YuMi Deadly Maths

Number

Prep to Year 9

Prepared by the YuMi Deadly Centre Queensland University of Technology Kelvin Grove, Queensland, 4059





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VERSION 3, 29/03/16

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The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

DEVELOPMENT OF THIS BOOK

This version of the YuMi Deadly Maths Number book is a modification and extension of a book developed as part of the Teaching Indigenous Mathematics Education (TIME) project funded by the Queensland Department of Education and Training from 2010–12. The YuMi Deadly Centre acknowledges the Department's role in the development of YuMi Deadly Maths and in funding the first version of this book.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at QUT which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

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ABOUT YUMI DEADLY MATHS

From 2000–2009, researchers who are now part of the YuMi Deadly Centre (**YDC**) collaborated with principals and teachers predominantly from Aboriginal and Torres Strait Islander schools and occasionally from low socioeconomic status (SES) schools in a series of small projects to enhance student learning of mathematics. These projects tended to focus on a particular mathematics strand (e.g. whole-number numeration, operations, algebra, measurement) or on a particular part of schooling (e.g. middle school teachers, teacher aides, parents). They resulted in the development of specialist materials but not a complete mathematics program (these specialist materials can be accessed via the YDC website, <u>http://ydc.qut.edu.au/</u>).

In October 2009, YDC received funding from the Queensland Department of Education and Training through the Indigenous Schooling Support Unit, Central-Southern Queensland, to develop a train-the-trainer project, called the **Teaching Indigenous Mathematics Education** or **TIME** project. The aim of the project was to enhance the capacity of schools in Central and Southern Queensland Indigenous and low SES communities to teach mathematics effectively to their students. The project focused on Years P to 3 in 2010, Years 4 to 7 in 2011 and Years 7 to 9 in 2012, covering all mathematics strands in the Australian Curriculum: Number and Algebra, Measurement and Geometry, and Statistics and Probability. The work of the TIME project across these three years enabled YDC to develop a cohesive mathematics pedagogical framework, **YuMi Deadly Maths**, that covers all strands of the *Australian Curriculum: Mathematics* and now underpins all YDC projects.

YuMi Deadly Maths (YDM) is designed to enhance mathematics learning outcomes, improve participation in higher mathematics subjects and tertiary courses, and improve employment and life chances. YDM is unique in its focus on creativity, structure and culture with regard to mathematics and on whole-of-school change with regard to implementation. It aims for the highest level of mathematics understanding and deep learning, through activity that engages students and involves teachers, parents and community. With a focus on big ideas, an emphasis on connecting mathematics topics, and a pedagogy that starts and finishes with students' reality, it is effective for all students. It works successfully in different schools/communities as it is not a scripted program and encourages teachers to take account of the particular needs of their students. Being a train-the-trainer model, it can also offer long-term sustainability for schools.

YDC believes that changing mathematics pedagogy will not improve mathematics learning unless accompanied by a whole-of-school program to challenge attendance and behaviour, encourage pride and self-belief, instil high expectations, and build local leadership and community involvement. YDC has been strongly influenced by the philosophy of the Stronger Smarter Institute (C. Sarra, 2003) which states that any school has the potential to rise to the challenge of successfully teaching their students. YDM is applicable to all schools and has extensive application to classrooms with high numbers of at-risk students. This is because the mathematics teaching and learning, school change and leadership, and contextualisation and cultural empowerment ideas advocated by YDC represent the best practice for **all** students.

YDM is now available direct to schools face-to-face and online. Individual schools can fund YDM in their own classrooms (contact <u>ydc@qut.edu.au</u> or 07 3138 0035). This Number resource is part of the provision of YDM direct to schools and is the second in a series of resources that fully describe the YDM approach and pedagogical framework for Prep to Year 9. It focuses on teaching number and covers number basics, whole number numeration, decimal number numeration, common fraction numeration, and percent, rate and ratio. It overviews the mathematics and describes classroom activities for Prep to Year 9. Because YDM is largely implemented within an action-research model, the resources undergo amendment and refinement as a result of school-based training and trialling. The ideas in this resource will be refined into the future.

YDM underlies three projects available to schools: YDM Teacher Development Training (TDT) in the YDM pedagogy; YDM AIM training in remedial pedagogy to accelerate learning; and YDM MITI training in enrichment and extension pedagogy to build deep learning of powerful maths and increase participation in Years 11 and 12 advanced maths subjects and tertiary entrance.

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Human thinking has two aspects: verbal logical and visual spatial. Verbal logical thinking, associated in some literature with the left hemisphere of the brain, is the conscious processing of which we are always aware. It tends to operate sequentially and logically and to be language and symbol (e.g. number) oriented. On the other hand, visual spatial thinking, associated in some literature with the right hemisphere of the brain, can occur unconsciously without us being aware of it. It tends to operate holistically and intuitively, to be oriented towards the use of pictures, and seems capable of processing more than one thing at a time – as such it can be associated with what some literature calls simultaneous processing.

Our senses and the world around us have enabled both these forms of thinking to evolve and develop. To understand and to modify our environment has required the use of logic and the development of language and number. It has also required an understanding of the space that the environment exists in and of shape, size and position that enables these to be visualised (what we call geometry). Mathematics is therefore a product of human thinking that has emerged from solving problems in the world around us and has **two aspects** at the basis of its structure: **number and geometry**.

This book focuses on number and covers number basics, whole numbers, decimal numbers, common fractions, and percent, rate and ratio. It precedes other books covering operations, algebra, geometry, measurement, statistics and probability. Placing this book first in the series of YDM teaching books reflects the importance of number in modern global culture and in the mathematics to be taught in that culture. Number directly underpins operations, algebra, measurement, statistics and probability, and is even crucial in geometry (for properties and for the algebraic representations of shape). All of modern global culture – trade, economics, politics, travel, computers, banking, defence, and so on – is based on it. Number itself is based on more basic mathematics such as attribute recognition and sets but its centrality places it as the focus of the first teaching book. This centrality is even more marked in school mathematics for the first 10 years of formal schooling (for Queensland, Years P to 9).

The purpose of this book is to provide the starting point for school mathematics in Years P to 9. It is followed by the books on operations and algebra that extend from number. The books then return to the other aspect of mathematics, geometry, before addressing what might be called the "applications" of number: measurement, statistics and probability.

To begin, this chapter looks at connections and big ideas (section 1.1), sequencing (section 1.2), teaching and cultural implications (section 1.3), and overview of book (section 1.4).

1.1 Connections and big ideas

The unit or the one is the starting point for number. It is the basis for all numbers. Units can be grouped to produce whole numbers (ones, tens, hundreds and so on), or they can be partitioned to produce decimal numbers (tenths, hundredths, and so on) or common fractions (e.g. sixths, ninths, and so on). Through this relationship, whole numbers are connected to decimal numbers and common fractions, and to the other versions of fraction, that is, percent, rate and ratio. This book begins with whole numbers, the counting numbers plus zero. They are the starting point for number. It then extends these whole numbers to decimal numbers by introducing fractional place-value positions, then looks at common fractions in more detail and ends with percent, rate and ratio.

1.1.1 Connections

In particular, number is excellent for portraying how to teach mathematics so that it is learnt in the way YDM believes it should be, as a **rich schema** containing knowledge of **when and why as well as how**. Rich schema has knowledge as **connected nodes**, which facilitates recall and problem solving. YDM argues that knowledge of the structure of mathematics, particularly of connections and big ideas, can assist teachers to be effective and efficient in teaching mathematics, and enable students to accelerate their learning. It enables teachers to:

- (a) *determine what mathematics is important to teach* mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present;
- (b) *link new mathematics ideas to existing known mathematics* mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem solving;
- (c) choose effective instructional materials, models and strategies mathematics that is connected to other ideas or based around a big idea can be taught with similar materials, models and strategies; and
- (d) *teach mathematics in a manner that enables later teachers to teach more advanced mathematics* by preparing linkages to other ideas and foundations for big ideas later teachers will use.

To teach for rich schema, it is essential that teachers know the mathematics that precedes, relates to and follows what they are teaching, because they are then able to build on the past, relate to the present, and prepare for the future. If connections and big ideas are the basis of the curriculum, the teaching and learning of whole numbers is accelerated; it also enables the acceleration of other number topics, such as decimal numbers, common fractions, and percent, rate and ratio.

The connections relating whole numbers to other topics are not simply linear; for example, the concept of fraction is the basis of the decimal place-value positions such as tenths and hundredths. These fraction positions have to be developed to allow whole number knowledge to be extended to decimal number knowledge. Further on in the learning sequence, percent has to be understood in terms of both decimal numbers and common fractions. These connections are diagrammatically represented on the right.



In turn, these number concepts are the basis for operations and lead into algebra. Finally, through the inclusion of geometry, number, operations and algebra lead on to measurement and, more directly, to statistics and probability. In particular, there are strong connections between decimal numbers and metric units (notably in relation to metric conversions), and between common fractions and probability. This can be diagrammatically represented as below:



1.1.2 Models, symbolic representations and connectivity

Models show the mathematical similarity or equivalence of whole numbers, decimal numbers, common fractions, and percent, rate and ratio, while symbolic representations tend to show difference. In fact, to see the similarities that assist in learning, differences in notation have to be discounted.

Models common to all the number types are the usual ones of *set* (e.g. 5 can represent five chairs); *area* (e.g. a square divided into three equal pieces can represent $\frac{1}{3}$; *number line* (e.g. 1.4 can be represented on a ruler as

1 m and 40 cm); *position* (e.g. a chart of three columns named hundreds, tens and ones from left to right with 2 in hundreds, 3 in tens and 4 in ones can represent 234); *pattern*, (e.g. rows of fractions such as $\frac{2}{3}$ $\frac{4}{6}$ $\frac{6}{9}$ $\frac{8}{12}$... can be used to show the rule for equivalent fractions); and *change* (e.g. 45% is \$270 can be thought of as "total × 0.45 = 270" and the total is, therefore, 270 ÷ 0.45 = \$600). Each of these models applies to more than one number type and **connects these types**.

Symbolic and language representations differentiate and **act against connections** – making students think that the different number types represent different things. For example, a whole partitioned into 5 pieces and 3 being taken is $3\div5$ ("three divided by five"), 3/5 ("three fifths"), 0.6 ("zero point six" or "six tenths"), 60% ("sixty percent"), and 3:2 ("ratio three to two"), all different notations and languages. Yet only one is necessary – decimals. Fractions such as 3/5 could still be thought of as a whole divided into 5 equal pieces and taking 3 pieces but written and said as 0.6; 60% could be thought of as 60 hundredths but written and said as 0.6 (and 6% as 0.06), and 3 sand to 2 cement (3:2) could still be thought of as 3 parts to 2 parts with 5 as whole but written and said as 0.667 (the cement is 0.667 times the sand). However, because of history, we have all these different number types to describe the same things, all looking syntactically different.

1.1.3 Major big ideas

There are five major big ideas for all numbers that apply in a similar way to whole numbers, decimal numbers, common fractions, and percent, rate and ratio. These are as follows.

1. Part-whole (includes Notion of a unit). The basis of number is the unit which is grouped to make large numbers and partitioned to make fractions. Thus everything can be seen as part and whole – a ten is a tenth of a hundred and a whole of 10 ones. The value of each digit in the number system is determined by its position relative to the ones' position, by its place value. The value of each digit is given by that digit number of place values and the value of a number is found by adding these (i.e. 204 is 200+4). The pattern also holds for fractions (e.g. sixths) and measures whose base is not 10 (e.g. time).

There is also a sub-big idea in that the pattern for the groupings and partitioning in numbers is a **pattern of threes**, that is, ones-tens-hundreds ones, ones-tens-hundreds thousands, ones-tens-hundreds millions and so on. This means that knowledge can be transferred from ones to thousands to millions and so on. The concept for this big idea is **place value**, and the processes are **reading and writing**.

2. Additive structure. A number's value is the sum of its parts, e.g. 234 = 200+30+4. Thus, place values increase and decrease by adding and subtracting. The consequence of this is that each place-value position counts forwards and backwards and values of place-value positions increase and decrease.

There is also a sub-big idea in that counting follows a pattern, the **odometer pattern** – forwards from 0 to 9 and back to 0 as the number on left increases by 1 (e.g. 274, 284, 294, 304, 314, ...); backwards from 9 to 0 and back to 9 as number on left decreases by 1 (e.g. 234, 224, 214, 204, 194, 184, ...). For fractions, this counting pattern differs, as it does for measures (e.g. time) that are not base 10.

The concept for this idea is **counting** and the processes are **seriation** and **counting patterns**.

3. Multiplicative structure. A number's value is a sum of the products of value and place, e.g. 234 = 2×100 + 3×10 + 4×1. This means that there is a multiplicative relationship between adjacent place-value positions – values increase ×10 when moving to the left and decrease ÷10 when moving to the right. Thus, the value of a number is found by adding the multiples of digit × place value (e.g. 204 = 2×100 + 0×10 + 4×1).

The concept for this idea is **multiplicativity** and processes are **renaming** and **flexibility**. Flexibility is thinking of things more than one way by changing the unit (e.g. 300 is 30 tens or 0.3 thousands and so on).

4. Continuous vs Discrete/Number line. Regardless of place values and digits, each number is a single quantity represented by a point on a number line, has rank and can be compared to other numbers. The number-line representation changes perspective of number – for example, the 0 is no longer nothing, it is the starting position of positive whole numbers (and of rulers and other measuring devices).

The concepts for this idea are **comparison/order**, **rank** and **density** and the processes are **comparing/ordering**, **rounding**, and **estimating**. Order just means to work out the larger/largest but rank means to place on line where they should be proportionally. For example, 91 being larger than 32 can be shown by students in a row with 91 after 32, but rank is shown by the 32 being on a line one third of the way between 0 and 100 and the 91 being near the 100. Density is how many numbers between adjacent numbers.

5. **Equivalence**. Sometimes a single quantity can be represented by more than one number. For example, 04 is the same as 4, 2.40 is the same as 2.4, $\frac{2}{3}$ is the same as $\frac{4}{6}$, and 3:5 is the same as 6:10. Equivalence often reflects adding zero (the additive identity) or multiplying by one (the multiplicative identity).

