s and look at the two fractions aligned: $\frac{15}{35}$ and $\frac{14}{35}$. These are equivalent to $\frac{3}{7}$ and $\frac{2}{5}$ respectively. This means that $\frac{3}{7} + \frac{2}{5} = \frac{15}{35} + \frac{14}{35} = \frac{29}{35}$, and $\frac{3}{7} - \frac{2}{5} = \frac{15}{35} - \frac{14}{35} = \frac{1}{35}$ (see below):

3	6	9	12	15	18	21	24	27	30
7	14	21	28	35	42	49	56	63	70

2	4	6	8	10	12	14	16	18	20
5	10	15	20	25	30	35	40	45	50

- Step 2: Use sticks to reveal pattern.** The fraction sticks are a **patterning** material in that their actions reveal the pattern behind unlike denominator addition and subtraction. We see that $35 = 7 \times 5$ is the common denominator because it is a multiple of both fractions' denominator; and so $\frac{3}{7} = \frac{15}{35}$ because both top and bottom are multiplied by 5, and $\frac{2}{5} = \frac{14}{35}$ because both top and bottom are multiplied by 7, thus:

$$\frac{3}{7} + \frac{2}{5} = \frac{3 \times 5 + 7 \times 2}{7 \times 5}$$

This means in general that $\frac{a}{b} + \frac{c}{d}$ can be added by common denominator $b \times d$ and that $\frac{a}{b} = \frac{(a \times d)}{(b \times d)}$ (multiplying top and bottom by d) and that $\frac{c}{d} = \frac{(b \times c)}{(b \times d)}$ (multiplying top and bottom by b), meaning that:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d + b \times c}{b \times d}$$

A similar pattern exists for subtraction.

- Step 3: Give up sticks.** When this pattern is known, the sticks can be removed.
- Step 4: Practise pattern.** Use the pattern to add $\frac{1}{6}$ and $\frac{3}{8}$ and to subtract $\frac{1}{2}$ from $\frac{7}{9}$.

Set of fraction sticks:

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

4.3 Multiplication and division of common and decimal fractions

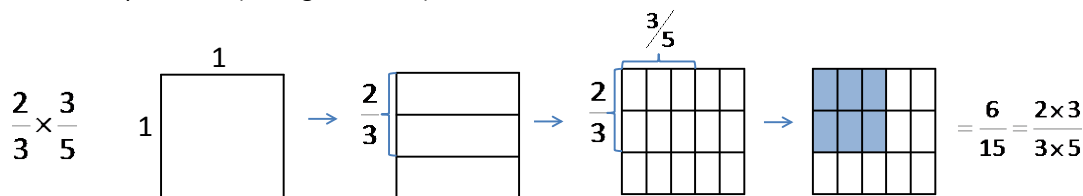
This section includes multiplication of common fractions, multiplication of decimal numbers, division of decimal numbers and division of common fractions.

4.3.1 Multiplication of common fractions

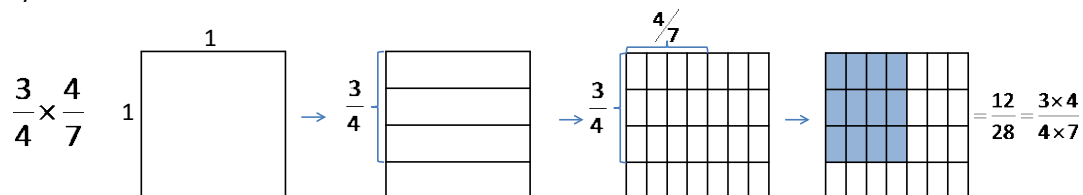
As stated in section 4.1, “lots of” is not a useful model for fractions as this requires a whole number, so we use the area model as an extension of arrays. There are two ways to model this.

1. Area in a square unit

For this method, think of $\frac{2}{3} \times \frac{3}{5}$ as the area of $\frac{2}{3}$ unit $\times \frac{3}{5}$ unit. Begin by drawing the square unit (must be square) and then constructing $\frac{2}{3}$ on one side and $\frac{3}{5}$ on the other. The rectangle $\frac{2}{3}$ by $\frac{3}{5}$ can be calculated as a fraction of one square unit (see figure below).

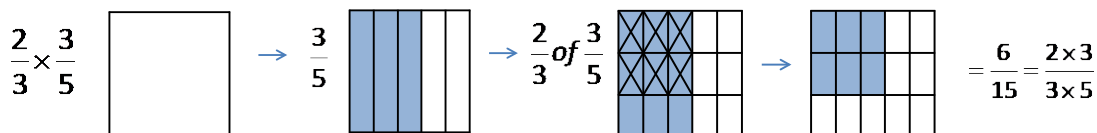


Similarly for $\frac{3}{4} \times \frac{4}{7}$:



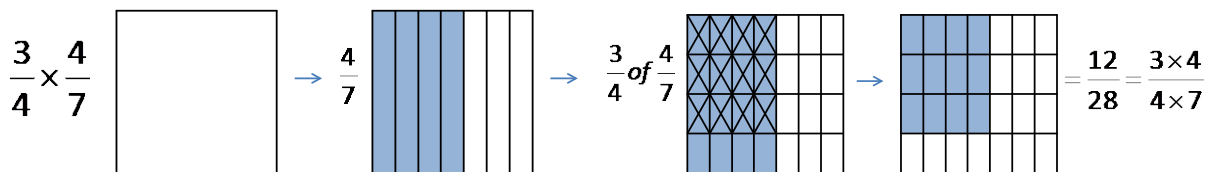
2. Part of part

Multiplying fractions involves finding a part of a part. In this understanding, it is helpful to think of multiplication in relation to fractions using “of” instead of “by”. Thus, for example $\frac{2}{3} \times \frac{3}{5}$, we first construct $\frac{3}{5}$ from a rectangle, then we find $\frac{2}{3}$ of this $\frac{3}{5}$, and calculate this as a fraction. Similarly for $\frac{3}{4} \times \frac{2}{7}$. See the figure below.



Finding the pattern

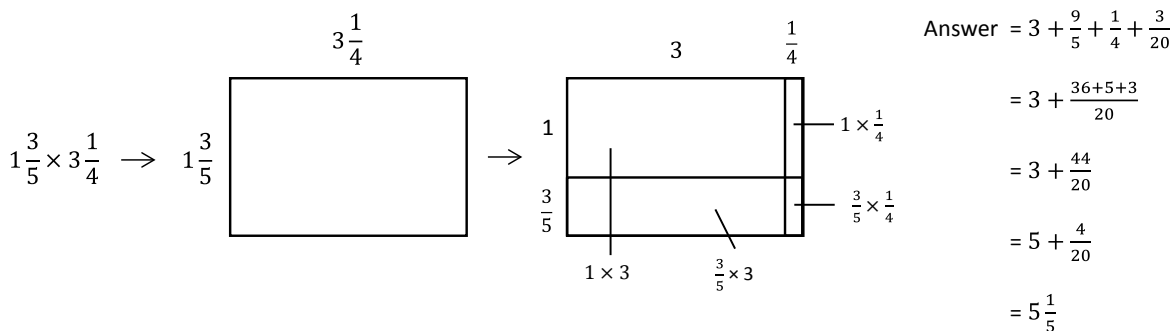
Either model above allows students to look for a pattern. It is fairly obvious that, for example, $\frac{2}{7} \times \frac{3}{8} = \frac{2 \times 3}{7 \times 8}$; in other words, that the numerators and denominators are multiplied. This pattern can then be used.



Mixed numbers

Mixed numbers can be multiplied two ways (e.g. $1\frac{3}{5} \times 3\frac{1}{4}$):

- changing mixed numbers to improper fractions and multiplying with pattern, so $1\frac{3}{5} \times 3\frac{1}{4} = \frac{8}{5} \times \frac{13}{4} = \frac{8 \times 13}{5 \times 4} = \frac{104}{20} = 5\frac{4}{20}$ or $5\frac{1}{5}$; and
- constructing an area of mixed number by mixed number, then dividing into rectangles based on whole numbers and proper fractions, as in the example below.



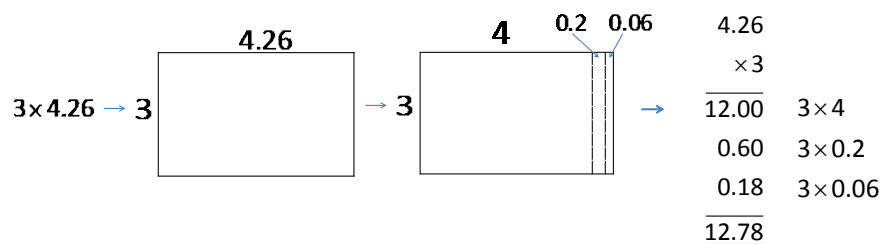
Notes: In these constructions, whether for mixed numbers or proper fractions, the following holds:

- a representation of the whole needs to be identified and then the fractions can be represented along each dimension and the multiplication can be modelled;
- using this model to represent the process works best with smaller fractions that are able to be shown this way;
- once the reasons why the methods work are established, then the generalised form (pattern) of the operation can be described and used with any examples; and
- it is always useful to place the problem in a real setting (e.g. how many square plaster boards are need to cover an area 1.6 m by 3.25 m, that is $1\frac{3}{5}$ m by $3\frac{1}{4}$ m?).

4.3.2 Multiplication of decimal numbers

Multiplication of decimal numbers follows the pattern of whole numbers but determining the ones position is the problem. The following steps are useful.

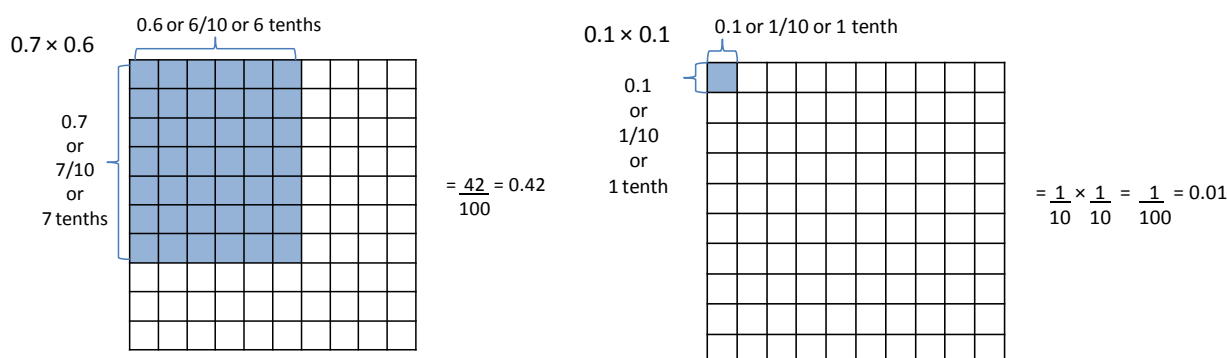
- Complete whole $\times$ decimal examples.** These follow the pattern for multiplication of whole numbers. See example below.



2. **Use area models for simple examples.** Look at 0.6×0.7 and 0.1×0.1 using area model – see below (can also do this for examples like 3.2×4.7 if students need to see this). This shows that tenths $\times$ tenths = hundredths. This is part of a pattern that:

tens $\times$ tens = hundreds $\rightarrow$ tenths $\times$ tenths = hundredths

tens by hundreds = thousands $\rightarrow$ tenths by hundredths = thousandths; and so on



3. **Relate decimals to whole numbers by naming in last place-value position.** 3.2×0.7 is 32 tenths $\times$ 7 tenths = 32×7 hundredths = 224 hundredths = 2.24.
4. **Use calculator patterns.** Use calculators to identify the pattern for the ones position

Use calculators:

$38 \times 6 = \underline{\hspace{2cm}} \rightarrow 3.8 \times 0.6 = \underline{\hspace{2cm}}$
 $74 \times 5 = \underline{\hspace{2cm}} \rightarrow 0.74 \times 0.5 = \underline{\hspace{2cm}}$
 and so on

Don't use calculators:

$27 \times 3 = 81 \rightarrow 2.7 \times 0.3 = \underline{\hspace{2cm}}$
 and so on.

State the pattern. The pattern/rule is that we have to add the decimal place-value position to find the ones position. That is, for example 0.36×0.6 , there is a total of 3 decimal place-value positions which places the 6 in the thousandths position. This in turn makes the ones position 4 from the right. Thus, if 36×6 by whole numbers is 216, the answer is 0.216.

Refrain from putting a rule forward that describes the pattern in terms of decimal points. Say that the two decimal place-value positions in 0.34 and the one decimal place-value position in 0.6 means three decimal place-value positions in the answer, which means the ones are the fourth position from the right. However, note that if an example has $4 \times 5 = 20$, $0.4 \times 0.5 = 0.2$ but this really is 0.20 – this can upset the pattern if not taken into account.

Use pattern and whole-number methods together. Complete any decimal multiplication using whole-number algorithm methods and then use the pattern to determine the ones. That is, 2.4×4.5 is $24 \times 45 = 1080$ in whole numbers, there are a total of two decimal place-value positions, so the ones position is three from the right, so the answer is 10.80 or 10.8.

4.3.3 Division of decimal numbers

As stated earlier, the problem for division is that sharing requires the number of sharers to be a whole number and grouping requires the number of groups to be a whole number. In this subsection we will look at this situation but spend most time looking at the situation where both factors are decimals.

One factor is a whole number

Division can be an extension of whole-number division if the divisor is a whole number. For instance, see the algorithms below for $7.32 \div 3$.

$\begin{array}{r} 3 \overline{) 7.32} \\ \underline{-6.00} \\ 1.32 \\ \underline{-1.20} \\ 0.12 \\ \underline{-0.12} \\ 0 \end{array}$	$7.32 \div 3 = 2.44$	$\begin{array}{r} 2.44 \\ 3 \overline{) 7.32} \\ \underline{6.00} \\ 1.3 \\ \underline{1.2} \\ 0.12 \\ \underline{0.12} \\ 0.00 \end{array}$
3 lots of 2 3 lots of 0.4 3 lots of 0.04		Share the ones Share the tenths Share the hundredths

Similarly, division can be an extension of whole-number division when the divisor can fit into the dividend. See the example on right for $43.2 \div 0.9$.

$$\begin{array}{r} 0.9 \overline{) 43.2} \\ \underline{18.0} \\ 25.2 \\ \underline{18.0} \\ 7.2 \\ \underline{7.2} \\ 0 \end{array}$$

20 lots of 0.9

20 lots of 0.9

8 lots of 0.9

Both factors are decimal numbers

This subsection explores examples like $0.378 \div 0.7$. To be able to do this using extension of whole-number techniques, it is necessary to change the example, without changing the answer, to where 0.7 is a whole number.