1.1.4 Application of big ideas to number

This application is shown in the figure below. This figure has been designed as follows: (a) whole and decimal numbers are combined for brevity (and because they are similar); (b) the topics in whole and decimal numbers, common fractions, and percent, rate and ratio have been placed, as far as possible, to approximately align with year levels shown on left; (c) the big ideas are in italics and the dotted arrows show their sequence and the topics which they underlie; and (d) the down and angled solid line arrows show sequences within topics and the sequence, approximately, across topics but within a big idea.



1.2 Sequencing

The secret of sequencing is to build early work around big ideas that also apply to numbers further along the sequence and in other areas. The five big ideas listed above apply to the four number areas. This means that once these are covered in whole numbers, the ideas they represent can be quickly transferred to other number areas and into larger whole numbers. The sequence for how ideas move from number area to number area is given in the figure below.

As for big ideas, this is designed to show the sequences in number as follows: (a) the four areas, whole numbers, decimal numbers, common fractions and percent, rate and ratio, are separated and show a line of sequence with the vertical arrows; (b) the topics in each line of sequence are placed, where possible, to show an approximate alignment with year levels from P to 9 - as indicated on the left of the figure; (c) the number

areas are placed left to right, as far as possible, to show a sequence one topic to another across areas by diagonal arrows. The problem was where to put fractions. The figure places fractions on the right of decimals. This indicates, in general, that fractions follow decimals but this is not always the case. In fact, fractions precede decimals in some topics (e.g. fraction part-whole concepts before decimal place values).

For sequencing **across the number areas**, the figure below shows that: (a) whole-number place values are extended to decimal place value and in turn to percentage; (b) early number ideas lead to partitioning a whole unit which leads to common fractions; (c) common fractions introduce the decimal place values (e.g. tenths); (d) fractions and decimals are extended to percent, rate and ratio; and (e) decimal numbers, common fractions and whole numbers are extended to the rational and real number systems.



For sequencing **in the whole number/decimal number area**, the secret is to build ideas through the **pattern of threes** – that ones, thousands and millions (and so on) all have same pattern built around H-T-O (Hundreds, Tens and Ones) as on right. The sequence for whole numbers is:



	early number
	understanding everything about H-T-O ones
	extending this understanding to H-T-O thousands and H-T-O millions (and so on)
	extending this to H-T-O thousandths and H-T-O millionths (and so on)
7	whole number systems and directed numbers

Note: Sometimes extending from H-T-O ones to H-T-O thousands and millions does not work for all big ideas. In that case, the big idea will need to be revisited for the larger numbers. **Thus, the sequence from H-T-O ones to H-T-O thousands and millions will vary depending on student learning**.

1.3 Teaching and cultural implications

This section looks at teaching and cultural implications, including the Reality–Abstraction–Mathematics– Reflection (RAMR) framework and the impact of Western number teaching on Indigenous and low SES students.

1.3.1 Teaching implications

The first teaching implication is that, interestingly, and fortunately, mathematical ideas that are structurally similar can be taught by similar methods. For example, fractions and division are similar and both are taught by partitioning sets into equal parts – except that the set is seen as one whole for fractions and a collection of objects for division. Similarly, early grouping, place value for whole numbers, place value for decimals and mixed numbers can all be taught with size materials and position charts. The second teaching implication is that number instruction is both diverse and generic. The particular ways to teach the varied number types are given in later chapters. The major generic method that holds for all topics is given below.

The **RAMR framework** (see figure below) is the basis of lesson planning in YDM because of the generic teaching ideas contained in the framework, and is applicable to number teaching. For a start, it grounds all mathematics in reality and provides many opportunities for connections, flexibility, reversing, generalisations and changing parameters, as well as body \rightarrow hand \rightarrow mind. The idea is to use the framework and all its components throughout the years of schooling and this will help prevent learning from collapsing back into symbol manipulation and the quest for answers by following procedures.



The YuMi Deadly Maths RAMR framework

The **Payne and Rathmell triangle** is one of the underpinnings of the Abstraction and Mathematics stages of the RAMR framework. It is important in teaching number because of its focus on the relationship between models, language and symbols. Activities and questions should be constructed that encourage students to connect and move flexibly between models, language and symbols in all directions.

Real-world problem



The Payne and Rathmell (1977) triangle for early number

1.3.2 Cultural implications

Aboriginal and Torres Strait Islander students may find the teaching of number confronting because of the differences of the number-oriented culture of the mathematics classroom and their culture, and because many students are from low socio-economic status (SES) backgrounds.

Aboriginal and Torres Strait Islander cultures followed a different path from number-oriented cultures (European, Indian-Arabic, and Chinese-Japanese) in the development of mathematics; for Indigenous cultures, people were seen as more important than number so their mathematics specialised in areas other than number. This different focus could be seen as emanating from their cultural beliefs with regard to group rather than individual ownership. Thus, the teaching of number, operations and measurement can bring Australian mainstream Eurocentric school teaching into conflict with Indigenous students; it can be a topic that can, in the terms of Indigenous mathematics and mathematics-education researcher Dr Chris Matthews, designate these cultures as primitive. It must be taught with care as part of a European culture that Indigenous people need to understand. It should not be celebrated as something that raises some people above others.

For low SES Aboriginal and Torres Strait Islander students in Australia, the outcome is exacerbated. As low income people, these students are sometimes considered as unsuccessful. The number systems created as part of Eurocentric mathematics have always benefited high SES people at the expense of low SES people, and promulgated the idea that bigger numbers (e.g. money, house cost, cars) are better, and mean that the person with the bigger numbers is more successful. The way the numbers function within Eurocentric societies achieves two outcomes simultaneously: (a) benefits one class of people at the expense of the other; and (b) puts the blame for their lack of benefit on the actions of the class that is not benefited. The mathematics of number, operations and measurement must be taught with care to low SES students because its teaching can designate these students as failures. And if the students are both Indigenous and low SES, even greater care must be taken.

1.4 Overview of book

This overview lists the chapters in the book and the role of big ideas and the RAMR framework throughout the book. The book consists of seven chapters and an appendix:

Chapter 1: Purpose and overview Chapter 2: Number basics Chapter 3: Whole number numeration Chapter 4: Decimal number numeration Chapter 5: Common fractions

Chapter 6: Percent, rate and ratio

Chapter 7: Teaching framework for number

Appendix A: Detailed descriptions of big-idea activities for thousands and millions.

All chapters are based on the five big ideas (see subsections 1.1.3 and 1.1.4) plus sequencing of number across Prep to Year 9 (see subsection 1.2) – this means that the chapters are broken into subsections based on the five big ideas as well as the sequence of increasing complexity and size. This gives a coherence to the book that also occurs in the YDM Measurement book.

Within the subsections, activities for teaching the various number types are provided using the RAMR framework:

Reality – something from students' world, kinaesthetic

Abstraction – body \rightarrow hand \rightarrow mind, creativity

Mathematics – language/symbols, practice and connections



Reflection – validation (back to students' world), applications and extensions (flexibility, reversing, generalising, changing parameters).

This enables these activities to act as exemplars of the RAMR framework. Other activities in this format are included throughout the book but not with every stage of the framework detailed. The beginning of each activity is identified with the symbol on right above. *The activities are written in italics to help distinguish them from the main text.*

2 Number Basics

This chapter consists of five basic ideas that underpin number. The first of these is **attributes** which covers identifying attributes, sorting and classifying, patterning, unnumbered comparison and order, and one-to-one and one-to-many correspondence. This involves sorting objects by attributes and putting different sets in one-to-one or one-to-many correspondence. For example, sorting students into classes and providing a pencil for each student and a bus for each class of students, or sorting people into families and providing a hat for each person and a car for each family.

The second of these basic ideas is **counting**. This requires the recognition of different attributes, the sorting of the objects (particularly into those to be counted and those that have been counted), ability to see patterns, notions of bigger and smaller, understanding what is more and less, and correspondence (particularly between number names and objects). In rational counting, it relates to putting rote number names in one-to-one correspondence with each object or in one-to-many correspondence with each group of objects (or each collection of groups of objects).

The third basic idea is **numerals**. This is learning to read and write the numerals 0 to 9 – to recognise what they are and to relate them to language, models and real-world instances.

The fourth basic idea is applying counting and number to **continuous** entities like length – that is the introduction of number tracks, number ladders and **number lines**. This enables a wider length view of comparison to be taken.

The fifth basic idea is **complex reasoning and inferences**. This is the mathematics for real-life situations which involve complex activities with many interacting attributes, such as grading a floor routine in gymnastics.

2.1 Attributes and classification

Number cannot be understood unless students can recognise when things are the same and when they are different – this is the first step. Difference is the basis of number because number tells how many different things there are. Same is the opposite of different and is essential for understanding difference. Hence, the first step to number is **attribute recognition**, that is, the ability to recognise likeness and difference; for example, recognising everything that has the colour red, or can be sat on, or rattles, and so on. It leads to recognising place values, fractions, percent, ratios and rates (and perceiving same and different).

The second step is **matching** objects on the basis of attributes. This focuses on the likeness of objects, matching those that are alike. Underlying this section is the **continuous–discrete principle**. Some things are continuous (unbroken), like height. Other things are discrete (in parts), like pencils. To be counted, something must be discrete. One of the earliest learnings is detecting when there is a difference – when something continuous has a break, becomes discrete (e.g. a door in a wall, a change of colour). This matching continues in the later years – it is the basis of similarities between common fractions, chance and percent, and decimal numbers and metric units. It is also the basis of the difference between fraction and ratio.

Attribute games are an effective way to teach advanced attribute recognition. They are useful when objects have three or more attributes. For example, pictures can be made of cars that are different colours, different sizes, have or have not passengers, and have or have not white wall tyres. However, the best material, and the one for which the games will be described, are logic attribute blocks. Using these materials helps attribute recognition, enables sorting through Venn diagrams and builds understanding of logical connectives.

2.1.1 Sorting and classifying

From matching, activities move on to sorting and classifying. This means identifying attributes and sorting sets of objects into subsets by these attributes (e.g. sorting by colour, shape, size or number). The idea is that sorting by colour, size or shape will underpin sorting by number and, therefore, identifying different numbers and giving one number to all collections with that number of objects. It will continue into place value, fractions and ratio, and is the basis of equivalence for fractions and ratios.

The sequence of activities should be to start with sorting by one attribute, going on to sorting by two attributes, and then to three or more. Reversing activities should be included – asking students to sort objects, and asking students to identify the attributes used with objects that have already been sorted. Remember also that sorting can be done in terms of no attribute (e.g. not red or not large). Again this is the basis of equivalence – identifying no change.

2.1.2 Patterning

Patterns are **rules that repeat**. In pre-number work this is more general, to seeing some regularity in the objects. However, this skill can be developed with classical repeating patterns. These patterns can be linear, two dimensional and three dimensional.

Early patterns are based on colour, size, and shape – not number. Finding and following patterns is the basis of mathematics. The idea is that patterns with other attributes (colour, size, shape) will precede and lay foundations for number patterns. Students often confuse patterns with designs that have some form of structure, particularly if that structure is symmetrical. Students often believe that the next item in this pattern, A B B A B B A B B A B, is A because the pattern started with an A. Patterning activities are discussed in detail in the YDM Algebra book. The build in complexity as the year levels pass by is one of the ways of introducing algebra.

2.1.3 Unnumbered comparison and order

The idea here is that comparing objects for attributes such as height and mass leads into comparing quantities in terms of size of number. **Comparing** is looking at two objects and seeing which has more of a particular attribute. **Ordering** is looking at more than two objects and putting them in sequence from the one that has the least of the attribute to that which has the most. Comparing should precede ordering. Ordering should begin with three objects. For three objects, the first activities should involve comparing two objects and then adding in a third object which is the smallest or the largest. Then, activities should move to comparing two objects and then adding in a third which is in between the first two.

The important feature in the move from two objects (comparing) to three (ordering) is the development of the notion of "betweenness" – that objects can have attributes which means that one is between the other two in terms of these attributes. Comparing means a lot of new language. In terms of length we have long/short, thick/thin, narrow/wide, short/tall, etc. This language is discussed in the YDM Measurement book. Comparing and ordering can also be applied to pictures (e.g. circle the picture that is longer). Obviously, comparing and ordering is a major component of number across all the year levels and is considered to be a number process.

Note: The use of unnumbered activities before numbered activities is very important and is a major part of the YDM Measurement and Algebra books. Unnumbered activities allow big ideas to be recognised much more easily than numbered activities and lay the foundation for learning that is connected and structural. Much of the ideas here are based on the work of Davydov.

2.1.4 The attribute of number

There are three aspects here. The first is that number is a special attribute in its own right – it identifies something (usually a collection or group) by the amount of objects (in terms of counting discrete objects). For

this attribute, other attributes do not matter. For example, the number of objects is two regardless of any differences between the two objects such as colour, size, difference in types, and so on. It just has to be two identifiable discrete objects.

The second aspect is that number can be applied to other attributes (e.g. length) and so is part of descriptions of that object even though the central attribute is not number (e.g. I took the 6m lengths of wood away from the steel rods). Of course, sometimes the attribute is number even though it is also lengths of wood (e.g. sort the wood by length).

The third aspect is that we can only show number by examples which have other attributes and this requires abstraction by the learner. Number is complex because it is not natural like wood, red, rabbits, and so on. It is something we invented to apply to these things. So, we can always show two hands, two marbles, two pencils, and so on, but the attribute of number as the **twoness** requires the learner to abstract a commonality from these examples.

Therefore, it is easier to understand number if attributes do not change across the objects making up the number – for example, counting red balls of the same size before balls of many colours and sizes.

2.2 Early counting

This section includes the earliest mathematical ideas and activities for counting that underlie the decimal number system, not as precursors to more advanced number ideas such as place value, but as areas in their own right that have a role across all year levels and all number ideas. These mathematical ideas have come from the early years of mathematics and should be used in the later years.

There are six important aspects of counting that students must internalise before they can be said to count reliably and consistently.