- Using reality.** Consider $0.378 \div 0.7$ as representing a reality situation, say metres (m) and millimetres (mm). Then consider $0.378 \div 0.7$ is in m and if we change to mm it becomes $378 \div 700$. This can be done by whole-number methods. (Please note, this method works best when translation to reality is simple. For example $0.21 \div 0.07$, changing it from metres to centimetres changes the example to $21 \div 7$, which is easy to solve.)
- Using the number-size compensation principle.** In this method, 0.7 changes to 7 by $\times 10$, so to compensate we change 0.378 the same way (see figure on right). That is, $0.378 \div 0.7 = 3.78 \div 7$, and this can be solved by whole-number methods.
- Using fraction equivalence.** The example $0.378 \div 0.7$ is the same as fraction $\frac{0.378}{0.7}$; multiplying top and bottom by 10, this is equivalent to fraction $\frac{3.78}{7}$ which is the same as $3.78 \div 7$.
- Using calculator patterns.** The same four steps for decimal division can be used as for multiplication (use calculators to relate $0.345 \div 0.5$ to $345 \div 5$ and find the pattern, then don't use calculators to practise the pattern, then describe the pattern, and finally use the pattern). This will give the pattern that for decimal division $0.345 \div 0.5$, it is possible to do the whole-number division and subtract the decimal place-value positions to find the ones in the whole-number answer. That is $345 \div 5$ is 69, it is 3 places – 1 place = 2 places, so ones are 3 from right, so decimal division answer is 0.69.

$$0.378 \div 0.7 = 3.78 \div 7$$

$\xrightarrow{\times 10}$
 $\xleftarrow{\times 10}$

4.3.4 Division of common fractions

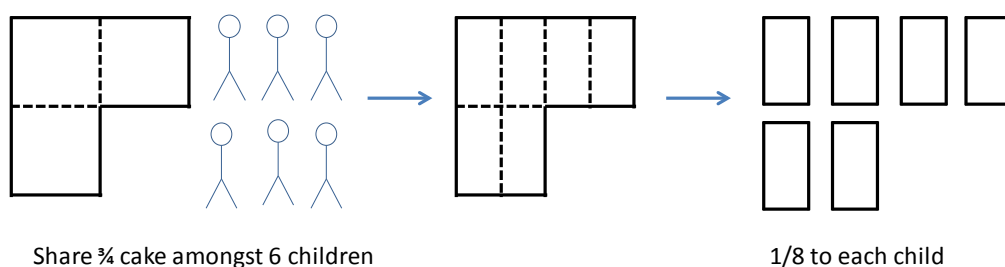
In division of common fractions the grouping and sharing meaning of fraction finally becomes unworkable. It is possible to share $\frac{3}{4}$ among 6 and find how many groups of $\frac{1}{4}$ go into $2\frac{1}{2}$, but how can materials be used to share or group an example such as $\frac{2}{3} \div \frac{3}{4}$? At this point it is necessary to return to the mathematical meaning of division, which is multiplication by inverse, that is, the reciprocal. This subsection looks at methods for doing fraction division and builds towards multiplication by reciprocal by using grouping and sharing methods where one factor is a whole number.

It should be noted that, for the division of fractions, students need a sound understanding of the operation of division as well as knowledge of fractions. Students who have been encouraged to write stories to model different operations should be able to think of real situations where the division of fractions makes sense.

Grouping and sharing

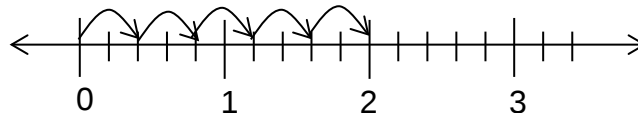
The following methods are available. Each is described and then, if it is applicable, how to use them to build towards reciprocal is discussed.

1. **Using reality.** Any work with fractions is best begun with reality. A division story for whole numbers could be to find how many boxes will be needed to put 100 pencils into boxes of 12 ($100 \div 12$). The same operation just needs a sensible context where fractions are used instead of whole numbers, for example: *How many times will I need to fill a bucket that holds $\frac{1}{8}$ of a litre to fill a tank that holds $35\frac{1}{2}$ litres of water?* This is $35\frac{1}{2} \div \frac{1}{8}$. Using a contextual problem like this helps students to see the connection to using the inverse when dividing fractions. It makes sense to realise that it will take 8 buckets to make 1 L and to work out how many buckets to fill $35\frac{1}{2}$ L will be able to be found by completing the computation $35\frac{1}{2} \times 8$ (which is multiplication by reciprocal).
2. **Using sharing.** A problem could be to share $\frac{3}{4}$ cake among 6 children. This could be done by cutting a diagram of the $\frac{3}{4}$ cake into 6 pieces and seeing that each child receives $\frac{1}{8}$ of the cake as below.



This process can be discussed as finding $\frac{1}{6}$ of an amount (here $\frac{3}{4}$), thus it is $\frac{1}{6} \times \frac{3}{4}$.

3. **Using models.** A more complex story could be: *How many pieces of ribbon that are $\frac{2}{5}$ of a metre long can I cut from 2 m of ribbon ($2 \div \frac{2}{5}$)?* This problem could be solved logically using repeated subtraction of $\frac{2}{5}$ if students can manage this. A number-line representation would be helpful – show a number line divided into fifths from 0 to 2 and count how many sections of $\frac{2}{5}$ can be made – this is shown on number line on right.



If the problem is more complex, say $10\frac{1}{2} \div \frac{2}{5}$, students can be encouraged to see that five ribbon lengths can be made from 2 m so for every 2 m you will be able to make 5 ribbons (inverse of $\frac{2}{5}$). So to find how many in $10\frac{1}{2}$ m they would divide $10\frac{1}{2}$ by 2 (to find how many 2 m lengths there were) and then multiply this by 5 because there are 5 ribbons able to be cut from each 2 m length. So the operation is $10\frac{1}{2} \div 2 \times 5 = 2\frac{1}{2} \times 5$, which is multiplication by reciprocal.

Multiplication by reciprocal

Examples like $\frac{2}{3} \div \frac{3}{4}$ require the construction of a new meaning of division that is the mathematically correct one, multiplication by reciprocal. To do this, go through these steps.

1. **Revise simple divisions.** Start with $6 \div 2$ for which the answer is 3. Ask if there is a $\times$ that will give the same result, or 6 lots of what give 3. Encourage the students to see that $6 \times \frac{1}{2} = 3$. Repeat this for other simple examples. Introduce $\frac{1}{2}$ as reciprocal of 2, and so on.
2. **Practise this new meaning of division with larger whole numbers.** Use a calculator to look at pairs of examples like $91 \div 7$ and $91 \times \frac{1}{7}$; $812 \div 24$ and $812 \times \frac{1}{24}$; $23\,456 \div 128$ and $23\,456 \times \frac{1}{128}$; and so on. Reinforce that division by number is the same as multiplication by reciprocal.
3. **Play the change game for multiplication.** Use the arithmetic excursions activity where students construct paths from one number to another by arrows with operations on top of them (see below); extend these to multiplication only examples where they have to go from a large to a small number (see below).

$\begin{array}{ccccccc} & \times 20 & + 60 & - 18 & & & \\ 2 & \longrightarrow & 40 & \longrightarrow & 100 & \longrightarrow & 82 \end{array}$ is a path from 2 to 82

Construct these paths:

$75 \longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow 11$
 $26 \longrightarrow \longrightarrow \longrightarrow \longrightarrow 1047$
 $423 \longrightarrow \dots \text{Any number of steps} \dots \longrightarrow 20\,002$

Try this path only using multiplication:

$632 \longrightarrow \dots \text{Any number of steps} \dots \longrightarrow 236$

4. **Find reciprocal of fractions.** The inverse of $\times 2$ is $\div 2$ because $2 \div 2 = 1$. We also know that $2 \times \frac{1}{2} = 1$ and so $\frac{1}{2}$, the reciprocal of 2, is also the inverse of 2 for multiplication. Now $\frac{3}{5} \div \frac{3}{5} = 1$ so that $\div \frac{3}{5}$ is inverse of $\times \frac{3}{5}$. Similar to the argument above, the reciprocal of $\frac{3}{5}$ (which is $\frac{1}{3/5}$) is the inverse of $\frac{3}{5}$ ($\frac{3}{5} \times \frac{1}{3/5} = 1$). However, $\frac{3}{5} \times \frac{5}{3} = \frac{15}{15} = 1$. Thus, $\frac{1}{3/5} = \frac{5}{3}$, and so the reciprocal of a fraction is the inversion of the fraction – numerator to denominator and vice versa. This means that the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$, of $\frac{5}{7}$ is $\frac{7}{5}$, of $\frac{11}{13}$ is $\frac{13}{11}$, and so on.
5. **Extend reciprocal to fraction division.** Now apply ideas from point 2 and point 4. For fraction $\frac{2}{3} \div \frac{3}{4}$, we use the proper mathematical form of division which is multiplication by reciprocal and we use the fact that reciprocal is the inversion of the fraction (i.e. the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$). This means:

$$\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$$

4.4 Problem solving and estimation

This section explores how fractions are used in problem solving and estimation exercises.

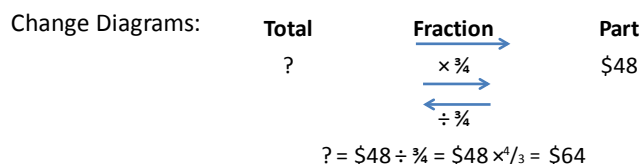
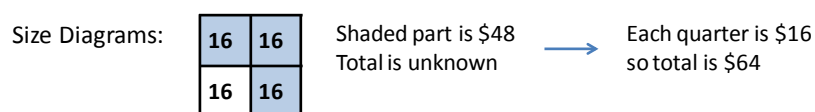
4.4.1 Problem solving with fractions

There are a few specific points to be made regarding fractions in problem solving.

1. **Fractions in problems.** The use of fractional amounts does not change normal word/operation problems. Whether the problem states “I took 5 cows from the paddock and this left 6 cows, how many in the paddock to start with?” or “I took $\frac{5}{8}$ of a tonne of sand from the pile, this left $\frac{1}{4}$ tonne, how much sand to start with?”, the problem remains the same – what operation is to be used, addition or subtraction, multiplication or division? Here it is addition because it is part-part-total with two parts known (the part taken away and the part left) and the total (the start) is not known (see sections 2.6 and 3.6 on problem solving).

Thus the problem-solving aspects of word problems remain the same:

- use the sameness of the parts and that one number is describing “lots of” to determine whether a problem is multiplication-division or addition-subtraction;
 - use part-part-total and factor-factor-product to determine whether addition-subtraction problems are addition or subtraction, and whether multiplication-division problems are multiplication or division; and
 - use breaking into parts, drawing/acting it out, systematic/exhaustion, given-needed-wanted, and simplify/restate the problem strategies to do multi-step problems and break them down into components where the earlier ideas apply.
2. **Fraction comparison problems.** There is one type of problem that does need some extra ideas and that is the use of multiplicative comparison fraction problems; for example, “if I paid \$48 for $\frac{3}{4}$ of the package, how much did it cost in total?”. These are best understood as size diagrams or change, as below. We will do a lot more of this in Chapter 5 (and add in the double number line):



3. **Equivalence problems.** Like ratio and proportion, some problems rely on equivalence. For example, “three-quarters of the material was to be taken away, there were 68 items, how many taken away?”. This is done by equivalence which says top and bottom are multiplied by the same amount (see below).

$$\frac{3}{4} = \frac{?}{68} \quad ? = 51$$

$\swarrow \times 17$ $\searrow \times 17$

4.4.2 Estimation with fractions

Once again, there are some specific points.

1. **Strategies.** No matter the number, the estimation strategies of *rounding*, *straddling*, and *getting closer* still apply (see sections 2.5 and 3.5). Front end is not applicable.
2. **Equivalence.** As in other number types, the above estimation strategies (particularly *getting closer*) for estimating fraction operations rely on being able to order fractions. This is because, for example, knowing that rounding has reduced a number means that the estimate is low. For unlike denominator fractions this requires equivalence. We can see this in example, $10\frac{2}{3} \times \frac{5}{7}$. We first expand the multiplication to $\frac{32}{3} \times \frac{5}{7}$. Then we round $\frac{5}{7}$ to $\frac{6}{8}$ (this is rounding to a close number which is easy to compute because of cancellation). The multiplication becomes $\frac{32}{3} \times \frac{6}{8} = 4 \times 2 = 8$. Now, equivalent fractions show that $\frac{6}{8} = \frac{42}{56}$ and $\frac{5}{7} = \frac{40}{56}$, so $\frac{6}{8}$ is larger than $\frac{5}{7}$. Thus, the estimate is too high, but not by much as $\frac{6}{8}$ is not much larger than $\frac{5}{7}$. Still, using *getting closer*, a better estimate might be 7.5 or 7.7 [the actual answer is 7.619].
3. **Benchmarking.** For rounding, we often need to know whether the fraction in a mixed number is less than or larger than $\frac{1}{2}$. For example, $7\frac{3}{7}$ is 7 when rounded to the nearest whole number because $\frac{3}{7}$ (3 out of 7) is smaller than $\frac{1}{2}$ ($3\frac{1}{2}$ out of 7).

4.5 Extension to indices and algebra

This section briefly covers the extension of fractions to algebra and indices.

4.5.1 Extension of fractions to indices

This subsection briefly extends on the properties of indices from section 3.7 to explore the effect on fractions. There are two points to be made.

1. **Fraction numbers.** Whether numbers are fractions or whole numbers, even whether they are irrationals or rationals (common fractions), makes no difference – they still can have indices with the same meaning. This means that, for example:

$$\begin{aligned} \left(\frac{3}{5}\right)^0 &= 1 & \left(\frac{3}{5}\right)^1 &= \frac{3}{5} & \left(\frac{3}{5}\right)^2 &= \frac{3 \times 3}{5 \times 5} & \left(\frac{3}{5}\right)^3 &= \frac{3 \times 3 \times 3}{5 \times 5 \times 5} & \text{and so on; and} \\ \left(\frac{3}{5}\right)^{-1} &= \frac{1}{3/5} = \frac{5}{3} & \left(\frac{3}{5}\right)^{-2} &= \frac{5}{3} \times \frac{5}{3} = \frac{5 \times 5}{3 \times 3} & \left(\frac{3}{5}\right)^{-3} &= \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} = \frac{5 \times 5 \times 5}{3 \times 3 \times 3} & \text{and so on.} \end{aligned}$$

Using the same techniques as in section 3.7, these results can be generalised to show that:

$$\left(\frac{3}{5}\right)^p = \frac{3^p}{5^p} \text{ and } \left(\frac{3}{5}\right)^{-q} = \frac{5^q}{3^q}; \text{ and } \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p} \text{ and } \left(\frac{a}{b}\right)^{-q} = \frac{b^q}{a^q}$$

2. **Fraction indices.** Section 3.7 showed that $n^{p+q} = n^p \times n^q$, thus:

$$n = n^1 = n^{\left(\frac{1}{2} + \frac{1}{2}\right)} = n^{\frac{1}{2}} \times n^{\frac{1}{2}}, \text{ thus } n^{\frac{1}{2}} = \sqrt{n} \text{ [square root of } n].$$

Similarly, we can show that:

$$\text{Unit fraction indices: } n^{\frac{1}{3}} = \text{cube root of } n; n^{\frac{1}{4}} = \text{fourth root of } n; \text{ and so on; and}$$

$$\text{Common fraction indices: } n^{\frac{4}{3}} = \text{cube root of } n \text{ raised to power of } 4 = (n^{\frac{1}{3}})^4.$$

4.5.2 Extension of fractions to algebra

This subsection explores fractions as unknowns and algebraic understanding of fraction processes.