- Each object must be touched once and only once as it is counted **one-to-one correspondence**.
- As each object is touched its corresponding number name must be said in the correct order rote counting.
- The objects can be touched in any order from any starting point without affecting the number of objects there are **robust counting**.
- Changing the arrangement of the objects does not change how many there are robust counting.
- The last number said when counting will always tell how many objects are in the collection **rational counting/cardination**.
- Groups and groups of groups can be counted as well as individual items and we have to move easily between different interpretations (e.g. both of these are correct John had 36 eggs; John had 3 cartons of eggs) **flexible counting**.

2.2.1 One-to-one and one-to-many correspondence

One crucial skill in number is putting number names in one-to-one correspondence with objects and one-tomany with groups of objects – one number name for each object or group of objects. One idea here is that this skill can be assisted by putting one set of objects in one-to-one correspondence with another set (e.g. one egg for each eggcup, one knife for each fork, one fish for each fishing line, etc.). The evidence for this particular approach is not strong. There appears to be evidence that putting two sets of objects in one-to-one correspondence with each other is more difficult than putting number names (words) in one-to-one correspondence with objects.

Understanding of whole and decimal number in terms of place value and fractions in terms of parts appears to be underpinned by correspondence. Certainly, understanding that three groups of 10 is both 3 tens and 30 ones is important, as is realising that in place value, a 3 placed one position to the left of the Ones is 3 Tens.

2.2.2 Saying numbers

Saying number names and understanding their meaning involves three main stages which students progress through to develop robust counting: *sorting and correspondence* (established in pre-number activities), *rote counting* and *rational counting*. Once they have developed robust counting, students are able to use their counting skills for further applications of numbers such as comparison, conservation, ordination and labelling, subitising and sight groups, counting on and counting back strategies and flexible counting.

Patterns in number names are not useful for learning number names up to thirteen in Standard Australian English. These first number names must simply be learnt through repetition in rhymes and stories. Following thirteen, students are able to hear the repeating pattern from four, five, six, seven, eight, nine within the number. From twenty onwards number names take a recognisable pattern which allows students to learn the name for the tens and then rely on the known pattern for the remaining numbers to the next ten.

2.2.3 Rote counting

Rote counting is achieved when students can say the **number names** in the right order without worrying about whether they touch one object as they say each name. For example:

one, two, three, four, five (good enough for starting Year 1) ...

six, seven, eight, nine, ten (even better for starting Year 1)

Rote counting includes counting for whole and decimal numbers, common fractions, mixed numbers and percent. Counting begins with rote counting (i.e. saying the number names in order), moves to using the names to count out a set, goes on to applications such as comparison, conservation, ordination and labelling, and then on to advanced counting skills such as subitisation, counting on, counting back, flexibility, and counting to 100.

If students cannot say the number names in order, then they need to sing a lot of nursery rhymes like, "One, Two, buckle my shoe ..." (traditional number rhyme) or "One, Two, wallaby stew ..." (a number rhyme made up by Indigenous teacher aides), and so on.

2.2.4 Rational counting

In rational counting students first combine rote and sorting/correspondence skills to be able to put number names one-to-one or one-to-many with objects or groups and are able to chart a path through the objects or groups so that none are missed or counted twice. When students can: (a) **count a set** – point to (or touch) each object in a set and say one number name for each object, as shown in the figure below; and (b) **count out a subset** – identify a given number of items from a larger collection – requiring knowing that the last number name tells how many (sometimes called cardination), they are rational counters. This should be applied to continuous (e.g. number line) as well as discrete situations (e.g. sets).



Last number name tells how many

When rationally counting, the student has to **keep track** of objects – remember which objects s/he has counted, and which objects s/he still has to count. It is much more difficult to do this when the objects to be counted are scattered and most difficult when the objects are arranged in a circle, as in the figure below. This often requires the student to be able to determine a **visual path or pattern** through the objects which includes all items, does not repeat any, and knows when to stop. Students need experiences counting along lines as well as objects (e.g. counting steps).



Rational counting and the difficulties of keeping track

Discrete materials should be used for younger students when they begin learning, starting from *real-world* materials and then replicas of the real-world materials, such as toys. When students are able to rationally count, *concrete* ("hands on") materials can be used, such as counters and then Unifix cubes. Following this, students should be asked to count things in *pictures* (see figure below). This is more difficult as students cannot pick up or move the objects in a picture. Finally, students have to know that they can apply number to even imaginary discrete things and to number lines.



Sequence of materials and pictures

The abstraction component of the RAMR cycle is where oral and written **number names** and written **numerals** are introduced. Start with a situation from the real world, then model this with sets and number lines. For example, count objects and count along number tracks and number lines – note that the number line, or length, model requires counting *jumps* not starts/ends of jumps. Sequence the numbers – do 1-4 first, 0 second, 5-9 third and all 0-9 finally. Move on to using numerals in activities focusing on one more/one less, comparing and ordering, and ordinal numbers (first, second, third, etc.). Then use numerals for ten, counting on/back, composite numbers (separate subgroups) and other advanced counting ideas.

2.2.5 Robust counting

Robust counting is also known as rational or point counting. To ensure counting is a robust knowledge for the students, give counting activities where the start and/or finish are identified beforehand and are not at the start or finish of any line or identified grouping. These counting activities (shown below) are more difficult than the scattered arrangement or the circle because students are given a starting or a finishing point that is not at the end of a line.



Activities to develop robust counting

It is important that students, from the start, count in a number-line sense as well as a set sense. That is, to count steps, to count along a number track, and to play games like Snakes and Ladders.

2.2.6 Counting on and counting back

This comprises counting forwards 1, 2 or 3 from a given number (essentially a strategy for addition facts) and counting backwards 1, 2 or 3 from a given number (essentially a strategy for subtraction facts). For example, 5 + 3 would be counted by recognising that one collection has 5 counters and then thinking: five, six, seven, eight; and 6 - 2 would be counted by recognising that the original collection has 6 counters and then thinking: six, five, four.

2.2.7 Flexible counting

These are counting forms that are the basis of the one-to-many correspondence and of place values, that is, when objects are not presented as a single collection. Some examples are: (a) counting imagined objects (called abstract counting); (b) counting when there are groups or groups of groups, for example, counting the number of lollies when they are in bags (with same or different numbers of lollies in each bag) – called composite counting; and (c) counting the number of groups of objects and ignoring how many in each group (e.g. forming bundling sticks into groups of ten and then counting only the number of tens) – called measure counting.

2.2.8 Complex counting

When students know the number names to "ten", they have all they need to count to trillions or thousandths, and to count fractions. This counting should be reinforced across all the years of schooling. At the beginning, once they know the number names to nine, students should be encouraged to count further. Initially, they need to do this often until they can say all the number names in the right order to "twenty" and "hundred". It is a good idea to have a number line running around the room and to use number boards as shown below.

0	1	2	3 4	5	67	8	9 10) 11	12 13	14	15 16	17	18	19	20
			0	1	2	3	4	5	6	7	8	9			
			10	11	12	13	14	15	16	17	18	19			
			20	21	22	23	24	25	26	27	28	29			
			30	31	32	33	34	35	36	37	38	39			
			40	41	42	43	44	45	46	47	48	49			
			50	51	52	53	54	55	56	57	58	59			
			60	61	62	63	64	65	66	67	68	69			
			70	71	72	73	74	75	76	77	78	79			
			80	81	82	83	84	85	86	87	88	89			
			90	91	92	93	94	95	96	97	98	99			
			100												

0 to 20 number line and 0 to 100 number board

It is important that students both count discrete objects (sets) and count along continuous lines (number lines). The number-line model can be experienced by counting steps and using a number track, and with games like Snakes and Ladders, Ludo, and any other board games where they have to count along lines, or games where they have to take the number of strides that they are told or work out (such as the number of times they have the letter "N" or "T" in their name). Body and hand activities with both discrete and continuous are used to encourage students to imagine and gain **mental models** of numbers as sets or steps on lines.

It is also important to ensure that the patterns in the digits and the language is reinforced and not lost because of teen numbers and zeros. For teen numbers, we recommend using words that relate to tens and ones, such as "onety-seven" for 17. We also recommend that the "zero" be put back into language, at least as an option at the beginning of teaching. This would mean that, initially, the following language would be used: 20 would be "twenty-zero" or even "twoty-zero", 350 would be "three hundred fifty-zero", 416 would be "four hundred onety-six", 310 would be "three hundred onety-zero", and 804 would be "eight hundred zeroty-four".

Note: The "and" often given between hundreds and tens, e.g. 324 is "three hundred and twenty-four", can be confusing as it can be interpreted to mean two numbers, 300 and 24.

2.2.9 Subitising

Subitising is recognising a quantity through sight alone. Most people can do this up to five (although there is some evidence that some cultures, for example, Aboriginal people, can sight count to much higher numbers). The way objects are arranged affects the ability to subitise – see the examples below.



Examples of counting by subitisation

Subitising is an important skill in relation to computation. The ability to subitise builds students' ability to identify how many is in a collection and their trust that this quantity will not change and therefore does not need to be recounted as part of an addition or subtraction computation.

Advanced counting activities are part of abstracting. A student needs to recognise "5" without counting so they can trust there is five. Then the five objects can be covered and the student will be able to count on or back without needing to count the unseen objects again.



Advanced counting

Counting all is an inefficient method of finding a quantity. Counting on is more efficient and assists students to develop early ideas about addition and computation strategies. The ability to subitise and trust the number in a collection is also necessary for understanding multiplication as repeated equal groups, for example:



By being able to recognise by sight that each group is 5, a student can find the total without needing to count all – either using addition or multiplication.

2.3 Introduction of numerals

Reading and writing numbers involves the use of language, materials, models and symbols (numerals). When students read and write numbers teachers can recognise student understandings about numbers. Writing numbers includes students' ability to recognise symbols as representing a specific quantity of items and to draw the related symbols. When number names are known and can be used to count a set and count out a subset, the next step is to attach the words and symbols for the number names and the numerals up to nine, namely, "zero" 0, "one" 1,"two" 2, "three" 3, "four" 4, "five" 5, "six" 6, "seven" 7, "eight" 8, and "nine" 9, to the oral names and

to any representation of that number of objects either with materials and pictures, or imagined in the mind. It is based on relating symbols to the real world, models (e.g. set, number line) and language. This learning of written names and numerals is basically completed through repetition of the associations.

The components of number - language, materials and models, and symbols/numerals - are discussed and related in the following subsections, followed by an exemplar RAMR lesson which provides examples of classroom activities to introduce counting, language and numerals.

2.3.1 Components

New mathematical concepts are introduced to students through the use of language. First, the ordinary language of the students can convey mathematical concepts; for example, "I am three", "my bear is next to the table". As students progress in their understanding and experiences with mathematics, the language they use also progresses to become more mathematical and symbolic. For example, the concept of subtraction can be described using lost, covered up, take away, subtract, and the symbol "-". The level of abstraction in the language used can develop over time and be varied to suit particular student or conceptual needs.

Having students pose stories that match mathematical concepts can be an enriching activity at all levels from stories about addition to stories about rate and percentage.

Physical materials are commonly used to support student learning of mathematics. As students develop their mathematical understanding the materials used to represent real-world contexts need to gradually decrease in the amount of contextual information they provide for students, as shown below.



Sequence of physical material representations

When using models for number it is important to include both discrete and continuous models to assist students with understanding the continuous vs discrete big idea, as well as providing enough examples of items in the real world to ensure students build flexible and robust understandings. Note that continuous models from the real world become discrete and countable when arbitrary measures are applied. For example, water in a jug with no markings is continuous; marking increments up the side breaks the continuous into discrete countable units. Similarly a streamer is continuous but its length becomes discrete and countable when placed alongside a ruler or measuring tape.

Discrete Continuous Continuous made countable 000 000 Set model Volume model Length model Area model

Examples of discrete and continuous models to support maths learning

Some real-world models for discrete, continuous, and continuous made countable (or discrete) are pictured below showing set model, volume model, length model and area model.









Discrete – set model

Continuous - volume, length, area models

Continuous made countable volume, length, area models

Real-world models for discrete, continuous, and continuous made countable

Examples of discrete models for whole number:

- actual objects buttons, toy cars
- models of actual objects counters, blocks
- structured models ten frames, bundling sticks, MAB.

Examples of continuous models for whole number:

- number boards hundred and 99 boards
- number lines numbered lines, unnumbered lines, empty number lines, double number lines.

A number line is a continuous line. To enable numbers to be applied to the number line, an interval of length one is used as a unit to divide the line into discrete parts. The placement of the number is based on this partitioning of the line. Since many measures are associated with lines, it is important all numbers are related to positions on lines as well as sets of objects.

2.3.2 Symbols/numerals and relationships

Mathematics contains many **symbols** which students need to interpret to be able to communicate mathematically and to understand mathematical concepts. Often the symbols used in mathematics are particular just to mathematics. Understanding maths symbols is best left until after students understand the concept the symbols represent. Some examples of mathematical symbols in number are the digits 0–9, the decimal point, the notation for common fractions and mixed numbers, and the percent and ratio notation.

The symbols for the **numerals** can be difficult for students in terms of visual distinction. For example, 2 and 5 have aspects that are reflections of each other, 7 and 1 can look similar and 9 and 6 are rotations of each other. As visual images, they need kinaesthetic experience. This is the same for numerals in larger whole numbers, decimals, common fractions and mixed numbers, and percent, rate and ratio.

Examples of kinaesthetic activities for developing the visual recognition of symbols include: walk numerals (e.g. draw large numerals on ground and have students walk them), make numerals with body (e.g. lie on floor and contort body to make a 2), trace numerals out with hands (e.g. trace in air with hand many times, have numerals on felt stuck to cardboard for students to trace out, detect numerals by shape with eyes blindfolded), and so on. See RAMR example at end of this section.

Reality, models/representations, language and symbols need to be effectively **connected** so that students can move back and forth flexibly between them when interpreting and solving mathematical problems. Of particular importance in abstracting number is the Payne and Rathmell triangle. The Payne and Rathmell (1977) triangle (see figure below) is an excellent vehicle for connecting real-world problems, representations, language and symbols. Activities and questions should be constructed that encourage students to connect and move flexibly between model, language and symbols in all directions.



The Payne and Rathmell (1977) triangle for early number

2.3.3 Teaching counting, language and symbol recognition

Counting teaching activities should be part of every year level. RAMR activities for sorting/correspondence, rote and rational counting, and symbol recognition are included here as exemplars for sequencing instruction and ideas to start from.



Materials: Everyday objects (e.g. toys, fabrics, pegs, cardboard cylinders, eggs, eggcups, spoons ... ; things that are twisty, curly, pointy, roll ... ; things that have many colours, different lengths and thicknesses ...); supermarket materials (e.g. lollies, packs of lollies, foods, cans ...); other objects (e.g. blocks, counters, Unifix cubes, sticks, plastic animals ...); pattern blocks, logic attribute blocks, number tracks, number lines, racetrack games, Snakes and Ladders games ... ; and computer visuals and paper pictures of collections of things.

Reality

Reality focuses on teaching and learning by (a) using the world of the student; (b) ensuring prerequisite and connected knowledge is known; and (c) using real-life kinaesthetic activity. Counting, symbols and early grouping activities for these three parts of this initial component of the RAMR framework are as follows.

Using local culture and environment. Teaching of pre-number and early counting appears to be advantaged by basing them on reality using materials from the students' local environment or culture. For example, sorting objects from home and environment (e.g. toys, fabric, pegs), counting objects from outside (e.g. seeds, nuts, sticks or rocks – rather than plastic counters), and putting everyday objects in one-to-one correspondences (an egg for every eggcup or a spoon for every plate using real eggs, eggcups, spoons, and plates). Any informal object that has more than one attribute (e.g. colour, size, shape, ...) is useful – from home, shed, outdoors and classroom or from trips and visits to other places. This includes photographs and pictures. Try to find out how things are compared, matched, and put together with other things (one-to-one) in the everyday life of the students.

Existing knowledge. There are many attribute words that have to be known: (a) for attribute recognition – colour words (e.g. red, green, dark, light), size words (e.g. big, little, tall short), shape words (e.g. circle, square, pointy), and other words (e.g. twisted, curly); (b) sorting and matching words – words that show similarity and difference (e.g. same, different, match); and (c) comparison words – words that show direction of difference (e.g. more, less, most, least, thicker, thickest). Aboriginal, Torres Strait Islander and low SES students may not have these words. Spend time in real situations acting out the meanings of these words. Photograph the results and display. Act out words with material or on posters (e.g. "place the cat on top of the tree"). Find everyday examples of objects to count and objects in packages to begin early grouping (e.g. packs of lollies or foods such as a box of four ice-creams).

Kinaesthetic activities. Use students' actions to introduce pre-number and counting activities. For example, identify attributes of each other's bodies, sort students by these attributes (e.g. hair colour), pair students (e.g. are boys and girls one-to-one in the class?), place students in families (one-to-many?), compare heights, count students, and so on. Develop counting **for number lines as well as sets** of objects – count steps, count tiles along a floor, count blocks up the wall, and so on. An important early kinaesthetic activity is to use movement of the body to introduce the numerals and groups and ones in early grouping. Organise students to walk the numerals, tracing them out with their hands, and make the numerals with their bodies.

Abstraction

Abstraction focuses on teaching and learning by: (a) developing sequences of representations to abstract ideas from reality to mathematics; (b) creating two-way connections from reality to mental models (from body to hand to mental or mind activities); and (c) creating students' own representations. Activities and materials for these three parts of this second component of the RAMR framework are as follows.

Sequences. Young students should begin learning with discrete materials, starting from real-world materials (e.g. biscuits, marbles) and then replicas of the real-world materials. When they are able to rationally count, the

teacher starts to use concrete ("hands on") materials such as counters and then Unifix cubes before asking students to count things in pictures, which is more difficult as students cannot pick up or move the objects in a picture. Finally, students have to know that they can apply number to even imaginary discrete things and to number lines.

The abstracting component of RAMR is where oral and written **number names** and written **numerals** are introduced. Start with a situation from the world, model this with sets and number lines (e.g. count objects and count along number tracks and number lines – note that the number line, or length, model requires counting jumps not starts/ends of jumps). Sequence the numbers – do 1–4 first, 0 second, 5–9 third and all 0–9 finally. Move on to using numerals in activities focusing on one more/one less, comparing and ordering, and ordinal numbers (first, second, third, etc.). Then use numerals for ten, counting on/back, composite numbers (separate subgroups) and other advanced counting ideas.

Note: Because we have a base of 10 in our number system, we only have numerals for 0 to 9. The numeral for ten is one ten and zero ones or 10. The problem for us in our numbers is that our language is based on a base of 20 ("score") and so we have language for 0 to 19 ("one", "two", ..., "nine", "ten", "eleven", "twelve", "thirteen", ..., "nineteen"). The last ten number names should be "onety", "onety-one", "onety-two", ..., "onety-nine". Many classrooms include the teens within early number. This book leaves the teens until whole numbers.

Advanced counting activities are part of abstracting. For example, see the methods below for teaching counting on and counting back.



Methods for teaching counting on and counting back

Mental models. It is important that students both count discrete objects (sets) and count along continuous lines (number lines). The number-line model can be experienced by counting steps and using a number track, and with games like Snakes and Ladders and Ludo and any other board games where they have to count along lines, or games where they have to take the number of strides that they are told or work out (such as the number of times they have the letter "N" or "T" in their name). Body and hand activities with both discrete and continuous are used to encourage students to imagine and gain mental models of numbers as sets or steps on lines.

For early grouping, begin with simply forming groups (e.g. "here are 8 counters, do you have enough to form a group of three, a second group of three, a third group of three – so how many groups of three and how many ones left over?"), and then move on to forming groups and ones on a Groups-Ones chart (moving the left hand from groups to ones as you say the number of groups and ones). This kinaesthetic activity enables a mental model of groups being on the left of ones to emerge. Activities include: adding and removing ones until new groups are formed or existing groups broken up, and using dice in a game to be the first to four groups. At all times, students should be saying and recording the number of groups and ones after each change.

Creating own representations. Give students opportunities to share the number names from their home culture and language if different to normal mathematics. Let students create their own number names and number numerals, and make up their own groupings and words for the groupings. Set up counting in another base.

Mathematics

Mathematics focuses on teaching and learning by: (a) appropriating formal language and symbols; (b) reinforcing and familiarising through practice; and (c) connecting new ideas to existing ideas. Counting, symbols and early grouping activities and materials for these three parts of this third component of the RAMR framework are as follows.

Appropriation. The end of the Payne and Rathmell triangle is the beginning of formal mathematics – the appropriation by students of formal number names and numerals. Ensure there are activities that relate language and symbols to each other in both ways but also to all models and real-world situations both ways. This means having lessons where:

- teacher gives symbols and students give language, models and situations;
- teacher gives language and students give symbols, models and situations;
- teacher gives models and students give language, symbols and situations; and
- teacher gives situation and students give language, symbols and models.

Practice. This is an essential part of learning. New mathematical ideas need prerequisite knowledge to be familiar not just known, so there must be practice of previous mathematical ideas before going on to new ideas. This means games, oral question and answer sessions, and simple worksheets. However, games and worksheets need not just be with symbols; excellent practice activities involve relating symbols to language, models and reality (in fact, this is the definition of learning for many mathematics educators – that students can connect different representations of the same idea). Some good activities are:

- **Tabular work sheets** have columns for some or all of reality, models, language and symbols, fill in one column and require the students to fill in the rest.
- Make the amount place numerals on plates; students have to place counters on plates to match numerals (can also place language cards); reverse the activity, put counters on the plates and students have to match with numeral and language.
- Act it out use kinaesthetic activities to act out numerals and even words with body and hand; make a set of numerals on cards with sandpaper or silk fabric for students to trace.
- **Mix-and-match cards** draw up cards (all the same colour) with some or all of the four options for different examples, cut cards up the same way, mix together, and students match to reconstruct cards.
- **Cover-the-board** make decks of cards (different colours) for some or all of the options, make a base board from some of the options; students take turns placing card on board or over another person's card that represents the same thing; the person with the most of their colour on the top at the end wins (can also play Snap with these cards).
- **Bingo** make up boards with different representations of different numbers, construct call cards; one player shows card, others cover with a counter the same representation on their board; the first player with three in a row, column or diagonal wins.

Calculators can be a great teaching aid at this point – they allow students to show numerals by pressing buttons. This is also the time to get out the students' number rhymes and counting books. YDC has a collection of counting rhymes developed by Aboriginal and Torres Strait Islander teacher aides – create your own using local language and environment.

Connections. This is the time also for building connections between the knowledge being learnt and other knowledges. One important example of this is linking counting forward with adding one and counting backward with subtracting one. Calculators can be very useful: (a) to reinforce counting forward and back, and (b) for recording through adding and subtracting the actions of counting forward and back.

Reflection

Reflection focuses on teaching and learning by: (a) validating mathematical ideas in terms of the students' reality; (b) applying mathematical ideas back to their reality; and (c) extending ideas through strategies (being flexible, reversing, generalising, and changing parameters). Counting, symbols and early grouping activities and materials for these three parts of this final component of the RAMR framework are as follows.

Validation. Once students have developed proficiency with numbers in terms of relating numerals, language and models, get the students to check their understanding of early number with their experience of the world (e.g. Does what they know about numbers make sense? Does it appear reasonable?). In particular the question of whether the knowledge would be useful in the students' everyday world is worth discussing.

Application. After validating students' knowledge of number, this knowledge can be applied back into their reality. For example, students can use numbers to help them live in their world, that is, count things, relate amount to counting (e.g. Who has more/less?), use numbers in sport, and so on. During these applications, the teacher can reinforce the variety of words that go with the use of number, particularly with respect to comparison.

Extensions. There are many ways in which reflection on mathematics knowledge can lead to extensions. The first of these is **being flexible**. For example, discuss all the ways in which the number 6 can be used: two groups of three – 1, 2, 3 and 4, 5, 6; 7 subtract one; 5 add one, half the eggs in an egg carton, and so on.

The second way is **reversing** – encouraging students to think both ways. For example: (a) looking at how many different ways we can set out counters so that the total is eight, going from numeral \rightarrow objects as well as objects \rightarrow numeral; and (b) acting out a grouping and getting students to state the group (e.g. I had 2 groups and 4 ones – I added 3 ones and this made 3 groups and 2 ones, what size are my groups?).

The third way is **generalising** – encouraging students to see past particular outcomes. For example: (a) when relating rote to rational counting, the number further along counts more; (b) two groups with the same number of objects can always be put into one-to-one correspondence; (c) number is left unchanged if no object is removed or added; (d) ordinals are made by adding "th" to number names, except for the first three and the fifth term (thus ordinals should be "twoths", "threeths", "fourths", "fiveths", "sixths", and so on); (e) the group number determines when ones are bundled into groups.

The last way is **changing parameters** – changing something in what we are doing. For example, what if we grouped in fives and our numbers were "fee" for one, "fie" for two, "foe" for three and "fum" for four; how would we say 2 groups and 3 ones?

2.4 Counting and numerals on number lines

This section looks at applying early number to continuous entities like length – it covers continuous vs discrete, introducing counting, teaching early number where there is a unit to give number, and early comparison.

2.4.1 Continuous vs discrete

The big idea of continuous or discrete focuses on one of the fundamental differences in the world around us. Things are either: (a) discrete, that is, can be identified as separate entities and counted (e.g. fingers, people, chairs, lollies); or (b) continuous, that is, flowing forever without being naturally broken into pieces that can be counted (e.g. line or length, coverage or area, heft or mass). Because of the importance placed on determining number and quantity, Western culture found a way of "discretifying" the continuous by inventing **units**. A unit is a small amount of the continuous attribute used to break up the attribute into equal pieces that can be counted so the attribute can be quantified (e.g. metres for length, hectares for area, and kilograms for mass). Thus, number can be applied to both discrete and continuous attributes, and this is essential for all number types. Whole numbers, decimal numbers, common fractions, rates and ratios must be understood in terms of discrete and continuous – that is, seen in relation to separate objects and seen in relation to continuous entities. In practice this means that number should be represented in terms of **sets**, **area** and **number lines**:



Set, area, and number-line models for two thirds

The idea here is to count jumps as well as objects – using number lines and number boards (e.g. 100 boards). One way to begin this is to count steps and this leads to games and activities that use number tracks. However, the number line places numbers at the ends of steps while the number track has numbers in the spaces. Therefore, a number ladder (like a track but numbering the lines) can be a useful intermediary (see section 2.4.3).

2.4.2 Introducing counting on continuous things

The RAMR activities below introduce the idea of extending counting and numerals from discrete to continuous things.



Reality

Gather existing knowledge. Discuss what things can be counted and what cannot. Point out that, normally, items have to be discrete (individual and separated) to be counted. Point out that the world is full of discrete things (chairs, people, animals, days, grains of sand, etc.) but that some things have had to be "changed" to be countable.

Discuss length. It is not countable unless a unit of measure is used. Length is continuous (as is area, volume, mass, time, and so on) but units make it discrete. Ask why we want to turn the continuous into the discrete? [So we can apply number to it.]

Abstraction

Body. Walk distances in big steps and count. Identify unit. Walk distances in small steps and count. Identify unit.

Hand. Explore units with concrete activities using discrete and continuous to practise.

Mind. Identify discrete units and continuous things broken into units to count. Students create classification and list.

Mathematics

Connect to counting measures with non-standard units. If appropriate connect to counting measures with standard units.

Reflection

Find other items around the classroom/school/home environment that are discrete and continuous. Identify how these are counted.

Extend to flexibility of unit by identifying collections as unit where practical.

Generalising. Students can make a rule for classifying discrete and continuous. For example, discrete can be counted, continuous needs a way (e.g. a unit of measure) to change it into discrete for number to be used on it.

2.4.3 Teaching early number on a number ladder

It is important that early number work includes things that lead to the number line as well as to groups and leftovers. Below is an example of a RAMR activity for a **number ladder**. A number ladder is a useful model between number tracks where numerals are *between* divisions, and number lines where numerals are *on* divisions – see figure below. The arrow shows a sequence of ideas and models.