1. **Fractions as unknowns.** In sections 2.8 and 3.8, we looked at how operations extend to where there are no numbers given, that is, to algebra. Fractions can be the unknown as well as whole numbers. For example: *Pizzas were divided up so Frank got a whole one and a quarter and the other 6 people got the same fractional amount. There were 5 pizzas. How much did each person other than Frank get?*

The story can be written as $6PP + 1\frac{1}{4} = 5$. This means that there were 6 part pizzas (PP) plus $1\frac{1}{4}$ to give 5. Backtracking, $6PP = 5 - 1\frac{1}{4} = 3\frac{3}{4} = \frac{15}{4}$ and $PP = \frac{15}{4} \div 6 = \frac{15}{4} \times \frac{1}{6} = \frac{15}{24} = \frac{5}{8}$ of a pizza.

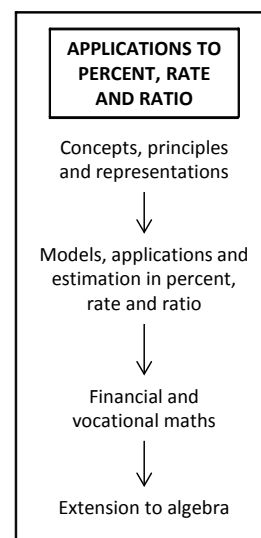
2. **Fraction processes and algebraic processes.** As seen earlier in this section, fraction processes can be generalised to algebra. For example:

- inverse or reciprocal: reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$; thus $\frac{b}{a} = \frac{1}{a/b}$
- mixed number to improper fraction: $A\frac{b}{c} = \frac{(Ac+b)}{c}$
- addition: $\frac{a}{b} + \frac{c}{d} = \frac{(ad+bc)}{bd}$
- subtraction: $\frac{a}{b} - \frac{c}{d} = \frac{(ad-bc)}{bd}$
- multiplication: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$
- division: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

5 Applications with Percent, Rate and Ratio

This chapter covers operations with respect to percent, rate and ratio. These topic areas are built around applications and are multiplicative (involving multiplication and division). The chapter is still based on the two components for operations, meaning and computation, and builds on the ideas developed in Chapters 2, 3 and 4. It deals with special numbers which relate to common and decimal fractions but are not quite the same. However, it covers an area that is important for computation with important applications in money and measurement.

The sequence for the chapter is based on the figure on right. The figure has four steps going from meaning (concepts, principles and representations) to computation (models, applications and estimation; and financial/vocational mathematics) and back again to meaning (extension to algebra). The chapter begins by revisiting the ideas from Chapters 2 and 3 to set up concepts (including representations), principles and processes that work for percent, rate and ratio (section 5.1). The chapter moves on to how models, applications and estimations work with respect to percent, rate and ratio (section 5.2). Common methods for all three areas are shown even though symbols and relationship to fractions are different for each area. Percent is directly related to common and decimal fractions in that it is hundredths. Rate is usually a decimal that is a result of division (e.g. \$9 for 6 L means \$1.50/L) and only has a weak relationship to fraction. Ratio is similar to fraction but it is part-to-part not part-to-whole (e.g. the ratio 2:3 is similar in action to the fraction $\frac{2}{5}$). The chapter then returns to meaning by looking at problems in financial and vocational mathematics (section 5.3) which relate to percent, rate and ratio, and concludes by looking at extensions to algebra (section 5.4).



5.1 Concepts, principles and representations

This section looks first at the concept of multiplicative comparison as it is the basis of all applications with percent, rate and ratio, relating it to change. It then looks at principles and symbolic representations, and concludes by looking at early activities in the three areas (percent, rate and ratio). This section needs to be looked at in relation to percent, rate and ratio from the YDM Number book and the concepts, principles and representations used in Chapters 2, 3 and 4 of this YDM Operations book.

5.1.1 Multiplicative comparison

In this subsection we will look at how applications in percent, rate and ratio are based on multiplicative comparison, how this can be seen as change, and how it leads to three types of problems.

Concept of multiplication as comparison

In their applications, the three topics of percent, rate and ratio are multiplicative; that is, they involve multiplication and division in their solutions. This is because **all three are a form of multiplicative comparison**:

- percent compares the original amount with the amount after the percent is calculated multiplicatively – for example, 200% of \$70 is $2 \times \$70 = \140 ;
- rate compares one attribute with another different attribute multiplicatively – for example, \$3/kg means that 4 kg costs $3 \times 4 = \$12$; and
- ratio compares the second amount with the first (same attribute) multiplicatively – for example, butter to flour in ratio 1:3 means that 200 g of butter is mixed with $3 \times 200 = 600$ g of flour.

Multiplicative comparison is a way that compares two amounts through multiplication (and, backwards, through division); it is best seen in relation to the other forms of comparison: (a) *numerical* – 24 is larger than 8; (b) *additive* – 24 is 16 more than 8; and (c) *multiplicative* – 24 is 3 times as large as 8.

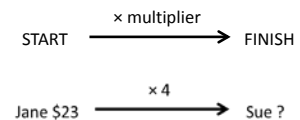
As we will see in early activities (subsection 5.1.3), using multiplicative comparison with percent, rate and ratio should follow:

- using multiplicative comparison with whole numbers and fractions and decimals (e.g. John has 3 times the money Jack has, Sue's house was $\frac{3}{4}$ the value of Jo's house, and so on); and
- developing the concepts of multiplicative comparison for multiplication of whole numbers (see Chapter 3).

As we will see in section 5.2, since percent, rate and ratio are similar in their mathematical structure, their applications and problems can be solved by using the same methods (we will show three models, all of which can solve all problem types). [Note: this is a **teaching big idea** – *mathematically similar ideas are taught using similar models and their problems are solved with similar methods.*]

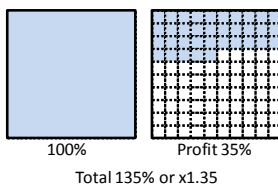
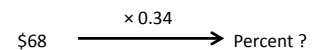
Multiplicative comparison as change

One way to think about multiplicative comparison situations is to think of them as change or transformation where a starting number is multiplied by a multiplier to get a finishing number (see diagram on the right). For the problem “Sue has 4 times the money of Jane, how much does Sue have, if Jane has \$23?” it is easy to see that the multiplier in the multiplicative comparison is $\times 4$ as on right (answer \$92).

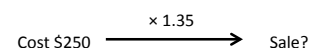


However, it is not so easy with percent, rate and ratio as the following shows.

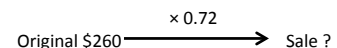
Percent. Percent is per 100 and is a fraction of a whole seen as 100 equal parts. This means that percents are hundredths (see decimal notation for percents in YDM Number book) and 34% is 34 hundredths is 0.34. So, if we have a percent, it must be changed to a decimal for the multiplier. Thus, 34% of \$68 has multiplier $\times 0.34$ as on right (answer \$23.12).



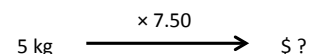
However, profit, loss and discount must be understood as well. For example, a profit of 35% on \$250 is 100% plus the 35% which is 135% (as shown on left). This means the multiplier is $\times 1.35$, and the change is as on right (answer \$337.50).



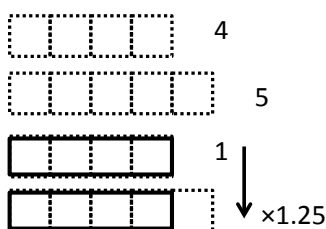
A discount of 28% (or a loss of 28%) is $100 - 28$ or 72% (see diagram on left). Thus, the sale or discounted price for a dress valued at \$260 discounted at 28% has a multiplier of 0.72 as on right (answer \$187.20).



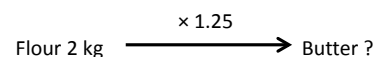
Rate. Rate is straightforward in terms of number. For example, \$7.50 per kg is to multiply by 7.5. However, one has to get the measure correct. \$7.50/kg means 1 kg changes to \$7.50, so the multiplier operates as on right for 5 kg (answer $7.50 \times 5 = \$37.50$).



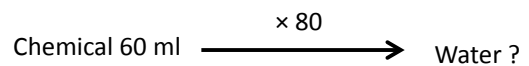
Ratio. Ratio is two numbers – these numbers must be changed to a single multiple.



For example, flour and butter mixed in a ratio 4:5 gives a multiplier of $5 \div 4$ or 1.25 (see on left) and, thus, the amount of butter for 2 kg of flour is as on right (answer is 2.5 kg of butter).



The situation is less straightforward if the example is more complex, for example, if a chemical is mixed with water in ratio 25 mL : 2L, then: (a) measures have to be the same (e.g. 25 mL : 2L is 25:2000 in mL); and (b) the second number has to be divided by the first to give the multiplier (e.g. $2000 \div 25 = 80$). Thus the change diagram is as on right if we have 60 mL of chemical to mix (answer is that we need 4800 mL or 4.8 L of water).



Practice. Students need a lot of practice **changing between story and change diagram**. One way to do this is to use a table, as below, with three columns – fill one column in at a time (a different one each time) and students complete the other two columns (this caters for reversing as well as the main idea). When filling the central column, students have to give multiplier and topic area – is it percent, rate or ratio?

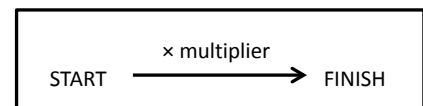
PROBLEM	MULTIPLIER AND TOPIC AREA	DRAWING

Types of multiplicative comparison problems

Problem types. As multiplicative comparison, percent, rate and ratio problems involve three components (Start, Change, and Finish):

- (a) for **percent**, it is the starting amount, the percent, and the finish percentage (e.g. in problem, *25% of \$48 is \$12*, the \$48 is the starting amount, the 25% is the percent, and the \$12 is the finishing percentage);
- (b) for **rate**, it is the amount of the second attribute, the rate, and the amount of first attribute (e.g. in problem, *30 L at \$1.50/L is \$45*, the 30 is the starting amount of litres [the second attribute of the rate], the 1.50 is the rate \$/L, and the 45 is the finishing dollars [the first attribute of the rate]); and
- (c) for **ratio**, it is the first amount, the ratio, and the second amount (e.g. in problem, *20 kg of sand at ratio sand:cement is 5:2 needs 8 kg of cement*, the 20 for sand is the starting amount, the 5:2 is the ratio, and the 8 for cement is the finishing amount).

When there are three components, there are three possible problems:



- | | | | | |
|--|--------|---------|---------|---------|
| (a) the finish can be unknown; | TYPE 1 | Known | Known | Unknown |
| (b) the start can be unknown; and | TYPE 2 | Unknown | Known | Known |
| (c) the change can be unknown. | TYPE 2 | Known | Unknown | Known |

As can be seen in the diagram above right and in the following table, this gives three problem types:

- (a) **Type 1** problems (finish unknown) are solved by multiplying the start by the multiplier;
- (b) **Type 2** problems (start unknown) are solved by dividing the finish by the multiplier; and
- (c) **Type 3** problems (multiplier or change unknown) are solved by dividing the finish by the start.

PROBLEM OVERALL	TYPE 1 finish is unknown	TYPE 2 start is unknown	TYPE 3 change is unknown
Percent: 25% of \$48 is \$12	What is 25% of \$48?	25% is \$12, what is the total?	\$12 is what percent of \$48?
Rate: 30 L at \$1.50/L is \$45	How much does 30L cost at \$1.50/L?	How many litres can you buy for \$45 at \$1.50/L?	What is the rate when you buy 30 L for \$45?
Ratio: 20 kg of sand at sand to cement ratio of 5:2 needs 8 kg of cement	How much cement for 20 kg of sand at sand to cement ratio of 5:2?	How much sand for 8 kg of cement at sand to cement ratio of 5:2?	What is the ratio for 20 kg of sand mixed with 8 kg of cement?

Knowing the start. One important component of translating stories to change situations is knowing which attribute/number is the start and, therefore, which attribute/number is the finish. This determines which operation is chosen for the solution, multiply or divide.

1. **Percent.** For percent problems, **the start is what the percent acts on** and this is always the whole/total or **the one which is 100%** in percent terms. This means that for the problem, *Find 45% of \$610?*, the \$610 is the start; while for the problem, *The down payment of 45% was \$610, what was the total cost?*, the \$610 is not the start but the finish, the start is the unknown total amount.
2. **Rate.** For rate problems, the **start is always the “per” attribute/amount**. For the problem, *Petrol costs \$1.60 per litre, how much will 50 L cost?*, the 50 L is the start as it is litres; while for the problem, *Petrol costs \$1.60 per litre, how many litres for \$50?*, the \$50 is the finish, the unknown litres is the start.
3. **Ratio.** For ratio problems, **the start is always the first amount in the ratio**. For the problem, *Cordial and water are mixed 2:7, how much water for 2 L of cordial?*, the start is the 2 L because cordial is first in the ratio; while for the problem, *Cordial and water are mixed 2:7, how much cordial for 10 L of water?*, the 10 L is the finish because it is the water, the unknown cordial amount is the start.

5.1.2 Principles and symbols

This subsection briefly discusses the integrating roles of multiplication principles and the separating effect of symbols in percent, rate and ratio. It concludes by looking at the equivalence principle in ratio and relating this to equivalence in division and fractions.

Application of multiplication principles

The principles (big ideas) for multiplication and division still apply. These include:

1. Inverse for multiplication and division – which is seen in the types of problems – finish unknown in multiplication but when multiplier or start is unknown, we have the inverse which is division.
2. Commutative for multiplication – for example, 25% of \$70 is 0.25×70 but can also be done in opposite order, that is, 70×0.25 .
3. Multiplication of fractions less than one makes things smaller – for example, 0.25×70 is less than 70.
4. Division of fractions less than one makes things larger – for example, \$0.65/kg means that \$50 can buy $50 \div 0.65$ kg which is more than 50.
5. The larger the amount means the larger answer – for example, even when multiplying by a fraction less than one like 25%, 0.25×70 is larger than 0.25×40 .

Thus, the **principles continue to provide similarity and integration** across the different topics (whole numbers, fractions, percent, rate and ratio) in this YDM Operations book – showing how things are the same and allowing the structure of mathematics to be revealed.

It should be noted that the following big ideas also apply:

- the *relationship vs transformation* principle in that both relationship and change approaches apply to percent, rate and ratio;
- the *interpretation-construction principle* that the best way to learn to interpret problems is to construct all the types;
- the *continuous vs discrete* principle in that both sets (e.g. hundredths through 10×10 grids) and number lines (e.g. double number lines) will be used; and
- the *triadic principle* that there are three problem types (see subsection 5.1.1).

The effect of language/symbol differences

In contrast to the principles which unite the different topics, the symbols differentiate and obscure similarities – 25%, \$1.50/L and 5:2 look very different but are **different notations for the same thing**. Only decimals are really needed, for example:

1. We can use decimals for percent which are in reality hundredths (see YDM Number book). That is, 25% can be 0.25 and 7.5% can be 0.075. We can say the interest rate is 0.075 (i.e. 7.5 hundredths), we do not need the special percentage word.
2. Decimals are already used for rate, so we can leave this as petrol is \$1.50/L, we travelled at an average of 95.5 km/h, or the cost of potatoes is \$2.65/kg.
3. Ratios can be determined by dividing second term by first term in the ratio. Thus, a ratio of sand to cement of 5:2 can be a ratio of $\frac{2}{5}$ or 0.4, a ratio of 25 mL of chemical to 2 L of water can be a ratio of 80, and a ratio of flour to butter of 4:5 can be a ratio of 1.25.

[Note: This change from ratio to rate has already happened in betting on racing – except that it is against 10 not 1. Instead of odds of 4:5 or 9:8, they are now just one number, e.g. 8, which is understood as being what you get from a \$10 bet, e.g. win \$8 per bet of \$10 (plus the \$10 returned as well) – thus, 3:2 = $\frac{3}{2}$ = 1.5 = 15 to 10 = 15.]

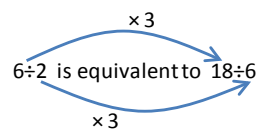
We have to know the different symbols and different languages and terms for percent, rate and ratio because they exist in our lives in daily usage and have come out of different parts of our history. However, we must try to ensure that the students do not see the topics of percent, rate and ratio as different just because the language/symbols are different.

Representations of mathematical ideas can be real-world situations, physical and virtual manipulations, pictures, language and symbols. In abstraction, it is common for one of the representations to cause difficulty. For example, in whole-number numeration, language is a difficulty. Instead of one ten and seven ones (17) being called “onety-seven” as it should, it is called “seventeen” because of the base 20 language system that existed before the base 10 Hindu-Arabic number system was adopted. Similarly, instead of 206 being called “two hundred and zeroty-six” as it should be, it is called “two hundred and six”, leaving out all reference to the zero. For percent, rate and ratio, the representation problem is symbols or notation – three different notations for what is really the same thing (i.e. multiplicative comparison).

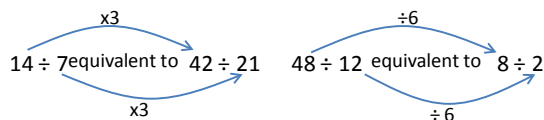
Equivalence principle applied to proportion

Ratio has equivalence (the same as division and fractions) – it has a special name, proportion. A brief revisit of equivalence in division and fractions will show how the ideas are similar for ratio/proportion.

Division. Division examples are equivalent (have the same answer) if they are related by multiplying/dividing numbers by same amount. This is true for examples $6 \div 2$ and $18 \div 6$. Both equal 3 so they are equivalent and we can get from one, $6 \div 2$, to the other, $18 \div 6$, by multiplying both numbers by 3 (i.e. $6 \times 3 = 18$ and $2 \times 3 = 6$) as on right. As described in work on the compensation principle earlier in this book, this means that $6 \div 2$ is multiplied and divided by 3, leaving it unchanged.

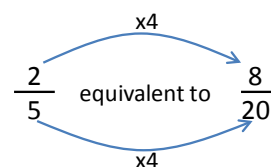


To teach this, students explore many division examples that have the same answer. As they look at these examples, we encourage students to find a pattern that two divisions are equivalent if the two numbers in the division are multiplied/divided by the same number to give the new division. For example $14 \div 7$, multiply both numbers by 3 and we get $42 \div 21$; multiply both numbers by 8 and we get $112 \div 56$. This means that $14 \div 7 = 42 \div 21 = 112 \div 56$ (and this is true because they all equal 2). Turning this around using another example, $48 \div 12$ can be changed to $8 \div 2$ by dividing/cancelling down both

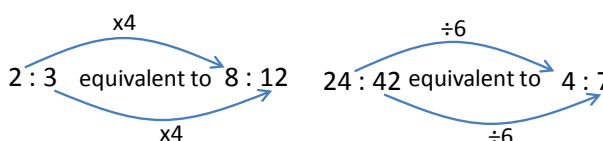


numbers by 6. This means that $48 \div 12$ and $8 \div 2$ are equivalent (and this is true as well because both = 4). Diagrammatically we can show this as on right above.

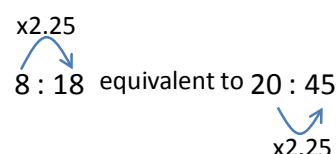
Fractions. Here, fractions are equivalent if they cancel down to the same starting fraction or they are the result of the numerator/denominator being multiplied/divided by the same amount. This can be seen diagrammatically as on right for the example $\frac{2}{5}$ is equivalent to $\frac{8}{20}$ because $\frac{8}{20}$ is $\frac{2}{5} \times \frac{4}{4}$. See YDM Number book section on equivalent fractions.



Ratio/Proportion. Similar to division and fractions, ratios have equivalence (equivalent ratio has its own name – it is called **proportion**). This equivalence is based, as for fractions, on multiplying by 1 (identity) or multiplying and dividing by the same number (compensation). However, for ratios this means that the two numbers are multiplied by the same number, that is, 2:3 is equivalent to 8:12 because both the 2 and the 3 have been multiplied by 4. This leads to two ratios being equivalent or in proportion if they can be cancelled down to the same starting ratio (e.g. 8:14 = 24:42 because 8:14 = 4:7 divided by 2 and 24:42 = 4:7 divided by 6). This is diagrammatically shown as on right.



Interestingly, the above rule also means that two ratios are equivalent, or in proportion, if the multiplication arrangement within each ratio (not across ratios) is the same. For example, 8:18 is in proportion with 20:45 because both cancel down to 4:9 **and** because $18 \div 8$ is 2.25 and $45 \div 20$ is 2.25. We diagrammatically show this meaning for proportion as on right.



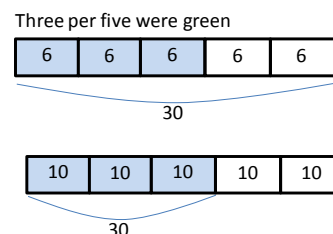
Both **division and fractions also have this equivalence rule**. For example, for fractions, $\frac{8}{10}$ is equivalent to $\frac{20}{25}$ because $8 \div 10 = 0.4$ and $20 \div 25 = 0.4$, as well as because they both cancel down to $\frac{4}{5}$. It is almost a “commutative” law for equivalence. What it means is that, for fractions, if A/B is equivalent to C/D , then two rules hold: (a) **the multiple across rule** – the multiple from A to C (the numerators) is the same as that for B to D (the denominators); and (b) **the multiple within rule** – the multiple from B to A is the same as for D to C. Both these rules mean that the fractions cancel down to the same starting fraction.

5.1.3 Early activities

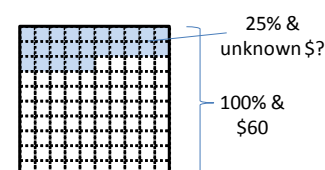
In the early years, rate can be simple things like 3 apples/bag, percent can be “per” 5, and ratio is the part-part component in simple fractions (e.g. $\frac{2}{3}$ is 2:1). Activities can be undertaken for simple problems (preferably with whole-number amounts) using the size diagrams.

Percent

One can do “per 5” type problems by shading the parts of 5 that are required. Consider the problem, *Three per five were blue. How many were blue if there were 30 items?* The diagram is drawn (blue is shown by shading) and students are encouraged to place the information from the problem on it, as on right. The 30 items total means 6 in each box which means 18 were blue. *Note:* If the problem was the other way around and there were 30 blue items, the total items is 50, also as on right (in the lower diagram).

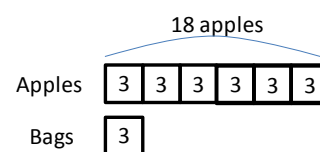


One can also do simple percent calculations for amounts that relate well to 100. For example, *find 25% of \$60*. This can be modelled on the 10x10 grid as on right. Students can draw the 25% on the 10x10 grid and be encouraged to use the drawing to record the information from the problem – the 100% is 60 dollars and the 25% is the unknown dollars. The \$60 for the whole 10x10 grid means each row is \$6 and as 25% is $2\frac{1}{2}$ rows, this gives $6 + 6 + 3 = \$15$ for the 25%.



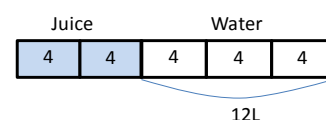
Rate

We can also do simple rates with size diagrams, as on right, with whole numbers. For example, the problem *The apples are packed 6 apples/bag, then how many bags for 18 apples?* can be solved with a size diagram that shows 6 apples to 1 bag. Once again, the size diagram is drawn and the students are encouraged to put information from the problem on the diagram. There are 18 apples for the 6 squares – so each square is 3 – so there are 3 bags.



Ratio

Finally, we can do simple ratio problems based on size diagrams for fractions. Consider the problem: *Juice is mixed with water 2:3, how much juice for 12 L of water?* We draw a size diagram as on right showing 2 beside 3 for parts of the whole. We encourage students to put information from the problem on the diagram. The 12L of water means each square is 4 so the juice is 8L.



Materials

We should spend time in the early years on activities such as this, using simple diagrams to act out the problems as we have above. However, before this, **act out the problems with real-life and physical materials** (e.g. Unifix instead of pictures of squares). The best is to find real-life materials to act out situations that are real to the students.

5.2 Models, applications and estimation in percent, rate and ratio

This section explores the three models that can be used to teach percent, rate and ratio problems and then shows how to use them to solve each of the three problem types for each of percent, rate and ratio. The section concludes with estimation and teaching. Each school has to make a decision on whether to teach: (a) all three methods for all three topic areas, (b) one method for all three areas, or (c) one method, but a different one, for each of the three areas. We recommend (a) because we believe that learning a repertoire of methods is what is important, and each of these methods is useful in future topics.

5.2.1 Size, double number line and change models

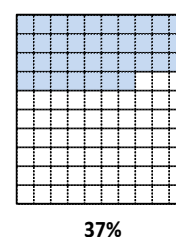
The models for percent, rate and ratio problems have to show the multiplicative relationship between the amount and percentage, the “per” attribute and the other attribute, and the first and second amount in the ratio. This can be done by:

- (a) **number-size diagrams** showing or representing size difference in terms of multiplication;
- (b) **double number lines** where above and below the line represent different amounts/attributes but are equivalent in the way they multiplicatively relate (either through fractions, division or proportion); and
- (c) **change diagrams** showing start, multiplier and finish.

Number-size diagrams

Number-size diagrams are using different-sized rectangles (usually) to show, pictorially and visually, the amounts and the attributes in a way that shows the multiplicativity of the relationship.

Percent. The need here is to show the relationship between 100% and the percent given in the problem. Since percent is per centum or per hundred, this can be done by shading the percent on a 10×10 grid which divides the whole into a hundred equal pieces or hundredths. See example of 37% on right – it shows three parts, shaded (37%), unshaded (63%) and whole (100%).

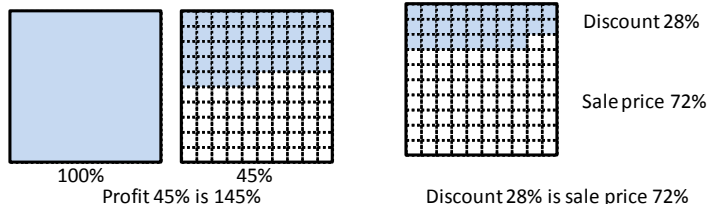


WHOLE (100%)	
SHADED PART 37%	OTHER PART 63%

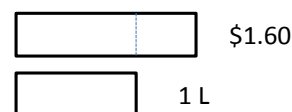
It is not necessary for diagrams to show the hundredths – any diagram that shows relative size can help the problems. So the percent can be represented by a plain square without the 10×10 grid. It can also be represented by a part-whole diagram as on left. This is particularly useful when the percents in

the problems are complicated, such as 17.45%. These percents are better suited to diagrams which do not show all the 100 squares.

It is necessary that students also understand how to shade in common percent situations such as profit and loss (or discount). Profit must be added to 100%, and discount or loss must be subtracted. Examples of these are shown on right.

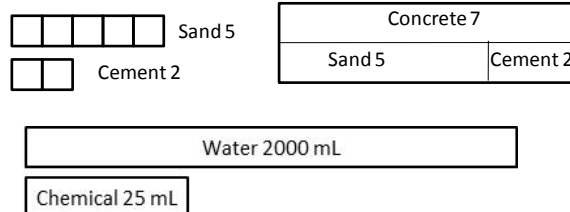


Rate. Rate is both easy and difficult to show with size diagrams – easy because one number is always 1; and difficult because the other can be a decimal which cannot be shown with simple squares. It is also not a part-part-whole situation. Still the diagram on right shows the relationship between the \$ and the L for rate \$1.60/L.



Ratio. Ratio is relatively straightforward to show with a size diagram. The example on right shows the two ways of representing ratio with size diagrams – using squares/rectangles and part-part-whole diagrams (where ratio is part-to-part). If the ratio is large like 25:2000 (e.g. 25 mL chemical mixed in 2 L water), the diagram would not be to scale – it would show a long rectangle beside a short one, as on right.

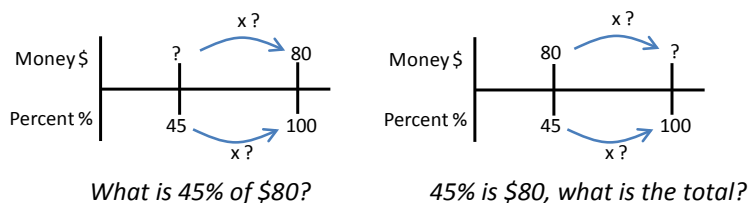
Ratio Sand:Cement is 5:2



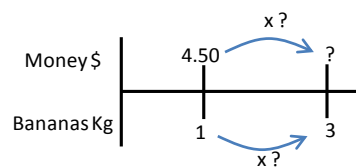
Double number lines

Double number lines are actually single lines on which numbers appear above and below for different attributes and amounts and using different scales. The numbers above and below are equivalent when on the same side of vertical lines that cross the number line.

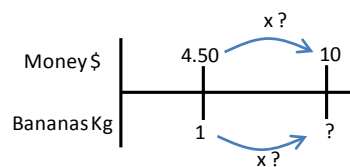
Percent. One side of the line is for percent and the other for what the percent is of. The 100% is the whole and is marked on the line, and any other percents or numbers are also marked. Steps are as follows: (a) draw the horizontal number line with a vertical line at the start; (b) label the attributes above and below the line (in this example, money and percent – place percent under the line); (c) mark little vertical lines for 100% and any other percents or amounts and put numbers on either side; (d) draw in the arrows above and below the number line; (e) mark unknowns with a ?; and (f) use the fact that the arrows have the same multiple to calculate the answers. See two examples below for: *What is 45% of \$80?* and *45% is \$80, what is the total?*



Rate. Rate is different because one attribute always relates to 1. Of course, this is really no different to percent where one number always relates to 100% which is also one, but it seems to be different. The same six steps are gone through. It is useful to always put the attribute that is 1 under the line. The diagrams below result for the two problems: *Bananas are \$4.50 a kg, what is the cost of 3 kg of bananas?* and *Bananas are \$4.50 a kg, how many kg of bananas can I buy for \$10?*

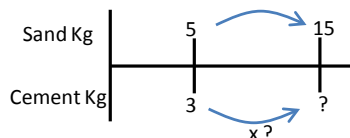


Bananas are \$4.50 a kg, what is the cost of 3 kg of bananas?

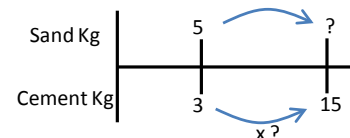


Bananas are \$4.50 a kg, how many kg of bananas for \$10?