Counting in track, ladder and line situations differs from counting of objects in two ways:

- counting focuses on jumps (or steps) between numbers/lines, not counting the numbers or lines; and
- the zero represents the start not nothing.

Materials: Number ladder large on floor, small number ladders with counters (with and without numerals).

Reality

Using local culture and environment. Look for things that count jumps – walking, games, kangaroos, etc. **Existing knowledge.** Check that students have counting and an understanding of counting and numerals for objects.

Kinaesthetic activities. Have students walk the large number ladder – focus on new role for zero and counting movements (e.g. jumps) not stopping points.

Abstraction

Sequences. Repeat counting activities using small number ladders and counters – without numerals and then with numerals – relate real world to model to language and finally to symbols using Rathmell triangle. **Mental models**. Students close eyes and imagine the number ladder and walk it in their minds. **Creating own representations**. Students make own number ladder with own symbols and walk it.

Mathematics

Appropriation. Ensure students understand formal language and symbols.

Practice. Relate numerals to positions on the ladder and vice versa. Count out to get positions. Play race games. **Connections**. Connect to set model – relate numeral, number name, drawing of a position on a number ladder and set of objects using physical and pictorial materials and games such as Bingo and Mix-and-match. **Reversing**. Ensure students go from position on number ladder to language and symbols.

Reflection

Validation. Discuss where the action of a number ladder is used (e.g. ruler).
Application. Apply to number ladder problems in world (e.g. house numbers).
Extensions. Flexibility – brainstorm number ladder or length applications (e.g. speedometers).
Generalising. Have students state what zero is for number ladder (start point); students discuss what the numbers do (tell jumps), show examples where it is necessary to count ends instead of jumps (e.g. 5 + 3 is 5, 6, 7, 8 which is 4 numbers but only 3 jumps).

Changing parameters. Set up ladder so that every 2 or 5 steps is bolded or coloured red – counts 2s and 5s. Extend ladder to 100s and count in 10s or 5s. Make each step more than one (e.g. to count as 2 or 5), discuss what this means.

2.4.4 Introducing comparison

There are two ways to introduce comparison. The first is to base it on students knowing that the number names and symbols have an order that relates to their relative quantity. They need to internalise these names and know their order to count collections of items. This is often achieved by rote learning the number names and applying them in one-to-one correspondence with objects when counting. In addition to knowing number names in order to represent quantities, numbers as symbols and names can be used in ways that do not necessarily refer to size or quantity. These are the ordinal and nominal numbers and serve only to indicate an order (for example, first, second ...) or name of an item (for example, house numbers or postcodes).

With respect to comparison, if a set of 4 objects is placed in one-to-one correspondence with a set of 5 objects, then the set with 5 objects has one more and therefore is larger in terms of number. This can lead to the generalisation that the number name **further along in the count** names the larger set (in terms of number). With respect to ordination, if situations where there is competition (e.g. a running race) are considered and the objects counted as they appear (e.g. as the runners pass the winning post), then the numbers will give the order of the objects. The ordinal version of the names can then be introduced (i.e. first, second, third, fourth, fifth, and so on).

The second way to introduce comparison is to use, for example, number ladders and lines to lead to activities that show that further away from the zero means the largest number. Thus, it is useful to have two people walk side by side on a number ladder and show that the person walking to 7 represents a larger number than the person walking to 5. This can be reinforced by using rope and pegs to place numbers on a line (0 to 10, 0 to 20, and so on).

2.5 Complex reasoning and inference

There are many advanced uses of mathematics that use the precursors to number. An example of this is measures of intelligence. Over the years, there have been many attempts to come up with an instrument that measures intelligence. Most have failed to produce universal acceptance not because of number or calculation difficulties (they mostly come up with excellent numbers which satisfy the most stringent statistics), but because of **attribute difficulties** (people do not think the instruments are measuring intelligence or all of what intelligence is).

There are even more difficulties when it comes to measuring wellbeing and happiness. Yet, decisions have to be made regarding benefits and responsibilities (e.g. access to health cover, who has to pay for this access, and so on) based on numbers (e.g. number of people, where they live, their healthiness or otherwise, and so on). And decisions have to be made as to how strong buildings have to be and what kind they should be (e.g. earthquakes, tsunamis, cyclones, fire risk, cost to heat and cool, and so on) based on numbers (e.g. these numbers for a tsunami, these for high winds, these for cost of construction, and so on). This requires applying numbers to complex situations with interacting attributes – the kind of work done by engineers or political advisors, risk assessors, or judges at Olympic Games, and so on.

To do this one has to know what the attributes are and this is often the basis of argument regarding how to combine them in a logical way and then infer a final number. It also means being logical and this is often not directly taught and often misunderstood. For example, what is the difference between "even and multiple of 7" and "even or multiple of 7"? Also, if sports stars are fit, does this mean that fit people are sports stars, unfit people are not sports stars, or people who are not sports stars are unfit?

This kind of complex reasoning and inference is beyond the scope of this book, but it does direct us to some activities that could be done in Years P to 9 that lead to this understanding of number as a measure of complex interactions; some of these activities will only be mentioned in terms of where they are in the other YDM books. These activities have to do with attributes, logical connectives and inference. Some of the ideas behind these activities are the basis of modern computing (particularly search engines and social media).

2.5.1 Identifying attributes

To become familiar with identifying particular attributes from among many attributes needs activities that do this. Games with logic attribute blocks are a powerful way to do this. Logic blocks (also known as attribute blocks) are a set of 60 coloured shapes – three colours (red, blue, yellow), five shapes (square, rectangle, circle, triangle, hexagon), two thicknesses (thick, thin) and two sizes (large, small). There are $3 \times 5 \times 2 \times 2 = 60$ shapes altogether. The logic blocks are used for:

- (a) learning about attributes (e.g. identifying attributes, describing attributes, relating attributes and changing attributes);
- (b) learning about logic, particularly understanding the logical connectives "not", "or", "and", "if ... then"; and
- (c) studying changes that are not numerical (e.g. red \rightarrow blue, thick \rightarrow thin, triangle \rightarrow square).

The first activities need to familiarise students with the description and use of the blocks:

- (a) describe the block teacher gives students a block and requires them to describe it in terms of colour, shape, size, and thickness;
- (b) find the block teacher gives a description (e.g. red, triangle, large, thin) and the students find the block; and
- (c) guess the block teacher holds a block behind back and students ask questions, teacher can only say "yes" or "no", students have to guess what the block is from the yes/no answers.

Domino games

An interesting way to use the blocks is through domino games. Here, all the blocks are put into a bag and students select 5-7 blocks (depending on number of players). A block is placed between players. Players place a block following the rules. Players who cannot place a block have to select another block from the bag. The player who first runs out of blocks is the winner. There are a variety of games as follows.

Domino game 1: Make a line. Players have to place their block at either end of a line, starting to the left or right of the first block, so that the block is one attribute different to the previously placed block.



One attribute different

Domino game 2: Make a cross. Players have to place their block at either end of two lines that cross where the first block was put. Block placement is one attribute different for one line and two attributes different for the other line.



Domino game 3: Three attributes. Players place their blocks similar to Game 1 but the placed block has to be three attributes different to the previous block.



Domino game 4: Matrix dominoes. This is played on a 10×6 grid. A block is placed near the centre. Then, players place blocks so that the block is one attribute different left to right, and two attributes different up-down.



Domino game 5: Dominoes solitaire. This is a one-player game. A block is placed in the centre of a large board and then other blocks are placed so that they are one attribute different along lines. The objective is to place all 60 blocks.



Venn diagrams

Another way to learn to recognise attributes is to be able to identify them when there is more than one involved (e.g. the difference between "even and a multiple of 7" and "even or a multiple of 7"). One way to do this is with attribute blocks and hoops. These can be done in discussions – where we talk about what goes where or as games and then discuss after. It is important that students **write down their own definitions** of what "and", "or" and "not" are. This is discussed in section 2.5.2.

Hoop game 1: Three-way. Place hoops in twos and threes intersecting. Label each hoop and then place blocks showing the ability to determine classifications and intersecting classifications.

To play as a game, put out three hoops and label (see an example on right). Divide the blocks among players, in turn place blocks, score 1 point for a hoop, 2 points for an intersection of 2 hoops and 3 points for an intersection of 3 hoops. Highest score wins.



Hoop game 2: Two-way. Play as above but with two intersecting hoops - this is a game for beginners.

Hoop game 3: Whysy. Play the games above but, to score a point, students have to correctly explain **why** they put the block where they do.

Hoop game 4: Notty. Play games 1, 2 and 3 but put "not red", "not thin", and "not circle" in the hoops. Could start with only one "not" attribute in one hoop.

2.5.2 Logical connectives

The logical connectives are "not", "and", "or", and "if...then". The blocks do a good job of introducing these also. However, just games is not enough. There must be discussion and recording of students' "definitions".

Hoop activities

The blocks with hoops are great for the logical connectives: "not", "and", "or", and "if...then".

"NOT". Place hoop and label it with an attribute (say, RED). Get students to place blocks inside and outside the hoop following the label. The ones outside are NOT the attribute.

Example:



(Then label the hoop NOT ... [e.g. NOT red] and get students to place blocks again.)

"AND" and "OR". Place two hoops so they intersect (overlap) and label them with different attributes (e.g. THICK and RED). Get students to place blocks inside the hoops, the intersection of the hoops, and outside the hoops, following the labels.

Example:



The materials in the intersection show "AND". The materials in both hoops and the intersection show "OR". This means AND means both attributes are present, while OR means one or both are present.

"IF...THEN". Make up cardboard strips with drawings of blocks, for example:



Then place blocks on these according to rules:

- "If red then blue"
- "If thick then thin"

Thus red gets changed to blue, thick gets changed to thin, and other attributes are unchanged.

Using trains

Train logic blocks have tracks as shown below right, and small cards on which attributes are written – the cards say which blocks go on which track.

Start all blocks on the top – the blocks that you want will be delivered at the top line, the "not" blocks will be delivered at the bottom line.



For example, "red and large" is as follows:



If...then can also apply to unnumbered work on Function Machine

The logic attribute blocks can be used for unnumbered Function Machine work. Have a student sit behind a box Function Machine with rules on top. Students hand in block on left and get changed block out of right.



Follow the rule. The student behind the box has a known rule. Other students input a block, the student behind outputs a changed block, and everyone else checks that the change is okay. Rules are given with arrows, e.g. $R \rightarrow B$ (red to blue), $L \rightarrow S$ (large to small), and can be forward or back, e.g. thick \leftarrow thin (thin to thick).

Guess the rule. The student behind the box has an unknown rule. Other students input a block, the student behind outputs a changed block, and everyone has to use the changes to guess the rule.

Reverse the rule. The student behind the box has a known rule. S/he outputs a block. Other students have to work out what block should have been inputted – reversing the rule (that is, in mathematics terms, finding the inverse).

2.5.3 Engineering problems, inference and Boolean algebra

There is a strong movement towards statistical inference using problems with large data sets and studying engineering problems like how and where to build a bridge – these are the problems for the Internet where there is access to a lot of data. We will look at inference and engineering problems in the YDM Statistics and Probability book. We now offer some ideas that have emerged from the work with logical attribute blocks.

Reversing

As a starting point, a lot can be done by **reversing the activities above** (where this is appropriate and possible). For example:

- Put out two intersecting hoops and start placing blocks given to you by students in the positions they should be. Students have to work out the labels that should be on the hoops. Do this for normal labels before "not" labels.
- The "guess the rule" game in the Function Machine section above, with students having to argue for their rule. Try to get more than one possible rule.
- Take blocks given to you by students and place them in the ending positions on the train logic blocks. Students have to work out the rule for the blocks going to the top position.
- Start putting out blocks for a domino game and students have to infer the rule and explain why their inference is correct.
Boolean algebra

Boolean algebra is a set of rules for "and", "or", "not" and "if...then". For these rules, special symbols have been produced:

and
$$\cap$$
, or \cup , not \sim if then \rightarrow

Thus, red \cap thick means red and thick, red \cup thick means red or thick, ~ red means not red and red \rightarrow thick means if red then thick. These symbols follow rules similar to algebra with letters and relationships. For example, assume red is R, thick is T and square is S. Then:

- (a) $\sim (R \cup T) = (\sim R) \cap (\sim T)$: not (red or thick) = not red and not thick
- (b) $\sim (R \cap T) = (\sim R) \cup (\sim T)$: not (red and thick) = not red or not thick
- (c) $(R \cap T) \cup S = (R \cup S) \cap (T \cup S)$: (red and thick) or square = (red or square) and (thick or square)
- (d) $(R \cup T) \cap S = (R \cap S) \cup (T \cap S)$: (red or thick) and square = (red and square) or (thick and square)
- (e) $\sim (R \rightarrow T) = \sim T \rightarrow \sim R$: not "if red then thick" = "if not thick then not red"

You might like to show that these are correct by placing logic blocks in hoops. To show (a), place blocks in lefthand side and the **not (red or thick)** blocks will be outside the two hoops, and place blocks in the right-hand side and the **not red and not thick blocks** will be in the overlapping part of the two circles. If these blocks are the same set of blocks, then the rule holds.



Note: Rule (e) is commonly misunderstood in argument. People find that "low income students get lower marks" and then argue that the opposite of this means that "students with low marks are low income", when the logical opposite is really that "students with high marks are high income".

Application of Boolean algebra

Double switch lights in a room work by the light being on when both switches are on or off together, and the light being off when switches are opposite, that is, when one is off and the other on. In Boolean algebra, this is as follows (when the two switches are called switch A and switch B):



This can be turned into a circuit: \cap means **sequential** and \cup means **parallel.**



This is how all circuits in houses are determined. Thus, if you wish to become an electrician, it is useful for you to study Boolean algebra.