Ratio. The double number line is excellent for ratio. There is no whole but it is normal to have the second attribute under the line. The six steps are as for percent. The diagrams below are for the problems: *Sand and cement are 5:3, how much cement for 15 kg of sand?* and *Sand and cement are 5:3, how much sand for 15 kg of cement?*



Sand and cement are 5:3, how much cement for 15 kg of sand?

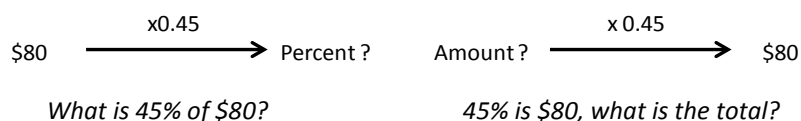


Sand and cement are 5:3, how much sand for 15 kg of cement?

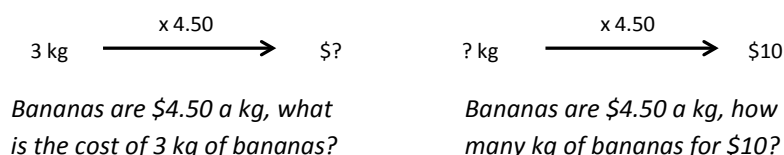
Change diagrams

We have already looked at change diagrams in subsection 5.1.1, so we will simply summarise by giving the diagrams for the problems from double number lines above. The answers are calculated by using multiplication if the finish is unknown and division otherwise.

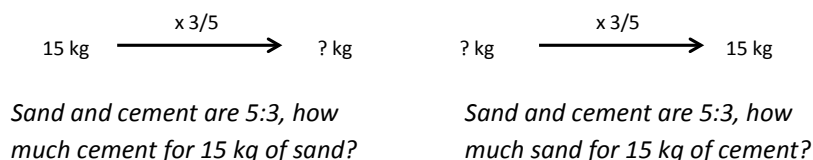
Percent. The multiplier is 0.45 for 45% and the start is 100% – the diagrams are as follows:



Rate. The change is from kg of bananas to \$ of money and the multiplier is 4.50 – the diagrams are as follows:



Ratio. The change is from sand to cement in kg and the multiplier is a number which changes 5 to 3 which is $\frac{3}{5}$ or 0.6 – the diagrams are as below:



5.2.2 Percent applications and estimation

The three models, number-size, double number line and change, can all be used with percent applications, and for each of the types of problems:

Type 1: Percentage unknown (e.g. find 45% of \$675)

Type 2: Total amount or 100% unknown (e.g. 45% is \$675, how much is the total?)

Type 3: Percent unknown (e.g. what % is the \$675 of \$1500?).

However, the aim in this section is for students to be able to go from problem to answer on their own. Thus, they need to be able to do two things **without the assistance** of the teacher:

- A:** take a percent problem and turn it into a diagram based on number-size, double number line or change, with numbers and unknown correctly placed on diagrams; and
- B:** use the diagram to solve the problem.

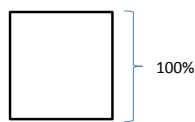
Therefore in this subsection, we will look at the steps for each of these things, once in detail and then in summary form for each model and each problem type (three models and three problem types means that there will be nine cases).

Number-size diagram method

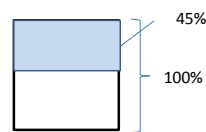
In the number-size model, we will use an abstract drawing for 100% – it will be square but the little squares will not be shown.

Problem Type 1: Find 45% of \$675

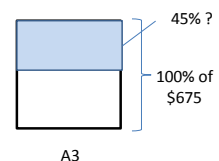
Step A1: Draw the 100% square.



Step A2: Mark in the 45%.



Step A3: Put in amounts and unknown.



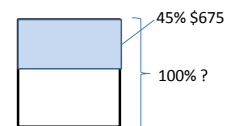
Step B1: Find some ways of getting from 100% to 45% – two ways are common (a) **unitary** – find the value of 1% ($\div$ by 100%) and then 45% ($\times 45$); and (b) **relationship** – think of 45 as $4\frac{1}{2}$ rows, find value of 1 row ($\div 10$) and $4\frac{1}{2}$ rows ($\times 4.5$). Note that you can also see 45% as $\frac{1}{2}$ a row less than 50%.

Step B2: Do the calculation (for unitary this is $675 \div 100 \times 45 = 303.75$).

Step B3: Give the answer – 45% of \$675 is \$303.75.

Problem Type 2: 45% is \$675, what is the total?

Steps A1 – A3: Draw and fill in diagram.

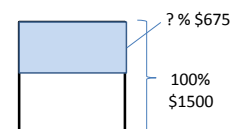


Steps B1 – B3: Calculate by finding a way from 45 to 100 (unitary $\div 45$ to get 1 and $\times 100$).

Answer – 45% is 675, total = $675 \div 45 \times 100 = \1500 .

Problem Type 3: What % is \$675 of \$1500?

Steps A1 – A3: Draw and fill in diagram.



Steps B1 – B3: Use \$675 and \$1500 to get a decimal, change to hundredths, for percent (675 is part and 1500 is whole, therefore fraction is $675/1500$ and this is $675 \div 1500$).

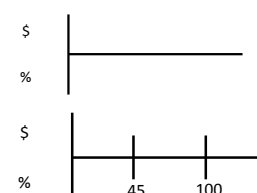
Answer – \$675 of \$1500 gives percent = $675 \div 1500 = 0.45 = 45\%$.

Double number-line method

In the double number-line model, we will look at \$ above and % under the line.

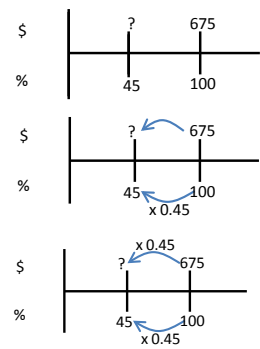
Problem Type 1: Find 45% of \$675

Step A1: Draw the outline of the double number line with \$ above and % below.



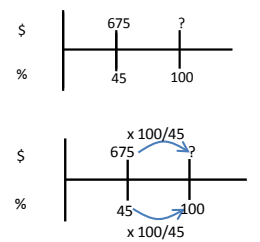
Step A2: Place in the 100% (the whole) and the line for 45%.

- Step A3: Put in amounts and unknown.
- Step B1: Work out the multiplication that connects the side with the two numbers; e.g. 45 and 100 – look for a multiplier that will change 45 to 100 or 100 to 45. This could be $\times 100/45$ to right or $\times 0.45$ to left. Left is the way to get from \$675 to ? so go this way.
- Step B2: Put same change on other side and calculate unknown.
 $? = 675 \times 0.45 = 303.75$.
- Step B3: Give the answer – 45% of \$675 is \$303.75.



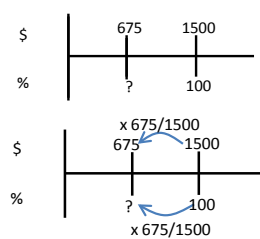
Problem Type 2: 45% is \$675, what is the total?

- Steps A1 – A3: Draw and fill in number line. (Note: arrow has to go to right.)
- Steps B1 – B3: Put in arrows, calculate $\times$ for arrow between numbers (here $\div 45 \times 100$) and put this on other side.
 Answer is $675 \times 100 \div 45 = \1500 .



Problem Type 3: What % is \$675 of \$1500?

- Steps A1 – A3: Draw and fill in number line.
- Steps B1 – B3: Put arrow to left and calculate $\times$ for side with numbers and use this on other side to calculate ?
 Answer is $675 \div 1500 = 0.45 = 45\%$.

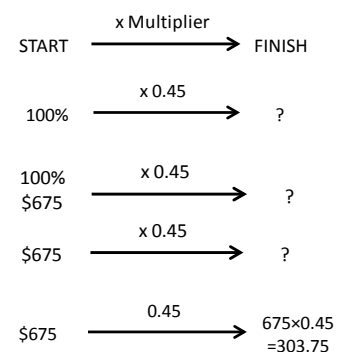


Change diagram method

In the change model we have start, finish and multiplier, where start is always the 100%.

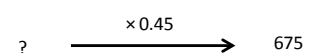
Problem Type 1: Find 45% of \$675

- Step A1: Draw the outline of the change model.
- Step A2: Place in the 100% and the multiplier, here 45% or $\times 0.45$.
- Step A3: Put in amounts and unknowns.
- Step B1: Determine whether ? is placed for $\times$ or $\div$ (here $\times$)
- Step B2: Calculate ?
- Step B3: Give the answer – 45% of \$675 = \$303.75.



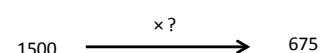
Problem Type 2: 45% is \$675, what is the total?

- Steps A1 – A3: Draw and fill in information on change diagram.
- Steps B1 – B3: Calculate answer (work out whether this is $\times$ or $\div$).
 The total is $675 \div 0.45 = \$1500$.



Problem Type 3: What % is \$675 of \$1500?

- Steps A1 – A3: Draw and fill in information on change diagram.
- Steps B1 – B3: Calculate answer (work out whether this is $\times$ or $\div$).
 The % is $675 \div 1500 = 0.45 = 45\%$.



5.2.3 Rate applications

Here we repeat the process for subsection 5.2.2 but only using summaries (full methods in 5.2.2). The three models are number-size, double number line and change. The three problem types are:

Type 1: First attribute unknown (e.g. petrol is \$1.64/L, how much in \$ for 52 L?)

Type 2: Second attribute unknown (e.g. petrol is \$1.64/L, how many L for \$52?)

Type 3: Rate is unknown (e.g. what is the rate when paying \$85.28 for 52 L?).

There are still the two steps, A and B, to teach.

Number-size diagram method

We again use an abstract drawing to show size in the diagram. For the first two problem types, the rate is \$1.64/L.

Problem Type 1: How much for 52 L?

Steps A1 – A3: Draw and fill in information on diagram.

\$  1.64 (\$?)

Steps B1 – B3: Calculate ?. Using a unitary approach, 1 is 52 L so 1.64 is $1.64 \times 52 = 85.28$. Thus, 52 L costs \$85.28.

L  1 (52L)

Problem Type 2: How many L for \$52?

Steps A1 – A3: Draw and fill in information on diagram.

\$  1.64 (\$52)

Steps B1 – B3: Calculate ?. Using a unitary approach, 1 unit = $52 \div 1.64 = 31.7$. So, fuel for \$52 = 31.7 L.

L  1 (L?)

Problem Type 3: What rate if pay \$85.28 for 52 L?

Steps A1 – A3: Draw and fill in information on diagram.

\$  ? (\$85.28)

Steps B1 – B3: Calculate ?. As a rate this is $\$ \div L = 85.28 \div 52 = 1.64$. So rate for \$85.28 for 52 L is \$1.64/L.

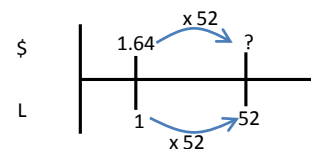
L  1 (52L)

Double number-line method

The first attribute is above the line and the second (the “per” one) is below the line. For the first two problem types, the rate is \$1.64/L (the price of petrol).

Problem Type 1: How much for 52 L?

Steps A1 – A3: Draw and fill in information on number line (the arrows go to the ?).

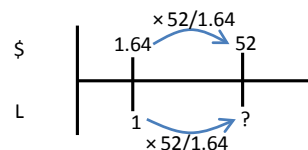


Steps B1 – B3: Calculate ? by determining the multiplication from the side with the numbers.

The cost of 52 L is $1.64 \times 52 = \$85.28$.

Problem Type 2: How many L for \$52?

Steps A1 – A3: Draw and fill in information on number line.



Steps B1 – B3: Calculate ? from the multiplication from the side with numbers.

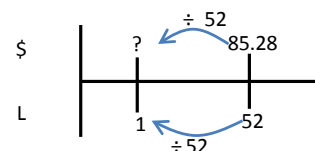
The litres of petrol for \$52 is $52 \div 1.64 = 31.7$ L.

Problem Type 3: What rate if pay \$85.28 for 52 L?

Steps A1 – A3: Draw and fill in information on number line.

Steps B1 – B3: Calculate ? using change on numbers side.

The rate is $85.28 \div 52 = \$1.64/\text{L}$.



Change diagram method

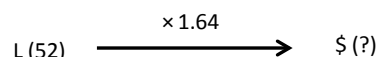
As for all rates, the starting number is always the second attribute (the “per” attribute) and the finishing number is the first attribute. For the first two problem types, the rate is \$1.64/L (the price of petrol).

Problem Type 1: How much for 52 L?

Steps A1 – A3: Draw and fill in information on change diagram.

Steps B1 – B3: Calculate ? (work out whether $\times$ or $\div$).

The cost of 52 L is $52 \times 1.64 = \$85.28$.

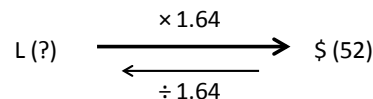


Problem Type 2: How many L for \$52?

Steps A1 – A3: Draw and fill in information on change diagram.

Steps B1 – B3: Calculate ? by inverting $\times 1.64$.

The litres for \$52 is $52 \div 1.64 = 31.7 \text{ L}$.

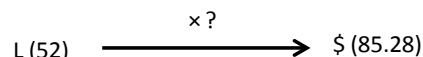


Problem Type 3: Rate if pay \$85.28 for 52 L?

Steps A1 – A3: Draw and fill in information on change diagram.

Steps B1 – B3: Calculate ? by division.

The rate is $85.28 \div 52 = \$1.64/\text{L}$.



5.2.4 Ratio applications

Here we again repeat the process from subsections 5.2.2 and 5.2.3 (but again using only summaries). Again the three models of size, double number line and change are used. The three problem types are:

Type 1: The second amount is unknown (e.g. red and yellow colouring are mixed in ratio 4:7, how many mL of yellow for 280 mL of red?)

Type 2: The first amount is unknown (e.g. red and yellow colouring are mixed in a ratio 4:7, how many mL of red for 280 mL of yellow?)

Type 3: The ratio is unknown (e.g. red and yellow colouring are mixed, 280 mL of red and 490 mL of yellow, what is the ratio?).

There are still two parts (A and B) that have to be taught, that is A to draw the diagram, and B to solve it.

Number-size diagram method

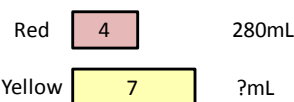
We again use an abstract drawing to show size in the diagram. For the first two problem types, red:yellow is 4:7.

Problem Type 1: How much yellow for 280 mL of red?

Steps A1 – A3: Draw and fill in information on diagram.

Steps B1 – B3: Calculate ? by, for example, unitary method (if 4 is 280, then 1 is 70, and 7 is $70 \times 7 = 490$).

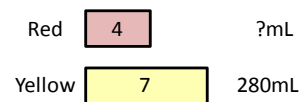
Answer: 490 mL of yellow is needed for 280 mL of red when ratio is 4 red: 7 yellow.



Problem Type 2: How much red for 280 mL of yellow?

Steps A1 – A3: Draw and fill in information on diagram.

Steps B1 – B3: Calculate ? by, for example, unitary method (if 7 is 280, then 1 is 40, and 4 is $40 \times 4 = 160$).