3 Whole Number Numeration

This chapter builds on the number basics described in the previous chapter. The big ideas of notion of unit, counting, multiplicative structure, discrete and continuous and equivalence provide the structure for this chapter as they do for the following chapters relating to decimals and fractions. Activities are provided based on the RAMR framework. The mathematical ideas for whole numbers have been chosen for the way they are relevant for both whole and decimal numbers and the way they allow teaching to form connections from one idea to the next idea without blurring distinctions. This chapter is based on the following components:

Part-whole: Place value – reading and writing Additive structure Multiplicative structure including renaming

Continuous vs discrete: Number-line model

Equivalence: More than one representation for one number

3.1 Part-whole/Place value H-T-O



This section looks at the applicability of the whole-part big idea to number, leading to early grouping, place value and reading and writing.

3.1.1 Notion of unit

The most important number in the system is 1 because it is the **unit** from which we make numbers larger or smaller. We can make numbers larger in two ways – by adding 1 more or by making groups (which is the basis of multiplying) – see figure below.



Counting is adding 1 and making groups

Flexibility

Anything can be a unit – a single object, a collection of objects, a section of a line, a collection of lines. Units can form groups and units can be partitioned into parts (e.g. if there are six counters, each counter can be a unit, making six units, or the set of six can be a unit, making one unit made up of six sixths). As well, the parts can be seen us the unit (e.g. if a sixth is a unit, then a one is six units). Thus, it is important to be able to flexibly change perception of unit and be able to perceive a situation in many ways, and to do this in an integrated way. For example, for three weeks and four days to be read and understood requires shifting from a unit of a day (for the four days) to a unit of a week (for the three weeks). Fortunately, there are many examples of unit change in reality (e.g. songs and CDs, eggs and cartons of eggs, beer cans and slabs of beer) on which we can build understanding of this big idea.

This means that perceiving number requires flexibility in changing unit from a single to a group to a group of groups. That is, we can count songs, CDs (each containing 12 songs) and packs (each containing 5 CDs with 12

songs each). We switch the unit of count depending on what we want to count. To understand (and read and write) numbers in terms of place value, students must continuously change their perception of unit (and gain a multifaceted perception of unit). For example, 324 requires students to realise that the 2 is two tens (where ten is the unit) and also twenty ones (where one is the unit).

Application to place value

For whole and decimal numbers, it is particularly important to flexibly change unit while moving through place values and parts and wholes. For example, for 428 to be read and understood as 4 hundreds, 2 tens, 8 ones, readers' perceptions of unit have to change from hundreds to tens and ones as their focus moves across the place values. This is because the 4 hundred does not refer to 400 single units which are ones, but to 4 units that are hundreds.

In the long run, sorting, correspondence and counting require focusing on groups and groups of groups as well as individual objects. In fact students need to be able to switch between different notions of units. For example, 234 is 2 hundreds, 3 tens and 4 ones – students have to be able to count hundreds for the hundreds position, left-over tens for the tens position and left-over ones for the ones position – or else, if they can only focus on individual objects they will see the 234 as 200304. In the longer run, students also have to be able to count fractions, tenths and hundredths.

It should be noted that this flexibility is also the basis of renaming. By changing the units on which we focus, 2 745 can change from 2 thousands, 7 hundreds, 4 tens and 5 ones to 27 hundreds, 4 tens and 5 ones. This idea will be discussed further in later sections as one of the main features and uses of multiplicative structure.

The notion of unit and place value begins with attribute recognition (recognising groups as something that could be counted) and correspondence (one-to-many as well as one-to-one). It requires students working with groups as well as individual objects in the very early years (e.g. forming groups, counting groups as well as individual objects, counting five steps as well as single steps, and so on).

3.1.2 Early grouping

This is an important mathematical idea that sets up the notion of place value by allowing students to experience groups and ones (with the groups on the left-hand side), and grouping and ungrouping. It also shows the power of kinaesthetic learning to teach a visual image (like place value) by moving the body to left and right as students put out groups and ones. The activity below describes how to teach it.



Materials: Groups/Ones board, counters, Unifix cubes or bundling sticks.

Reality

Using local culture and environment. Look for examples of groups in local shops, everyday life and culture. **Existing knowledge**. This activity extends one-to-one counting to one-to-many and relies on students knowing left and right (L-R) positions.

Kinaesthetic activities. Sort students into groups of three, four or five. Students count how many groups and how many individual students are left over.

Abstraction

Sequences. Make different groups out of a number of loose ones. For example, make groups of 3 out of 8 loose ones. Say: "Put out 8 ones (counters). Do you have enough ones to make a group of three?" [Students should say, "Yes".] Say: "Show me your group of three. How many groups of three do you have?" [1] "How many loose ones left?" [5] Repeat these questions until no more groups can be made. Repeat with other groups (such as groups of 5, 4, 8, 10).

Play the Zurkle game (Baratta-Lorton, 1981)

- Choose a number between 2 and 9, say, 4. Make up a nonsense name for this number, say "zurkle". Count four things a few times "one, two, three, zurkle". Put left hand on Groups on Groups/Ones board and say "this side is for zurkles", move left hand to Ones and say "this side is for ones". Start by moving left hand from Groups to Ones saying as you go: "zero zurkles and zero".
- Then ring a bell, students add one Unifix cube to the Ones side, move left hand and say "zero zurkles and one". Keep ringing bell, adding another one, moving left hand and saying, until you reach "zero zurkles and three". Then ask predictive question "what do we do when the bell rings again?" [Add one, make a zurkle by connecting Unifix, move to left-hand side, move hand, say "one zurkle and zero"]. Continue counting forward in zurkles and ones, moving hand, saying, and asking predictive questions, until four zurkles (can ask what this could be). Then count back removing a one and asking predictive question when there are zero ones.
- Play race game where students spin a 1-2-3 spinner and add ones shown, always stating how many zurkles and how many left-over ones first to 4 zurkles wins. Can race backwards too.
- Repeat game for other grouping numbers and nonsense names (2 to 5 before 6 to 9).

Show groups and left-over ones using virtual materials – counters, groups of counters and Groups/Ones boards. Interpret and draw pictures in terms of groups and ones.

Mental models. Get students to shut eyes and imagine a Groups/Ones board. Give a starting number of groups and ones. Then say to add or remove ones and get students to state how many groups and how many left-over ones. Ensure they have a picture in the mind.

Creating own representations. There is creativity here with nonsense names. You could also have students make up a counting system when only using one hand. It can be a lot of fun to use "fee" as one, "fie" as two, "foe" as three, and "fum" as four and have the giant in Jack and the Beanstalk counting his golden goose eggs.

Mathematics

Connections. Ensure that these early grouping activities are connected to the first tens and ones activities.

Reflection

Extensions. Early grouping activities can lead to many ideas that later will be introduced with tens and ones: (a) position – that groups are on the left of ones; (b) groups of groups – that four zurkles (or zurkle zurkles) needs a new name (like ten tens is one hundred); and (c) renaming – that groups are formed and broken up. One could also introduce numerals for groupings – for zurkles, this would mean that 2 zurkles and 3 ones is 23.

3.1.3 Early place-value ideas

Place value is an important aspect of number. The nature of our number system means that the value of a digit in any number is determined by its place and we only need 10 symbols to represent any number in our number system from very small numbers to very large numbers.

Hindu-Arabic number system

Before the Hindu-Arabic system, there had been many number systems invented. However, although all of these systems used names and symbols to indicate numbers and all of them had an adding property, they had no grouping/place-value property like the Hindu-Arabic system. The Hindu-Arabic system came to the English-speaking world from India via the Arabic culture. It was a purely positional system and therefore much more efficient than the additive Roman system and the multiplicative Chinese system. It represents larger numbers in terms of place values that rise as exponents of the base which is 10 (i.e. $10^0 = \text{ones}$, $10^1 = \text{tens}$, $10^2 = \text{hundreds}$, $10^3 = \text{thousands}$, and so on). Thus, this number system came to dominate the world.

However, its translation to English was **fraught with problems**. First, Arabic writing goes from right to left, so 243 is supposed to be read "3 ones, 4 tens, 2 hundreds". However, English writing is left to right. So the Hindu-Arabic right-to-left number system was placed into an English left-to-right writing system (making the English language say the numbers in the opposite direction to what they were meant to be said). This means that scanning whole numbers requires running eyes from left to right across the digits to find how many digits, working out place values by running eyes right to left from the ones place, and finally saying the number as eyes move left to right again.

Further, the Hindu-Arabic number system is base 10, so it only uses 10 symbols from 0 to 9. However, the English oral language number system was base 20 (e.g. people said "four score and seventeen years ago" where score was 20), so it used 20-language number names (i.e. "zero" to "nineteen"). Thus the Hindu-Arabic base-10 symbols were placed into a base-20 language. This causes a contradiction between language and numerals for the teens (e.g. 17 should be "onety-seven" but instead it is "seventeen"). The English oral number names also did not use zero and had slightly varied names for 20, 30 and 50 (i.e. "twenty" not "twoty", "thirty" not "threety" and "fifty" not "fivety"). This exacerbates the confusion. For example, 230 is "two hundred and thirty" not "two hundred and threety zero", and 511 is "five hundred and eleven" not "five hundred and onety-one" as the pattern in the digits indicates.

Place value and grouping

Before students can understand how our number system works they need to learn that a number means "how many" or is representing a quantity. To do this, students need to focus on single objects. Then oral and written number names and numerals can be associated with representations of numbers and can be used to count sets and to count out sets from larger sets (e.g. OOOO is "four" or 4 objects). However, for larger numbers, the focus must move from single objects to groups of objects, and students must determine amount from what numeral is used and from the position of the numeral according to other numerals. For example, the 6 in number 6 means six objects but the 6 in 64 means 6 groups of 10 objects. This means student perception must be expanded to move between one-to-one and one-to-many. To assist this move from individual objects to groups of objects, from counting to place value, early grouping activities are essential because grouping leads to place value. Thus, it is very important that young students have lots of practice making groups of 2, 3, 4, etc., finally ending up at groups of 10 when place value is the focus of teaching (e.g. 10 ones = 1 ten; 10 tens = 1 hundred; 10 hundreds = 1 thousand; and so on). Because place value is based on position of numerals and the size of the base, early grouping develops the subsequent mathematical ideas.

Students can gain experience with groups through early multiplication activities. Highlighting the difference between the groups and the ones can lead to early understandings of place value. The figure below shows the making of groups of four and the use of a Place Value Chart (PVC) for placing the groups and the ones. When there are enough ones a new group can be formed and moved across to the Groups side of the board.



Groups/Ones chart and grouping/renaming for grouping in fours

Grouping/renaming. When the number of ones reaches the group size, the objects are grouped and moved to the left position (see figure above). This prepares students for what happens when the number of ones passes 10 in numbers (e.g. 2 tens and 14 ones is reconstructed as 3 tens and 4 ones), when the number of parts passes the total number of parts in fractions (e.g. 2 ones and 9 fifths is reconstructed as 3 ones and 4 fifths), and when the number of units passes the conversion to the next unit in measurement (e.g. 2 m and 124 cm is reconstructed as 3 m and 24 cm).

Decomposition/renaming. When there are no more ones to remove, a group can be broken down into ones and shifted to the right to the ones position (e.g. 3 tens is 2 tens and 10 ones). This also prepares students for place value, fractions and measures, similarly to grouping above.

Place value and digit position

Value is shown by the **position** or **place** of the digits with ones on the right-hand side (the convention for showing where the ones are for whole numbers), the tens to the left of the ones, and the hundreds to the left of the tens. The positions increase in size to the left with a ten being 10 ones, and a hundred being 10 tens or 100 ones. Zeros are place holders, that is, if there are tens but no ones, or hundreds but no ones and/or tens, these positions are filled with a 0 (e.g. 4 hundreds and 2 ones is 402) to ensure that the right place values are indicated (e.g. without the zero in the tens place, 4 hundreds and 2 ones would be 42 and confused with 4 tens and 2 ones). However, zero on the left is unnecessary, but can be used as it does not change the number (e.g. it is common to write a date such as 4 May 2014 as 04-05-14 where 04 and 05 are the same as 4 and 5 respectively).

3.1.4 Teaching place value and to read and write numbers

This is the most central activity – teaching students how to understand place value for hundreds, tens and ones. The following RAMR activity contains steps to do this – for tens and ones first, then hundreds, tens and ones, then thousands. (These materials and ideas were taken from numeration materials developed under the leadership of Associate Professor Annette Baturo for training Indigenous teacher aides.)



Materials: Bundling sticks or straws and rubber bands; MAB units, longs and flats; money (\$100 and \$10 notes, and \$1 coins); Hundreds/Tens/Ones (H/T/O) PVC.

Materials have to show two things: (1) that position determines value and value moves $R \rightarrow L$ starting at ones, and going through tens and hundreds; and (2) digit in tens position is 10× larger than digit in ones position. The PVC shows position and the bundling sticks/straws, the MAB and the money show size or base. The money does not show size as completely as bundling sticks/straws and MAB, but it may be a more appropriate model for older students. Use MAB as first option unless students think that it is for "little kids", then move on to money.

Reality

Kinaesthetic activities. Get students to group themselves into groups of a number less than 10, say four or six, and then get them to count how many groups and how many are left over. Try this for tens and ones.

Abstraction

Sequences: Tens/Ones. Use materials (e.g. bundling sticks, MAB, money) to represent tens and ones. Record on PVC. Enter on calculator and write on paper. For normal numbers, introduce language "four tens and two ones" before "forty-two"; for zeros start with "four tens zero ones", and "forty-zero" before just "forty"; for fourteen use "one ten four ones" before "onety-four" before fourteen. Do not relate "four tens" to 40 otherwise students can see "four tens and three ones" as 403. A sequence of activities with bundling straws is given below.

<u>1. Getting to know the Place Value Chart and bundling straws</u>. Show the PVC. Say: "Read the place names as I point to them" (pointing from right to left). "Which place is largest in value?" [If students don't know, say: "Would you rather have 100 dollars, 10 dollars or 1 dollar? Why?"] "Which place is smallest in value?" [If students don't know, say: "What would be the smallest amount of money – 100 dollars, 10 dollars or 1 dollar?"] Show some loose bundling straws. Say: "These are called ones. Where would you put them on the Place Value Chart?" [Ones place] Show a bundle of 10. Say: "These are called tens. Where would you put them on the Place Value Chart?" [Tens place] Make sure the students know where to put the tens and ones by doing the following. Hand the student some ones, ask them what they're called and to put them on the PVC in the right place. Hand

the students another bundle of 10 and ask them what they're called and to put it on the PVC in the right place. Repeat until the students know that the ones go in the Ones place and the tens in the Tens place.

<u>2. Making groups of 10</u>. Pick up a handful of ones. "We're going to see how many tens we can make out of these ones. Do you have enough ones to make a group of 10? Show me. Put a rubber band around them to hold them together. Where will you put the 10 on the Place Value Chart – the Tens place [pointing to the place] or the Ones place [pointing]? Do you have enough ones to make another ten?" Repeat until no more tens can be made. Then, say: "What number have you made?" Count by tens first [10, 20, 30, 40, etc.]. Repeat the activity above a couple of times with a different number of ones.

<u>3. Making numbers</u>. Say: "Show me the number that has 2 tens 7 ones on the Place Value Chart. Show me the 2 tens part. Show me the 7 ones part." Repeat with these numbers: 4 tens 2 ones, 3 tens 8 ones, 1 ten 5 ones, 5 tens 0 ones. Say: "Show me the number, twenty-seven, on the PVC. Show me the twenty part. How many tens make twenty? Show me the seven part. Which is worth more – the twenty part or the seven part?" Repeat for the numbers 31, 43, 60.

<u>4. Writing numbers</u>. Use the bundling straws and PVC to make numbers like the one shown below. Ask the students to write the number and then read it.



Moving from materials and PVC to writing 2-digit numbers

<u>5. Counting and trading ones for tens</u>. Show 36 with bundling straws on the PVC. Ask the students to write the number. Put out another one straw, saying: "I'm adding 1 more one. Write the number now." Repeat until you get to 39. Put out 1 more and ask: "Do I have enough ones to trade for a ten?" [Trade] "What number do I have now? Write it." Keep going for a few more numbers. Ask the students to read the numbers they have written. Swap roles. Let the students make the numbers while you write the numbers. Repeat the above but have the students use the calculator to count as you build the numbers with bundling straws.

<u>6. Ungrouping tens to make ones</u>. Put 2 tens 7 ones on the PVC. Say: "How many tens do you have? How many ones? Roll the dice and take that number of ones away." [Suppose 4] Say: "Do you have enough ones to take away 4 ones?" [Yes] "Show me. Roll the dice again." [Suppose 5] Say: "Do you have enough ones to take away 5 ones?" If No, then say: "How can you get more ones from this number?" [Ungroup a 10 and put the 10 ones in the Ones place.] "How many tens do you have now?" [1] "How many ones?"

Repeat until the students have no more straws left. Play "Lose 5 tens 6 ones" if you have two or more players. Each player starts with 5 tens 6 ones of bundling straws on the PVC. Take turns in rolling the dice and taking off that many ones, trading tens for ones when necessary. First player to lose all the straws is the winner.

Note: The strength of size material like MAB on a PVC is that manipulation of materials can kinaesthetically do two things at once – build position (larger place values on left) and build size (the left-hand side digit is worth 10 of the right-hand side digit). However, the weakness is that material for 4 tens has 40 ones in it, so 4 tens and 3 ones can appear to be 40 on left and 3 on right (i.e. 403). The way to prevent this is to never just say that 40 is 4 tens – instead, say it is 4 tens and 0 ones – and to always stress that the students write the number of tens (not the number of ones in the tens) on the left-hand side and the number of left-over or loose ones on the right-hand side.

Sequences: Hundreds/Tens/Ones. The sequence below shows activities for H-T-O.

<u>1. Repeat the above but for three digits</u>. Do 456 before 450 before 406 before 400 before 416 type numbers. Note: 400 is "4 hundreds 0 tens and 0 ones" – it is not "4 hundreds" (otherwise "4 hundreds 2 tens and 3 ones"

may be written as 40023). Stress that the LHS is the number of hundreds, the middle is the number of left-over tens and the RHS is the number of left-over ones. A sequence of activities with MAB is given below.

<u>2. Check whether your students know that 10 MAB ones = 1 MAB ten</u> by having them line up the small blocks beside the long block (see figure below). Then check that they know that 10 MAB tens = 1 MAB hundred. Play "Lose 6 tens 6 ones" with MABs on the PVC. Put out 66 on chart with MAB. Throw two dice, students add the two numbers and remove that number of ones from the chart, regrouping as they go. Students record the number of MAB with a calculator as they play the game. First to 0 wins. Don't forget to ask the questions: "Do you have enough ones to take away XXX ones?" Make sure students are familiar with the word "trade" and its meaning.



Trading with MAB

<u>3. Introducing hundreds</u>. Ask students to show 99 with MABs on the PVC. Say: "Show me the ninety part. How many tens? Show me the nine part? How many ones? Add 1 more one. How many ones do you have now? Do you have enough ones to trade for a ten?" [Students should trade] "How many tens do you have now? Do you have enough tens to trade for a hundred?" [Students should trade] "What number do you have now? Keep adding ones [make sure students put them in the ones place] until you get to 112." Ask students to read the number now. Ask: "How many hundreds? How many tens? How many ones?" Ask students to show each of these numbers with MABs on the PVC: 574; 832; 333; 740; 200; and 409.

<u>4. Calculator counting</u> (students using PVC, MAB, pen and paper, and calculator). Start with 364 on the PVC. Ask students to read the number [three hundred and sixty-four] and write the number on the paper. Pick a place value, say hundreds, Say: "Add one hundred" – students put out one more hundred, write the new number under the starting number, and add 100 on the calculator. Keep doing this until students have 964. Ask students to read their calculator number and to check whether it's the same as their blocks number. Ask students what place changed first when they were counting by hundreds. Check the written numbers to see if only the hundreds changed. Repeat this for adding 1 in the tens or ones place.

<u>5. Reading numbers with zeros and tens</u>. Give a real-world story: "Malcolm counted 216 cars". Show the number with MAB (see figure below). Students say the number: "two hundred and sixteen".



Number of cars shown with MAB

Say: "Show me the two hundred part; show me the ten part; show me the six part." Discuss what the number should/could have been called: "two hundred and onety-six". Compare the number with 260 and 261. Put out MAB for both these numbers. Say: "How many ones in two hundred and sixteen, how many in two hundred and sixty, and how many in two hundred and sixty-one?" Repeat for tens and hundreds. Repeat for other numbers, for example: 311; 518; 470; 407; 200. Swap roles – the teacher says the number and the students show it with MAB. Don't forget to ask the students to show the hundreds part of the number, the tens and the ones.

<u>6. Writing numbers with tens and zeros</u>. Give a real-world story: "Malcolm counted 217 cars". Build the number with MAB (see figure below). Students say the number: "two hundred and seventeen".



Number of cars shown with MAB and digit cards

Say: "Show me the two hundred part; put a digit card underneath to show this part. Show me the tens left over (the ten part); put a digit card underneath to show this part. Show me the left-over ones (the seven part); put a digit card underneath to show this part." Ask the student to make a small PVC on paper and then write the number shown on the large PVC with MAB. Compare with 270 and 271. Make with MAB and add digit cards. Discuss the differences in the hundreds, tens and ones and the similarities in the language. Repeat for other numbers, for example: 412; 813; 630; 702; 300. Swap roles – write a number on the small PVC and say the number; the students show it with MAB. Don't forget to ask the students to show the hundreds part of the number, and so on.

Thousands. Repeat the above but for four digits. Once again leave zeros and teens to last. Look up history of numbers and show students other systems – e.g. Roman, Chinese , Mayan.

Mental models. When placing material on PVC, put your left hand on left-hand position and, as you move your left hand to the right, say "four tens and five ones", or "six hundreds, seven tens and two ones"; then repeat hand movement, saying "forty-five", or "six hundred and seventy-two". Do hand movements over and over as you change between hundreds, tens and ones (e.g. add 3 ones, take away 2 ones, add 5 tens, take away 2 hundreds, and so on). Use calculators – have students say the number out loud, record the number on small PVCs, on paper and enter on a calculator (get students to add the relevant number on the calculator when adding material to PVC). Give special attention to zeros and teens.

Creating own representations. Have students construct their own 3-digit number system for a 12-fingered alien race – make up symbols and number names.

Mathematics

Appropriate. Make sure students have mathematics symbols and language.

Practice

<u>Blockbuster</u>. Play games such as Blockbuster or make a flat/lose a flat. Throw one die and add/subtract number of ones shown, making a new ten from 10 ones or trading a ten for 10 ones when required, first to or from four tens wins; or throw two dice and stop at 100 or go to 0 from 100. Always ask "how many tens", "how many ones left over" and "how many more to next ten". Stress the number of tens goes in LH place-value position and the number of left-over ones goes in RH place-value position.

<u>Number expander</u>. Use a number expander to relate, in both directions, numbers to their expanded form (e.g. 4326 \rightarrow 4 thousands 3 hundreds 2 tens 6 ones, and 4 thousands 3 hundreds 2 tens 6 ones \rightarrow 4326). Pleat fold the expander at the coloured section so that it becomes, when folded, just four spaces and, when opened, the four spaces plus the expansion – you can, of course, only fold open some sections at a time.

Thousands	Hundreds	Tens	Ones

<u>Read-write calculator</u>. Call digits, students enter on calculator and say the number; say a number, students enter on calculator and then say the digits they used. Play "Wipe-out" – give a number and call out a digit – students enter number on calculator and reduce digit to zero with a single subtraction.

<u>Trading games</u>. Play "Win three hundred". Start with 6 tens. Roll the dice and put out that number of tens (make sure students put the tens in the tens place and trade when possible). Ask students to say the number each time (e.g. "one hundred and twenty"). Play "Lose two hundred". Start with two hundred, roll the dice and take away that number of tens, trading when necessary.

<u>Cards/jigsaw games</u>. Bingo: have one student with flash cards with numbers on them, show numbers in turn, other students look on their sheet and cover language or pictures of number, first to get three in a row wins. Cover-the-board has card decks with symbols, language and pictures and a base board with, say, PVCs with numbers – students in turn place cards on top of board numbers or other player's cards which are different representations of their cards; the person with the most number of their cards on top at the end, wins. Mix-and-match is placing different representations of numbers on cards, cutting cards the same way, leaving piles of pieces for student to put back together so all representations are of the same number. (Note: Can also make up dominoes).

<u>Game "Close, closer, closest"</u>. Put 200 on PVC with MAB for each player. Each player enters 200 on calculator. Remove 10, J, Q, Ks from a card deck. Shuffle cards. Deal two cards to each player. Players, in turn, form a twodigit number from their cards, add this MAB to PVC (completing all trading) and add number to calculator. They read the number and say how many tens and ones at each play. The game ends when the first player passes 600. The winner is the player closest to 600 at that point. <u>Option</u>: Start at 600 and remove the two-digit number at each turn. Closest player to 200 wins.

<u>Worksheets</u>. Prepare column worksheets with one column filled in (students fill in others) as follows.

Picture (MAB on PVC)	Language	Symbol
	Three hundred and seven	
		5302

Practise with virtual materials.

Connections. Ensure school maths plans connect this work to early grouping and to decimal number work.

Reflection

Flexibility. Discuss how many ways there are to get 61: 6 tens 1 one, 7 tens minus 9, ½ 100 + 11 and so on (look out for one hour and one minute). Think of everywhere numbers are used (money, cards, addresses, imaginary things, on the backs of sportspeople, board games, measuring tapes, codes on goods bought in shops, credit cards, temperature, names of TV and mobile phone companies, etc.). Try to think of every possible use.

Reversing everything. You say numbers/write numbers, ask students to show material; you show material, ask students to write/say numbers. Reverse with many materials, for example, lead students through MAB on Hundreds/Tens/Ones PVC \rightarrow saying hundreds, tens, ones \rightarrow saying number properly \rightarrow recording on small PVC \rightarrow entering on calculator; then reverse calculator \rightarrow small PVC \rightarrow proper language \rightarrow saying hundreds, tens, ones \rightarrow MAB on PVC. Then go backwards and forwards from materials $\leftarrow \rightarrow$ digits, giving examples that have zeros and teens.

Generalising. Say: "If we have fred tens and five ones, what is the number?" ["fredty-five"]. Say: "If we have Sue hundreds, Jean tens and Pat ones, how do we say the number ?" ["Sue hundred and Jeanty-Pat"].

Changing parameters. Give the place values in the wrong order: "What number is 6 tens, 8 hundreds and 7 ones?". Do a lot of these – it is important that students see that a number is determined by hundreds, tens and ones and it does not matter in what order the place-value positions are given. Reverse this – get students to give all the different ways that 587 could be given – 5 hundreds, 8 tens, 7 ones; 8 tens, 7 ones, 5 hundreds; 7 ones, 8 tens, 5 hundreds; and so on.

3.2 Additive structure H-T-O

3.2.1 Additive structure

The additive structure of place value relates to the ability to **count** in each place value. We can count on and back in the Ones, Tens, Hundreds, Thousands etc. places and each count is in units of the place value. It is possible to determine one more or less in each place value (e.g. 10 more than 53, 1 less than 419, 100 more than 345, 10 000 more than 53 000, 100 000 less than 409 000). Each large place-value position counts like the ones. For example, if you are counting on in tens each count is ten, i.e. 10, 20, 30, 40. You have counted 4 tens and arrived at 40. This is closely related to counting and odometer and is also the basis for patterning.

The odometer principle states that, in any place-value position, numbers count the same as in the ones place. Counting forwards, they go up to 9 and return to 0 with the digit to the left increased by 1; counting backwards they go down to 0 then return to 9 with the digit on the left reduced by 1. For fractions, the same happens but the number to which the counting goes in counting forward or changes in counting backward is one less than the number of parts in the fraction.



Students need to develop an understanding of the odometer principle in whole-part situations. It includes whole and decimal numbers, where the grouping is 10, and common fractions and mixed numbers, where the grouping may be any whole number. It is particularly important for whole and decimal numbers where each place value (PV) counts forward 7, 8, 9 and then goes to 0 with the PV on the left increasing by 1; and each PV counts backwards 2, 1, 0 and then goes to 9 with the PV on the left decreasing by 1. It should also be noted that odometer, along with place value, rank, and multiplicative structure, is one of the four meanings of whole and decimal number numeration.