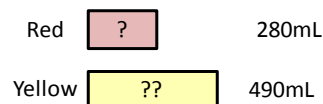


Answer: 160 mL of red is needed for 280 mL of yellow when ratio is 4 red: 7 yellow.

Problem Type 3: What is ratio if 280 mL of red is mixed with 490 mL of yellow?

Steps A1 – A3: Draw and fill in information on diagram.

Steps B1 – B3: Calculate the unknowns by equivalence or proportion: $280:490 = 28:49 (\div 10) = 4:7 (\div 7)$.



Answer: red and yellow are in ratio 4:7.

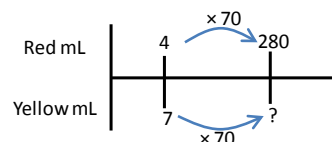
Double number-line method

For the double number-line method, the normal way is to put the second value in the ratio under the line. For the first two problem types, the ratio is red:yellow = 4:7.

Problem Type 1: How much yellow for 280 mL of red?

Steps A1 – A3: Draw and fill in information on number line.

Steps B1 – B3: Calculate ? using multiplication or division from side with all the numbers.

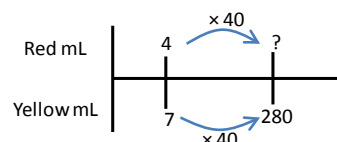


Answer: amount of yellow is $7 \times 70 = 490$ mL.

Problem Type 2: How much red for 280 mL of yellow?

Steps A1 – A3: Draw and fill in information on number line.

Steps B1 – B3: Calculate ? using multiplication or division from side with all the numbers.

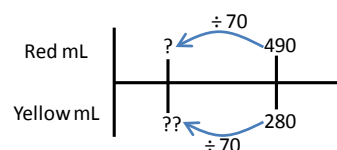


Answer: amount of red is $4 \times 40 = 160$ mL.

Problem Type 3: What is ratio if 280 mL of red is mixed with 490 mL of yellow?

Steps A1 – A3: Draw and fill in information on number line.

Steps B1 – B3: Calculate unknowns ? and ?? by looking for common multiple or divisor that applies to both and reduces numbers to where they can no longer be reduced.



Answer: red and yellow are in ratio 4:7.

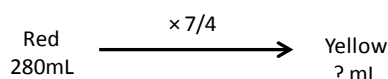
Change diagram method

For ratio, the starting number for change is the first amount or attribute and the finishing number is the second number or attribute. The multiplier is a fraction, second number $\div$ first number. Once again the ratio for the first two problem types is 4 red : 7 yellow.

Problem Type 1: How much yellow for 280 mL of red?

Steps A1 – A3: Draw and fill in information on change diagram.

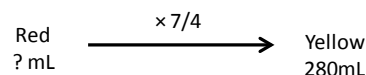
Steps B1 – B3: Calculate ? by working out whether $\times$ or $\div$.



Answer: amount of yellow is $280 \times 7 \div 4 = 490$ mL.

Problem Type 2: How much red for 280 mL of yellow?

Steps A1 – A3: Draw and fill in information on change diagram.

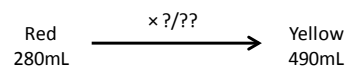


Steps B1 – B3: Calculate ? by realising this is $\div$ as reversing the arrow.

Answer: amount of red is $280 \div \frac{7}{4} = 280 \times \frac{4}{7} = 160$ mL.

Problem Type 3: What is ratio if 280 mL of red is mixed with 490 mL of yellow?

Steps A1 – A3: Draw and fill in information on change diagram.



Steps B1 – B3: Calculate ? by realising this is cancelling down situation
for $\frac{490}{280} = \frac{7}{4}$

Answer: red and yellow are in ratio 4:7.

5.2.5 Estimation and teaching

This subsection looks at the role of estimation and teaching approaches in percent, rate and ratio applications.

Estimation

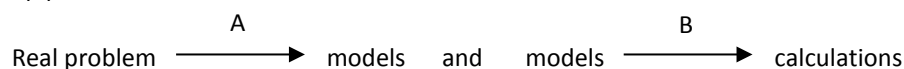
All calculations are multiplication and division. Thus, the section in Chapter 3 of this YDM Operations book (section 3.5) on estimation of these two operations should be useful. As the following will show, this means that students need (a) good estimation of multiplication and division; and (b) an understanding of what percent, rate, and ratio mean in terms of the numbers they multiplicatively compare.

1. **Estimate first.** Always try to estimate the answer before calculating. For example 52×1.64 is larger than $52 \times 1 = 52$ and smaller than $52 \times 2 = 104$. As 1.64 is a little over halfway between 1 and 2, then the estimate is a little over halfway between 52 and 104 (around about 80 to 85). As another example, $52 \div 1.64$ would be a little under halfway between $52 \div 1 = 52$ and $52 \div 2 = 26$ (under because dividing by more than halfway). (Note: this is using straddling estimation strategy from section 3.5.)
2. **Inverse relation.** As some of the numbers can be decimals or fractions that are less than 1, students need to know that multiplying by a number less than 1 reduces the answer while dividing by a number less than 1 increases the answer. For example, 30% is 0.3 and 30% of \$70 is 70×0.3 and this will be less than 70. Similarly $120 \div 0.3$ will be larger than 120 (and more than double as 0.3 is less than $\frac{1}{2}$).
3. **Relate estimation to what percents, rates and ratios are.** For example, a profit of 40% will make a number larger, while a loss of 25% will make a number smaller. Similarly red 4 : yellow 7 means yellow is always nearly double red, and red is always a bit more than $\frac{1}{2}$ of yellow. Finally, a rate of \$1.64/L means number of dollars are more than number of litres and number of litres are less than number of dollars, while a rate of \$0.45/kg means dollars are less than number of kg and number of kg are more than number of dollars.
4. **Realise difference between additive and multiplicative comparison.** Number multiplication is different and **faster acting** than addition: 8 being 4 more than 4 is the same additively as 21 being 4 more than 17; however, multiplicatively, 8 is twice 4 whereas it requires 34 (instead of 21) to be twice 17.

Teaching for mental models

It is important to remember that the teaching objective of what we have covered in this section is to have mental models of percent, rate, and ratio that students can use to get answers to problems and applications. This means students need both of the following learnings: (a) to translate real-world problems to mental models (we called this A); and (b) to use the models to calculate answers (we called this B). Thus, what we have denoted by A and B has to be the focus of our teaching – to have: (a) lessons and worksheets that **relate real problems to models as diagrams**, and (b) lessons and worksheets that **use the models to determine the calculations** (the actual calculations can be done with calculators).

This is a two-step process:

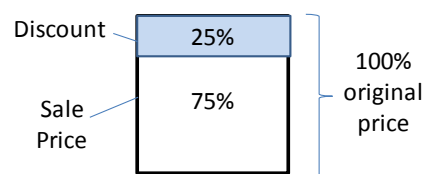


Finally, our aim is for students to be able to do A and B in their **minds** (using no drawings, or with just quick doodlings). So there is a need to ensure that students move away from formal drawn models to their own quick way of drawing things or to visualisation in the mind.

Body → Hand → Mind is a good way to do this:

- act out problems so students see what is involved;
- use drawings of models on which to place information so students can work out what calculations to use (and do not make incorrect suppositions); and
- get students to visualise the models and put problems into these imagined models to determine the calculations that have to be done.

For example, many students when given the problem, *There was a 25% discount, the dress sold for \$60, how much was the pre-sale price?* will find 25% of \$60 = \$15 and add this to \$60 to get \$75. However, a model such as that on right will show that 75% is the sale price and 100% is the original price so \$60 has to be divided by 75 and multiplied by 100 (which gives \$80 using a calculator).



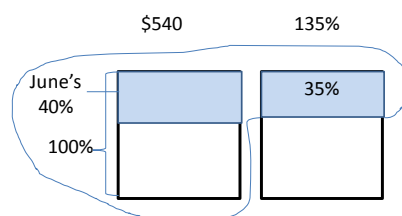
Teaching to practise variety

Finally two points need to be made concerning effective teaching in this area.

- Use a variety of methods with a variety of examples.** First, it is important to ensure we practise both problem → models and models → calculation for a variety of situations. These would include the following.

Percent. Ensure that students practise doing percent examples where there is profit, loss, and discount in all the three problem types.

Ensure that students practise more-than-one-step problems, where you move from a profit to a percent (e.g. *John made a profit of 35% on the sale selling an antique desk for \$540, how much did he pay back to Jane who had given him 40% of the price of the desk?*). Note that our diagrams help here (see on right) – \$540 is 135% and June gets 40% which is $\frac{40}{135}$ of the \$540 = \$160.



Rate. Give examples that relate a lot of measures, not just \$ and L or kg. Give scales that are large, 850 kg/m², and small, 0.07 g/m³. Get students used to seeing what is the one or the unit.

Ratio. Give a variety of ratios – examples with the same units and examples with different units. Have large unit differences, e.g. 20 mL : 5 L which is 20:5000. Explore examples of scale in maps as well as mixing things.

Practise using a variety of models, do not just practise calculations. Spend time practising **problem → model or diagram**. Also reverse everything, practise **model or diagram → problem**.

- School decisions about variety taught.** Second, each school has to make a decision on what to do about the variety of methods with respect to percent, rate, and ratio. These three are the same thing mathematically but different in symbols and words used. We have shown three models. So a school decision has to be made between three options as follows:

- Teach all models in all situations.* Teach the full variety of ways to solve all percent, rate, and ratio problems. This is based on believing that giving students a repertoire of strategies/models is the important outcome of mathematics teaching.

- (b) *Teach one model for all situations.* Schools choose to use one method for all situations. The size model, particularly using the part-part-whole model for percent and ratio, is the easiest to teach – it is not powerful but quick to teach. The double number line has been successfully used in vocational education and training in TAFEs for low-performing students. Finally, the change model, if taught successfully, is the quickest and most powerful to use.
- (c) *Teach one model, but a different model, for the three situations.* There is some argument that percent is best through number-size diagrams, rate through change diagrams, and ratio through the double number-line method.

5.3 Financial and vocational maths

Percent, rate and ratio have many applications in financial mathematics (e.g. interest, profit, loss) and vocational mathematics (e.g. mixing chemicals, making concrete, working out how much fuel to take). In this section, we look at applications in finance and applications in trades and other vocations, and briefly explore larger and more complex applications in investigations and rich tasks.

5.3.1 Financial maths

In this subsection we will look at applications of percent in simple and compound interest, and profit and loss plus applications of rate and ratio in buying and selling and gambling.

Percent and simple interest

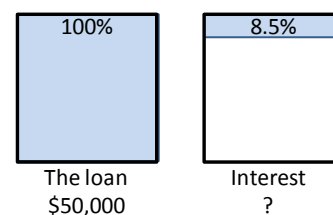
Interest is charged when an amount is borrowed; it is a percent of what is borrowed to be paid over a period of time (in normal finance this is usually a year). For example, \$50 000 may be borrowed and the interest rate is 8.5%/year. This means at the end of the year the borrower must pay back \$50 000 plus 8.5% of \$50 000.

There are two types of interest, simple and compound. Compound interest is when the interest is calculated and charged more than once in the year. For example, the 8.5%/year on the \$50 000 may be charged (we say compounded) every 3 months. This means that, for compound interest, the amount on which the interest is charged increases by the interest at each compound (see next subsection). The interest rate is pro-rata – the monthly interest rate would be $8.5\% \div 12$.

This subsection looks at simple interest, the next subsection at compound interest. Some points regarding working out simple interest on the problem *Joan borrowed \$50 000 at 8.5% per year, how much did she pay at the end of the year?* are given below.

1. **Size diagram.** We can draw a size diagram as on right – the interest is extra to the \$50 000, so we have two squares. The payment in % terms is 108.5. Calculating 1% (the unitary method) gives $50\,000/100 = 500$. So $108.5\% = 500 \times 108.5 = \$54\,250$. This is the amount Joan has to pay.

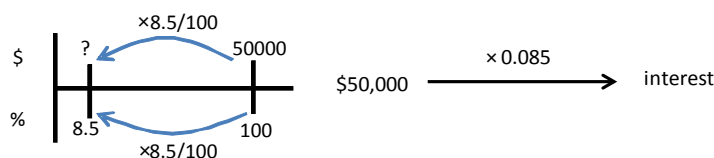
(Note: We should also discuss with students the similarity between profit and interest, that is if profit is 25%, then selling price is 125% and if interest is 8.5%, the repayment is 108.5%.)



2. **Other diagrams.** We can also do this as a double number line or a change diagram as on right.



(Note: If we need only to find the interest, then double number-line and change diagrams can be used as on right)



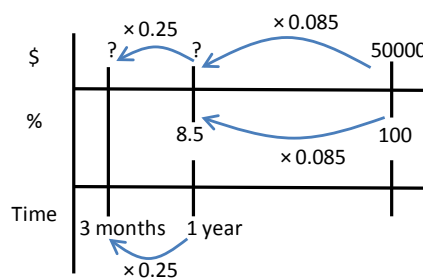
3. **Patterns.** If we look at interest problems in order to find a pattern or rule that we can use to replace the need for diagrams, it becomes evident that there are two rules:

- (a) simple interest is amount borrowed $\times$ interest rate as a decimal; and
- (b) repayment is amount borrowed + (amount borrowed $\times$ interest rate as a decimal).

4. **When time is different to a year.** Since interest is a rate per year, then the interest and the repayment depend on how long the amount is borrowed. For 3 months or $\frac{1}{4}$ of a year, the rate is $\frac{1}{4}$ of 8.5% or $8.5\% \div 4 = 2.125\%$. Thus amount paid back is:

$$\begin{aligned}
 \$50\,000 + \$50\,000 \times 2.125\% &= \$50\,000 + \$50\,000 \times 0.02125 \\
 &= \$50\,000 + \$1\,062.50 \\
 &= \$51\,062.50
 \end{aligned}$$

5. **Triple number line.** More complex problems like that in 4 above can be done with a triple number line, which works similarly to the double number line. In these diagrams, there are two lines and three parts – above, middle and below. We again enter all the data on the triple number line – the three parts are \$, % and time. We find that we have two unknowns but that we have sufficient knowns to work out all the change multiples (note that the change still always works towards the unknowns).



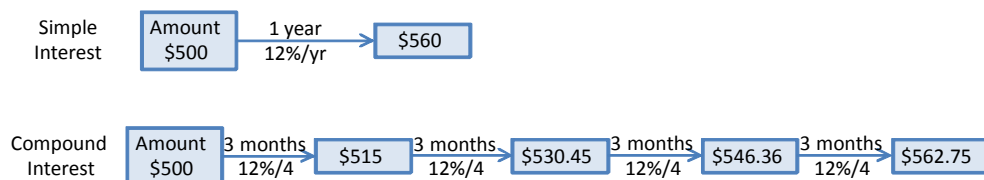
In the time part, 1 year to 3 months is $\frac{1}{4}$ so the 1 year is multiplied by 0.25. In the % part, interest is 8.5% so the 100 is multiplied by 0.085 (which is 8.5% as a decimal). Thus, in the \$ part, the interest on \$50 000 for 3 months is $\$50\,000 \times 0.085 \times 0.25 = \$1\,062.50$.

Percent and compound interest

Compound interest is when the interest from a previous period is added to the amount for the next period and the new interest acts on a larger number. Thus it is possible to have a 12% interest but this interest to be charged each month at 1% and the amount on which the interest acts to continually get larger. Consider the following comparison.

Simple interest 12%/year	Compound interest 12%/year compounded every 3 months
Amount borrowed \$500	Amount borrowed \$500
After year, amount owing is \$500 + interest	After 3 months, amount owing is \$500 + interest
$= \$500 + 12\% \text{ of } \500	$= \$500 + 12\%/4 \text{ of } \$500 = \$500 + 3\% \text{ of } \$500 = \$515$
$= \$500 + 0.12 \times \500	After 6 months amount owing is \$515 + 12%/4 of \$515
$= \$500 + \60	$= \$515 + 3\% \text{ of } 515 = \$515 + \$15.45 = \530.45
$= \$560$	After 9 months, amount owing is \$530.45 + 12%/4 of \$530.45
	$= \$530.45 + 3\% \text{ of } \$530.45 = \$530.45 + \$15.91 = \$546.36$
	After 12 months, amount owing is \$546.36 + 12%/4 of \$546.36
	$= \$546.36 + 3\% \text{ of } \$546.36 = \$546.36 + \$16.39 = \$562.75$
Interest is $\$560 - \$500 = \$60$	Interest is $\$562.75 - \$500 = \$62.75$

This simple interest – compound interest difference can be seen diagrammatically as follows.

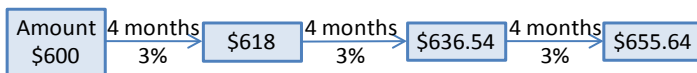
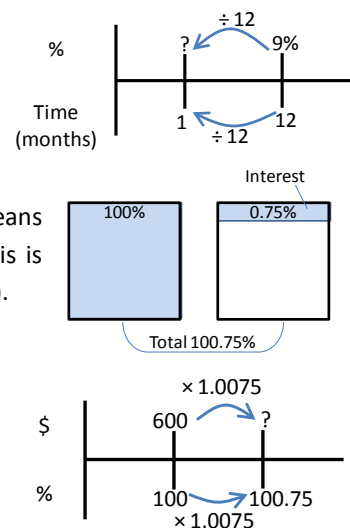


The compounding adds another \$2.75 to the amount to be paid and increases the real interest rate by $2.75/500 = 0.0055$ or 0.55%, that is more than an extra $\frac{1}{2}\%$. For credit cards, the interest rate used to be 18% compounded daily which was equivalent to 19.2% simple interest.

Note: Both simple interest and compound interest lead to formulae. These will be discussed under extension to algebra in section 5.4 that follows this section.

To teach compound interest, go through the following steps:

- Length less than a year.** Spend time teaching that interest rates are normally per year and have to be reduced when only acting for part of a year. For example, a rate of 9% per year is $\div 12$, $9/12 = 0.75\%$ for one month. The double number line helps: $? = 9 \div 12 = 0.75\%$.
- Changing interest to total amount.** Practise that an interest rate of 0.75% means that the repayment is $100\% + 0.75\%$ (amount + interest). Practise that this is 100.75% or the same as 1.0075 of the loan. A size diagram helps (as on right).
- Make students familiar with calculations of interest for a short length.** Spend time practising calculation of interest for short periods (the double number line helps again). For example, a loan of \$600 for a month at 9%/year is interest rate per month, which is 0.75% and 100.75% as total amount. This gives final amount of \$604.50.
- Do series of calculations across short periods.** For example, \$600 at 9%/year compounded every 4 months is 3 calculations of 3% from \$600 to final amount as the figure on right shows:



- Patterns.** Look for a pattern from simple interest to get a formula – see section 5.4. The formula is:

$$\text{Repayment} = \text{Amount} \left(1 + \frac{I}{M}\right)^N$$

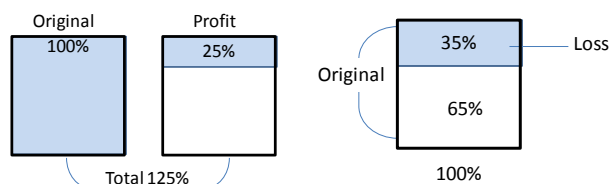
where I is interest, M is number of compounds per year and N is total number of compounds until repayment.

Profit and loss

Profit and loss require the percent to be calculated by adding profit to 100% and taking loss from 100%.

Thus using size diagrams:

- Profit of 25% means have original 100% plus the 25% = 125%. As a decimal this is 1.25.
- Loss of 35% means reducing original by 35% = 65%. As a decimal this is 0.65.



The usual practice is to look at profit and loss across a collection of cases. For example, *If I had \$5000 in shares and made a loss of 8% in a year, had \$3000 in property and made of profit of 12% in a year, and had \$3000 cash*

and made a profit of 5% in a year, what was the loss/profit of the whole amount? Taking each part separately, the table below gives the answer.

Starting Amount	Profit/Loss	Finishing Amount
\$5 000	Loss 8% = \$400	\$4 600 (0.92×5000)
\$3 000	Profit 12% = \$360	\$3 360 (1.12×3000)
\$3 000	Profit 5% = \$150	\$3 150 (1.05×3000)
\$11 000 TOTAL		\$11 110 TOTAL

The totals mean that a profit of \$110 was made out of the \$11 000. This is a $\frac{110}{11000} = \frac{1}{100} = 1\%$ profit.

The use of tables becomes very important in problems of the above type as the strategies of “break the problem into parts” and “exhaust all possibilities systematically” are used.

Rates and best buy in supermarket

A good application of rates is to visit a supermarket, look at costs and calculate the best buy. Supermarkets have already done a lot of the work for us by stating cost/weight on their displays.

There are **two rates** that can be used to determine best buy in a supermarket as the example of two cans of beans (A and B) in the table below shows.

A	B
Mass: 440g, Cost: \$2.10	Mass: 300g, Cost: \$1.55
Rate 1: Cost/weight (\$/g)	Rate 2: Weight/cost (g/\$)
A gives $\$2.10 \div 440\text{g} = 0.0048 \text{ $/g}$	A gives $440\text{g} \div \$2.10 = 210\text{g}/\$$
B gives $\$1.55 \div 300\text{g} = 0.0052 \text{ $/g}$	B gives $300\text{g} \div \$1.55 = 194\text{g}/\$$
Thus B is more expensive because each gram costs more.	Thus B is more expensive because you get less grams for each dollar.

(Note: Lower is better for Rate 1, i.e. \$/g; while higher is better for Rate 2, i.e. g/\$.)

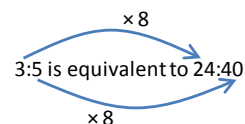
One could look at a lot of other situations and use rates to work out best value (although value is more than money – something may be more expensive but tastes better). For example, what is the costs of events (e.g. football games, concerts) in relation to how long they go for. This gives possibilities of working out cost/time (lower the better) and time/cost (higher the better).

Of course often there is more than one thing involved in determining value. For instance, a mobile phone might have higher text costs and lower call costs or vice versa. This leads to more complex investigations – see subsection 5.3.3.

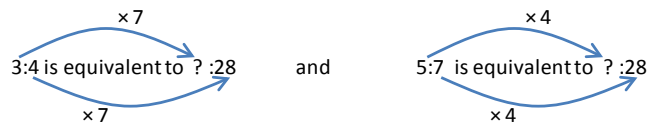
Ratio and gambling

Many gambling situations have odds. For example, *Winning on one game (game P) could be 3 chances in 7 (i.e. win:loss is 3:4) and in another game (game Q) 5 chances in 12 (i.e. win:loss is 5:7). Which of these gives better value?* There are two ways to do this, as follows.

1. **Proportion.** Similar to fractions, whether two ratios are equivalent, or in proportion, is determined by cancellation down to smallest possibility, and determining whether they are the same ratio. For example 24:40 and 33:55 both cancel down to 3:5. So they are equivalent or in proportion. This is because equivalence is a result of multiplying both numbers by the same amount (see example on right).



To compare two ratios that are not in proportion, we do the same as we did with equivalent fractions – we look for a common second number. Similar to equivalent fractions, the common second number for ratios 3:4 and 5:7 is found by multiplying the existing two second numbers (e.g. 4 and 7) which gives 28. We now use the rule for proportion to find ratios equivalent to 3:4 and 5:7 with a second number of 28 (as below).



We see that $3:4 = 21:28$ and $5:7 = 20:28$, so game P (the first game) is better value.

Note: This way of relating ratios by using a common second term has been adopted (as we have said before) by race betting. Race betting odds are now always in relation to a fixed second number, 10. So 3:4 would be 7.5:10 and 5:7 would be 7.1:10, thus evaluating the first odds to be a better chance.

- Percent.** The ratios can be turned into a percent by dividing the second number by the first and changing the decimal to a %. Since the ratio is win:loss, the second number divided by the first gives percent loss while the first number divided by the second gives percent win:

2nd by 1st (% loss)	1st by 2nd (% win)
Game P: $4 \div 3 = 1.33 = 133\%$	Game P: $3 \div 4 = 0.75 = 75\%$
Game Q: $7 \div 5 = 1.4 = 140\%$	Game Q: $5 \div 7 = 0.71 = 71\%$

Putting both these together, game P has the lowest loss % and the highest win %.

5.3.2 Vocational mathematics

Percents, rates and ratios are very important in vocational education and training. Nearly all trades and vocations have some role for them and, thus, a requirement for students to know how to use them.

In particular, ratio is widespread in its application:

- construction has different types of concrete to make from sand and cement;
- beauticians have to mix colours for hair colouring;
- nurses have to provide medicines in relation to body mass; and
- sprayers have to mix chemicals to remove pests or kill plants.

However, rates are also prevalent, for example using speed to work out time of travel, using fuel usage ratios to work out how much fuel to put in a boat to travel a certain distance, and so on.

Some examples of vocational uses are now given. For the spraying, the examples are from actual labels from bottles and cans of chemicals.

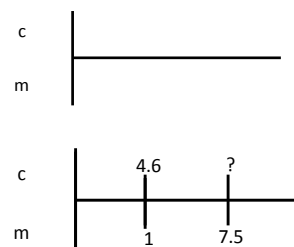
Rate examples from construction

Example 1: A mobile phone call costs 4.6 cents/minute, how much does a 7.5 minute call cost?

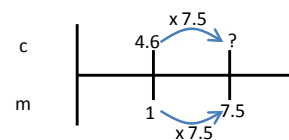
Step A1: Set up the double number line (c = cost, m = minutes).

Step A2: Mark in 1 minute and 7.5 minutes.

Step A3: Put in amounts and unknowns.



- Step B1: Work out the multiple. Use the other value to develop a multiplier.
- Step B2: Calculate the unknown: $? = 4.6 \times 7.5 = 34.5$.
- Step B3: Give the answer: The 7.5 minute call would cost 34.5 cents, \$0.35.



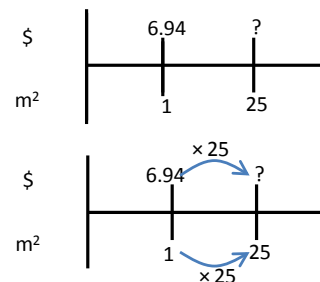
(Note: Once the double number line is identified, the multiplication of rates is straight forward.)

Example 2: Building materials cost \$6.94 per square metre. How much does it cost to build a building that covers 25 m² with the same materials?

Steps A1 – A3: Draw and fill in information on diagram.

Steps B1 – B3: Calculate ? using multiplication or division from side with all numbers.

Answer: Cost of building materials is $6.94 \times 25 = \$173.50$.



Ratio examples from spraying

Example 1: Chemical is mixed with 30 mL to 2 L of water, how much chemical for 5 L of water?

Step A1: Set up the double number line, with chemical above, and water below the line.

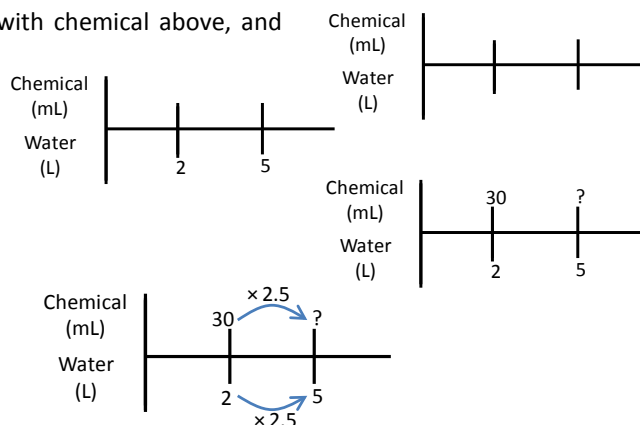
Step A2: Mark in 2 L and 5 L.

Step A3: Mark in amounts and unknowns.

Step B1: Work out the multiple. Use other values to develop a multiplier.

Step B2: Calculate the unknown:
 $? = 30 \times 2.5 = 75$.

Step B3: Give the answer: 75 mL of chemical is needed for 5 L of water.



Example 2: How much chemical has to be mixed with how much water to spray 85 m² of wall if the label says 150 mL of pesticide has to be diluted with 5 L of water to cover 100 m².

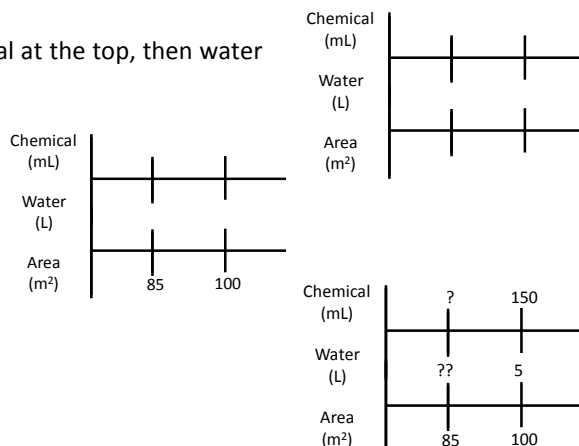
Most labels require the spray to be prepared in relation to both (1) a chemical-to-water ratio and (2) a mixture-to-area ratio. So this means both of these have to happen together. This means combining two double number lines into a triple line as follows:

Step A1: Setup a triple number line with chemical at the top, then water and area.

Step A2: Mark in 85 m² and 100 m².

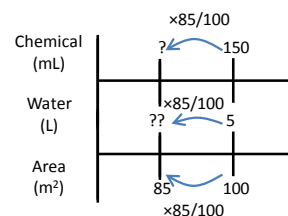
Step A3: Put in amounts and unknowns.

Step B1: Work out the multiple. Use other value to develop a multiplier.



Step B2: Calculate the unknowns:
 $? = 150 \times 85 \div 100 = 127.5$
 $?? = 5 \times 85 \div 100 = 4.25$

Step B3: Give the answer: To spray 100 m² of a wall with water requires 127.5 mL of chemical to be mixed with 4.25 L of water.



Note: The above examples come from booklets prepared to teach mathematics to TAFE Vocational Education and Training students. These books are on the YuMi Deadly Centre website <ydc.qut.edu.au> under [Vocational Learning resources](#) and are free to download. Look in particular for the Pest Management booklets.

5.3.3 Rich tasks and investigations

The ideas in subsections 5.3.1 and 5.3.2 above are just a start with regard to what is possible in financial and vocational mathematics in terms of percent, rate and ratio. They can be the basis of longer investigations or **rich tasks** that explore in an open-ended way. The following are examples of such tasks.

1. **Using the internet.** Use the internet to gather information to make decisions about something that is motivating to the students. For example:
 - (a) buying a smart phone and working out the best plan/agreement to be on;
 - (b) borrowing money to buy a car; or
 - (c) the difficulties with using money on a credit card.
2. **Reversing the procedure.** For example, in many applications, we start with a percent problem and end with an answer, so why not start with an answer and end with a problem? To reduce difficulty, we can limit the problem to having a % between 10 and 90 (and to not being a multiple of 10 if want to make it more difficult, and starting with an amount of 50 or over to make it simpler). A particular activity could be: *Find a type 2 problem with an answer of \$68?*

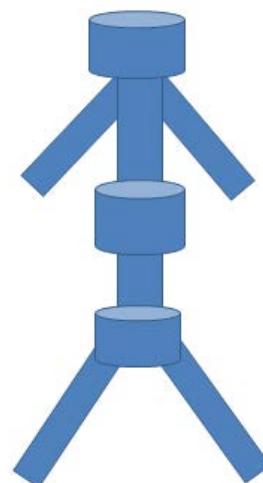
We have to start somewhere so we could say “what if \$51 was the % paid and then you had to find the total?”. As $\$51 \div \$68 = 0.75$, the problem could be “Jack paid 75% of the cost, which was \$51, what was the total cost?”

Other reversing problems could be to write a problem for: (a) Type 1 problem; answer \$124; (b) Type 2 problem; answer \$245; (c) Type 3 problem; answer 26%.

3. **Everyday activities.** Take an everyday activity for a truck company such as driving. Look at average speeds (km/hr) for truck drivers and average distances driven per hour. Consider how long a driver can drive until they have to rest.
 - (a) How long can a driver travel on average before they have to rest?
 - (b) How long would it take for a truck to travel to Melbourne and return with one driver?
 - (c) If you have to have 4 trucks per week travelling from Brisbane to Melbourne, how many drivers do you need?
4. **Other rich tasks.** Activities like the following, *Cheap houses* and *Look out for baby!* These are investigations into why square houses are cheaper and why you have to worry about your children and babies when the temperature is hot or cold. The reasons for this are as follows:
 - (a) The area/perimeter ratio for houses – you get the most area for the least perimeter in a square house. This makes it cheaper to build.
 - (b) The volume/surface area ratio for babies – small objects have a much smaller volume to surface area ratio meaning babies have more skin area to lose/gain heat and a smaller body volume to ensure there are no large changes in temperature.

5. **Ratio activities** like the following:

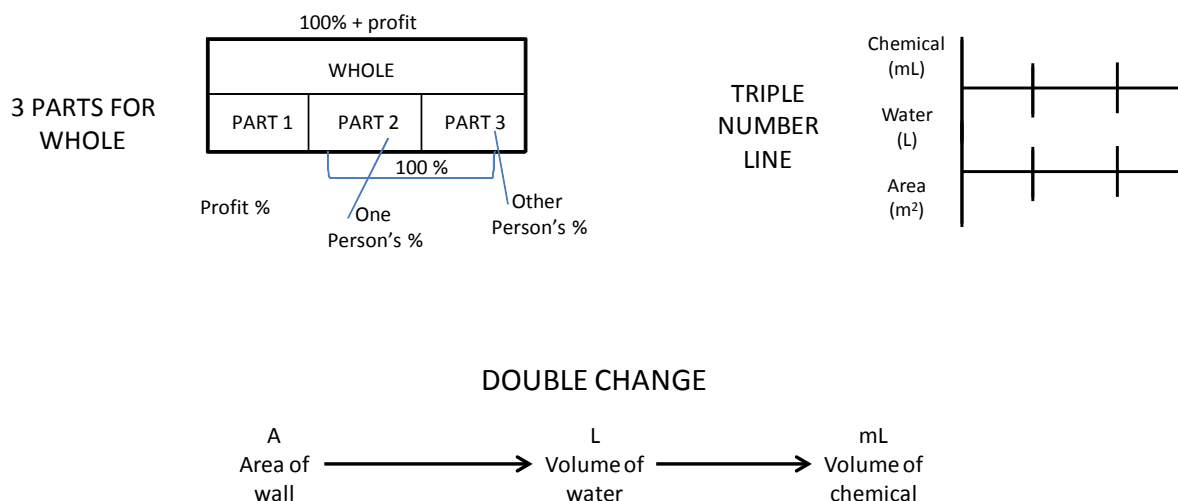
- Drawing faces.* Get students to draw their face. Then get students to measure the position of their hair, eyes, mouth and nose in relation to the sides of their face. For example, they will find the eyes are halfway down the face. Use this data to place the hair, nose, eyes, and mouth in a drawing of their head. Is this a more accurate drawing than the face they drew at the beginning?
- Making skeletons.* A skeleton can be made with strips of paper for backbone, arms and legs, and circular strips for head, chest, hips, etc. (as shown on the right). Pick someone in your group and make a $\frac{1}{2}$ and $\frac{1}{3}$ size skeleton. Compare the $\frac{1}{2}$ size one with the actual measurements of a one to two-year-old baby. Are there differences between the proportions between babies and students? (Yes there are, the head is larger proportionally in a baby.)
- Barbie monster.* Get a Barbie doll (preferably an older version) and measure her proportions. Make an adult size version of the doll (think of a way to do it). How do its proportions relate to ordinary people? Is it a monster? Discuss.
- How tall is the thief?* A house got robbed. The only clue is the footprint of a shoe in the ground beside the window used to break in. The shoe print is 34 cm long. Prepare a plan to determine the height of the thief. Make sure you have many trials to find a relation between height and foot length.



The YuMi Deadly Centre has a set of 12 Prevocational Mathematics books built around rich tasks like the above. These are available for download free from the YuMi Deadly Centre website <ydc.qut.edu.au> under [Vocational Learning resources](#). Many of the rich tasks in these booklets are built around rate and ratio.

5.3.4 Extension of models

Finally, it should be noted that, as problems become more complex, the models can extend as follows:



5.4 Extension to algebra

Algebra is the generalisation of arithmetic; thus, this extension, in particular, seeks to generalise the material so far in this chapter. This will be done in two subsections, the first on generalisations of models and procedures, and the second on formulae.

5.4.1 Generalisations

The chapter contains many options for generalisation. These often do not involve letters. They are a result of discussing examples and asking students for patterns – they are the final R in RAMR (reflection). **It is very important that this is done, and students are encouraged to develop their own “correct” generalisations even if they are informal and idiosyncratic.** Some useful ones are given below.

1. **Percent ↔ Number.** Percent is hundredths, therefore 27.8% becomes 0.278 as a decimal number. This change looks like as follows for PVC:

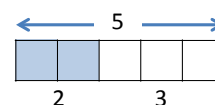
Percent (%)							Decimal Number						
H	T	O	t	h	th		H	T	O	t	h	th	
	2	7	.	8		↔			0	.	2	7	8

Thus the **general rule** for percent to decimal conversions is (a) numerals shift two places to right ($\div 100$) for percent to decimal number; and (b) numerals shift two places to left ($\times 100$) for decimal number to percent. The decimal number form is best for calculating percentage; for example, 27% of \$60 = 0.27×60 .

2. **Rate.** Rates are given as “number” “attribute”/“second attribute” (e.g. 24 km/hr, 6 Litres/km and \$3/kg). Rate problems, therefore, deal with these attributes. Looking at a lot of problems will show that attributes “cancel” as well as numbers (e.g. 3 hr @ 40 km/hr = 120 km; the hr/hr seems to become 1). This means the following general rules:
 - (a) if rate and first attribute is given, the operation is usually multiply; and
 - (b) if rate and second attribute is given, the operation is usually divide.

The generality (b) above comes from division, and its relation to multiplication by reciprocal. So, for example 160 km @ 40 km/hr, the calculation is $160 \text{ km} \div 40 \text{ km/hr} = 160 \text{ km} \times \frac{\text{hr}}{40 \text{ km}} = 4 \text{ hrs}$ (the km seem to divide away).

3. **Ratio and Proportion.** Ratios are part-to-part and fractions are part-to-whole. For example, in diagram on right, the 5 components give a ratio of 2:3 (part to other part) and two fractions $\frac{2}{5}$ and $\frac{3}{5}$.



In general, this means that ratio A:B gives two fractions ($\frac{A}{A+B}$ and $\frac{B}{A+B}$), while fraction $\frac{C}{D}$ gives another fraction $\frac{D-C}{D}$ and a ratio C:D–C.

Proportion divides ratios in sets equivalent to a starting ratio, that is, $A:B = 2A:2B = 3A:3B$ and so on (e.g. 2:3, 4:6, 6:9, and so on). This means that the rule for proportion is that ratios are in proportion (equivalent) if they cancel down to the same starting ratio (e.g. 12:16 cancels down to 3:4 and 51:68 cancels down to 3:4, so 12:16 and 51:68 are in proportion).

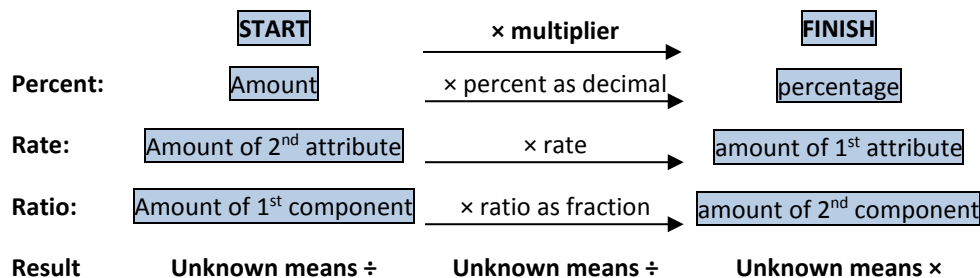
4. **Triad and Change.** All percent, rate and ratio examples have three components as follows:

Percent: Amount, percent and percentage

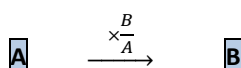
Rate: Amount of second attribute, rate, and amount of first attribute

Ratio: Amount of first component, ratio, and amount of second component

These translate to change as follows:



5. **Calculating multiplicative change.** If we have two numbers, say 56 and 32, then how we get from the first (56) to the second (32) by multiplication is $\times \frac{32}{56}$. In general:



5.4.2 Formulae

There are two formulae for interest.

1. **Simple interest.** Looking at examples, we can see that the interest is, as a decimal, Interest $\times$ Amount. This can become a formula:

$$P = IA \text{ where } P \text{ is interest to be paid, } I \text{ is interest rate as a decimal, and } A \text{ is the amount.}$$

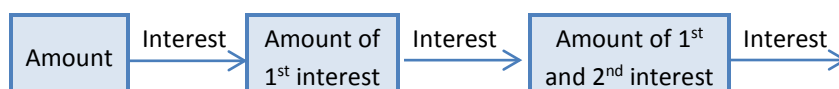
The amount to be repaid therefore is Amount + Interest $\times$ Amount and the formula is:

$$R = A + IA = A(1 + I) \text{ where } R \text{ is repayment and } A(1 + I) \text{ is } A \times (1 + I).$$

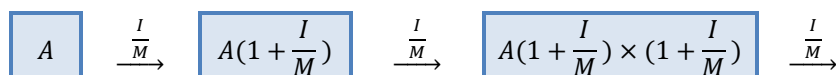
2. **Compound interest.** Compound interest is a series of simple interests. If the interest rate (I) is set for a year and the amount (A) is borrowed and interest is compounded M times a year, then, for the first period, the amount owing is:

$$A\left(1 + \frac{I}{M}\right)$$

Now compound interest acts as follows – as a series of $\frac{I}{M}$ interest amounts added to the starting amount:



However, for the second and subsequent compounds, the interest is not on A but on A plus the previous interest. So the compounding goes as follows:



and so on. If there are N compounds, then the final step gives:

$$\underbrace{A\left(1 + \frac{I}{M}\right) \times \left(1 + \frac{I}{M}\right) \times \left(1 + \frac{I}{M}\right) \times \dots \times \left(1 + \frac{I}{M}\right)}_{N \text{ times}}$$

Thus, the compound interest formula is:

$$R = A \times \left(1 + \frac{I}{M}\right)^N$$

In this formula, R is repayment, A is starting amount, I is yearly interest rate in decimal number form, M is number of compounds per year and N is total number of compounds before repayment.

6 Teaching Framework for Operations

The teaching framework organises the content for operations into a framework of four topics, namely concepts and principles, calculation, problem solving, and extension to algebra. Each of these topics is partitioned into sub-topics, chosen so as to represent ideas that recur across all year levels. The resulting framework is given in Table 1. This overall framework can be compared to the Australian Curriculum to produce year-level frameworks.

Table 1. Framework for teaching operations

TOPICS	SUB-TOPICS	DESCRIPTION AND CONCEPTS/STRATEGIES/WAYS	BIG IDEAS
Operation concepts and principles	Meanings	Forward, change, part-part-total/factor-factor-product Connections: real world $\leftrightarrow$ actions $\leftrightarrow$ models $\leftrightarrow$ language $\leftrightarrow$ symbols Special representations	Symbols tell stories Relationship vs change Interpretation vs construction Part-part-total
	Principles	Global properties Number size Field and extension of Field Equals (Equivalence class)	Identity, inverse, commutative, associative, distributive, inverse relation, compensation, equivalence, triadic
Calculation	Basic and extended fact strategies	+/-: counting and counting on, near doubles, near tens, think addition $\times/\div$: skip counting, patterns, connections, think multiplication $+/-\times/\div$: turnarounds, families Multiples of 10 facts, extension facts	Field (identity, inverse, commutative, associative, distributive); Extension (compensation, equivalence); Commutative, associative, distributive
	Automaticity	Practice games and drills $\rightarrow$ instant recall	
	Computation strategies	Separation (do parts separately) Sequencing (take a part and bring in rest in sequence – including additive subtraction) Compensation (do an easier but equivalent operation)	Field (identity, inverse, commutative, associative, distributive); Extension (compensation, equivalence)
	Estimation	Strategies: front end, rounding, straddling, getting closer Using strategies to estimate Determining required accuracy needed	Accuracy vs exactness, compensation, inverse relation
	Fraction	+/-: common denominators, mixed numbers, unlike denominators $\times/\div$: numerators-denominators, reciprocals, mixed numbers	Field (identity, inverse, commutative, associative, distributive); Extension (compensation, equivalence)
	Percent, rate and ratio strategies	Set/Area Double number line Change	Inverse, triadic, equivalence
Problem solving	Strategies	Visual (act, model, draw); Language (read, identify parts, restate, notation); Checking (check, find another solution, learn); Organising (pattern, table, possibilities, remove); Restructuring (guess, break into parts, backwards, hidden assumptions)	
	Thinking skills and plans	Thinking skills (e.g. visual, logical, creative, flexible) Plans of attack (e.g. See, Plan, Do, Check)	
	Word problems	Choosing an operation Constructing problems	
Extension to algebra	Variable	Unknowns, variables Connections: real world $\leftrightarrow$ actions $\leftrightarrow$ models $\leftrightarrow$ language $\leftrightarrow$ symbols Generalisations	



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