Seriation is the ability to determine one more or less in each place value (e.g. 10 more than 42, 10 less than 56, 1 more than 359, 1 less than 420, 100 more than 601, and so on). This ability is related to counting and odometer as 10 more is one count forward in the tens position and 10 less is one count backward in the tens position. It also is the basis for patterning (e.g. What are the next three numbers? 276, 286, 296, ___, ___, ____ is solved by 10 more, 10 more, and so on, using the odometer pattern when 9 is reached).

3.2.2 Teaching seriation and patterning

The RAMR activities below have many teaching ideas using the 99 board.

Materials: 99 boards (starting at 0, 100, 200, etc.), cards, counters, 99 board windows and worksheets.



Abstraction

Sequences. The 99 board represents number in terms of rows down for tens and columns across for ones. Activities focus on building understanding of position of numbers so that students can easily determine one more and less and ten more and less. A sequence of possible activities follows.

Getting to know the patterns of numbers. Have students read columns and rows. For example, 4, 14, 24, ...; and 60, 61, 62,, and notice the patterns.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Reading a column:

It can be useful to have students read the column as "four, onety-four, twoty-four", and so on. Then the pattern in the column can be seen – it is that "four" is said each time with the tens going up, that is, the ones stay the same and the tens increase.

Reading a row:

The row is read as "sixty, sixty-one, sixty-two", and so on. Then the pattern in the row becomes apparent – it is that the "sixty" is said each time with the ones going up, that is, the tens stay the same and the ones increase.

Cut 99 boards into jigsaw puzzles and get students to re-form them. Get students to make puzzles for each other. Hand out 99 boards with parts missing and students have to complete the numbers.

Knowing where numbers are – placing numbers by tens and ones. Always start at zero. For position to number – get students to start at zero and move down and across, encouraging students to see the pattern, e.g. that three down and seven across is 37. For number to position – move the tens down and ones across, e.g. 54 is five down and four across (starting at zero).

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
\$0	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

In the first teaching direction, three down and seven across is given and students find they reach 37. In the second teaching direction, 54 is given and students find the movements (five down and four across) that reach this number.

This is an example of the generic strategy of **reversing** – teaching in both directions.

These activities practise finding and placing numbers.

Teaching seriation. Use the board to identify the numbers on left, right, above and below chosen number – show how left and right is one less and one more, above and below is 10 less and 10 more.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Look at 78: one less is 77, one more is 79, 10 less is 68 and 10 more is 88.

Start with the number and look at the left, right, above and below numbers – use a calculator to determine relation of these numbers to the original number (circle the number as for 46 on the left).

Start with the number and determine (use a calculator) one less, one more, 10 less and 10 more and then find these numbers on the board and their relation to the starting number.

Three-digit numbers. Extend these ideas to 99 boards that start with 100, 200, etc. – that have three-digit numbers.

Patterns. Construct patterns of the type 356, 366, 376 and so on for different place values (also have decreasing patterns). After constructing, then get students to decipher the patterns – they have to find the changing place value. Make sure students know what happens as place-value position goes up to 9 or down to 0.

Mental models. Ask students to close their eyes and imagine the 99 board. Get them, with eyes shut, to find numbers, to state numbers that are one more and less or 10 more and less. Ensure they have a picture of the board they can use for seriation and patterns.

Mathematics

Practice. Play "Three in a row". Players in turn take two cards from a pack with 1 to 9 in it (A is 1 - K, Q, J and 10 removed) and cover any number they can make (e.g. 4 and 3 could be 43 or 34) with a counter (students can remove opponent's counter to place theirs). The first player to get three in a row (row, column or diagonal) wins.

Other practice activities

1. Construct a 99 board window with a hole in the middle; place over a number so only that number is visible and ask students to write numbers one less and one more, above and below. Examples of windows are:



- 2. Give 3×3 squares with number in middle and ask for other numbers.
- 3. Give 3×3 squares with numbers on outside and ask for number in middle (reversal of step 2 above).

36	

4. Give a section of a number board and provide some numbers. Vary the difficulty or number of squares. Also extend to use larger numbers. See examples A to G below:



3.2.3 Teaching counting, number and odometer

The RAMR activities below have many examples of good teaching in this area.

Materials:

- MAB, bundling sticks and rubber bands
- Unifix, PVCs
- Flip cards (numbers 0 to 9 that can be flipped over 4-ring folder will do can make 3- and 4-digit numbers then flip over any place value)
- 99 boards (normal and 3-digit versions)
- Calculators, pen and paper.

Reality

Using local culture and environment. Look for things that follow odometer pattern in local environment and culture – e.g. an odometer from a car, clock or boat.

Existing knowledge. Check that the students have counting and an understanding of place value for the number of digits being considered.

Kinaesthetic activities. Get the students to act out an odometer. For example, set up three 0–10 tracks, three students at 0 on each track, ones place student walks along track. When ones place students get to 9, step off and another student starts at 0, while student in tens place steps forward one space. When tens place student makes it past 9, the hundreds place student steps forward one place. Could have a hopping relay race to see who gets to 500 first.

Abstraction

Sequences. Move through bundling sticks on PVC, MAB on PVC to flip cards to calculator. For example: (a) put MAB on PVCs, pick a place value, add MAB one at a time to this value, state number, write number, add on calculator as you go along – also count backwards, removing one MAB piece at a time (remember to write and record on computer); (b) repeat above with flip cards, counting and writing as you go – also count backwards, flipping cards backwards (remember to write and record on computer); and (c) use calculator, enter a number, say 254 and +10, clap hands, students press =, state the number – get them to write the numbers down to see the changes; get them to -10 as well and count backwards.

Repeat the activities from (c) above but this time get the students to state only the place-value position that is being counted, e.g. 324 – "two", 314 – "one", 304 – "zero", 294 – "nine", and so on – going forward and backward. This activity is particularly strong for the calculator. Extend the ideas to normal 99 boards and 99 boards that have 3-digit numbers.

Mental models. Get students to shut eyes and imagine the odometer pattern – get students to call out the numbers as you clap hands, without a calculator and with eyes shut. If you can, obtain a virtual sequence of an odometer turning over and get students to imagine this with eyes shut. Do this for an odometer turning backwards. Get students to think of numbers in place-value positions as **being able to turn over or back** – so the 4 in the tens position in 847 turns over to 5, 6, 7 if counting forwards, or to 3, 2, 1 if counting backwards.

Creating own representations. Ask students to make an odometer of another base – what would a 12-fingered Martian odometer look like (they can make up own symbols). Get odometer to work both ways.

Mathematics

Appropriation. Ask students to write down the sequences as they use calculators so that they can see how numeral patterns of odometer counting work.



Practice. Use completing patterns such as 376, 386, 396, ___, ___, ___ and 343, 342, 341, ___, ___, ___, ___, to practise odometer pattern. Construct patterns of this type for different place values (also have decreasing patterns) and get students to decipher the patterns (have to find the changing place value). Make sure students know what happens as place-value positions go up to 9 or down to 0. Show that all place-value positions count.

Connections. Connect odometer work to place value and counting. Also connect to seriation – being 10 or 100 more is the same as one step in odometer counting; for example, 10 more than 348 is the second term in counting sequence 348, 358, 368, ____, and so on.

Connect to comparison and order – get students to use mental model of digits rolling forward or back to understand what it means to be within 30 of 847 – get students to see this as within 3 in the Tens position, as below: the 817 and the 877 give the ends of the interval that is within 30 of 847.

____, ___, **817**, 827, 837, 847, 857, 867, **877**, ____, ____

Roll the 5 in 654 up 4 tens to 694 and roll back 5 tens to 604. Discuss what it means to be close. Look at being within 30 of 812.

Reversing. Get students to construct odometer patterns, not just decipher them.

Reflection

Validation. Discuss the action of an odometer – ask students to find where these are in their communities (e.g. reading electricity use). Discuss if it is a sensible way to record change in numbers.

Application. Apply to problems of odometer readings before and after travel – working out how far travelled. Apply odometer to counting situations with large numbers.

Flexibility. Extend the odometer activities above to find all situations where this odometer action occurs in the world – include ones that are not in students' experience.

Generalising. Try to get the students to state what is happening in place-value positions as a generalisation. Get the students to think about the change as they count to 9 or back to 0 (i.e. every position counts forward from 1 to 8, 9, and then back to 0 with digit on left increasing by 1, and backward from 9 to 2, 1, 0, and then back to 9 with digit on the left reducing by 1).

Changing parameters. Extend to other bases or groupings – count weeks and days, years and months, m and cm, hours and minutes – note what happens as you near the grouping number. Count fractions, e.g. $1^{3}/_{5}$, $1^{4}/_{5}$, $2^{0}/_{5}$, $2^{1}/_{5}$, and so on. Try to get students to see that every position counts forward to one less than the grouping, and then back to 0 with the digit on left increasing by 1; and backward to 0, and then back to one less than the grouping with the digit on left reducing by 1.

3.3 Multiplicative structure H-T-O

This section looks at the multiplicative relationship between adjacent place-value positions – this is not often taught yet is the basis of metrics and their conversions.

3.3.1 Nature of multiplicative structure

Multiplicative structure is both a principle and a concept. It is the understanding that adjacent place-value positions relate to each other multiplicatively, that is, one position to the left is 10× larger (multiplication by 10) and one place to the right is 10× smaller (division by 10). It is important that students understand that it is the change in numbers in relation to the place-value positions. These changes to numbers should not be remembered as the adding or removal of zeros.

This understanding is important when students work with decimals and when converting measures, for example converting cm to m and vice versa. The addition or removal of zeros are consequences of the place-value movement not the focus. The figure below shows the multiplicative structure of the place-value system. Moving the number 3 two places to the left changes the number from 3 to 30 then to 300. $3 \times 100 = 300$. Moving 3 two places to the right changes it from 300 to 30 and then to 3.



Students can be helped to see that in general whenever a number is moved one place to the left it is $\times 10$, two places to the left is $\times 100$ etc. The same works in reverse. A number moved one place to the right is $\div 10$, two places to the right is $\div 100$ etc.

Students need to develop flexibility to recognise that numbers can be renamed in relation to their place values. Renaming is the ability to understand (and calculate) that numbers can have more than one meaning in terms of PV positions. There are two "types" of renaming – simple ones that rename all of a PV position (e.g. 362 is 36 tens 2 ones, 3 hundreds 62 ones, or 362 ones) and more complex ones that rename only part of each place value (e.g. 362 is 2 hundreds 14 tens 22 ones). This is aligned with multiplicative structure because of the need to understand that, for example, 6 tens is $6 \times 10 = 60$ ones. Students who understand this flexibility of numbers based on sound knowledge of the structure of numbers based on their place value will be able to use this knowledge in computations where they separate numbers flexibly to make computations more manageable.

3.3.2 Teaching multiplicative structure

The RAMR activities below have many body-hand-mind activities to teach multiplicative structure.



Materials: Cardboard place-value names and some digits, digit cards and PVCs, slide rules, calculators, pen and paper.

Place-value (PV) car	rds:	One Thousands	Hundred Ones	d Ten Ones	One Ones	
Digit cards:	0	1 2	3 4	56	7	8 9
Slide rule						
		One	Hundred	Ten	One	
		Thousands	Ones	Ones	Ones	
						J



These materials can be used to study what happens when digits change place-value position – to make students aware of the multiplicative (\times 10 or \div 10) relationship between place-value positions. The following is a sequence of activities for 3-digit numbers that cuts across reality and abstraction. It can also be easily extended to 4-digit numbers.

Reality

Kinaesthetic. Make up large copies of digit cards and place-value positions, e.g. three or four place values; give students calculators and undertake the following activities.

<u>Using students' bodies</u>. Give three students PV cards and organise them to stand in correct positions. Students set up their digit cards and PVCs (or their slide rules), pen and paper, and calculator.



Give another student a digit card, say 6, and get them to stand in front of each position. Add zero cards to show what each number means. Press buttons to place numbers on calculator (see below, e.g. for 6 in tens position); write numbers on paper.



Repeat this for 2- and 3-digit numbers on cards in front of PV cards, e.g. 230, 604, 14, 824, and 615. Move from cards to calculator and calculator to cards (reversing). Say numbers in terms of hundreds, tens and ones and properly. Reverse everything: movements to x, \div ; and x, \div to movements.

<u>Exploring the movement</u>. Put a digit card in front of PV cards, move card left and right, use calculator \times and \div buttons to show relationship in moves (e.g. 6 tens going to 6 ones is \div 10 and 6 ones going to 6 hundreds is \times 100). Put a number in calculator, e.g. 40 and multiply or divide by 10, move cards to show these multiplications and divisions (note that the place-value cards could be stuck on wall); write numbers down.



As the students in the front of the classroom move along the PV cards, the other students copy the movements with their digit cards/slide rules, write down the changes on paper, and make the changes on their calculator with an appropriate \times or \div . Make sure that: (a) activities go both ways along cards (to lower and higher PVs); (b) activities are reversed – that you move the student and ask for the \times or \div ; and give a \times or \div and ask for where the student should move to; and (c) moves are one, two and three PV places.

Abstraction

Sequences

<u>Translation to pictures</u>. After acting out the above in front of the class, all students should focus on their own small digit cards and PVC materials, or their slide rule. They should be directed to move digits across the PV positions left and right (note that this material can also be in virtual form). As they do this, they should write numbers on paper and follow the movements with calculator.

Set up PVC and place a 4 on the ones of the PVC (or use a slide). Ask students: "How many ones, how many tens, how many hundreds?" Ask students to write number on a small PVC and to put on a calculator. Say: "Move the 4 to the tens of the PVC". Repeat questions and ask students to write number on the small PVC. Ask students how they change 4 ones on calculator to make 4 tens. Show ×10.

Repeat moving 4 from tens to hundreds. Ask: "How do you change 4 tens on calculator to make 4 hundreds?" Show ×10. Ask: "Can you see a pattern? What happens if a number moves from ones to tens to hundreds?" <u>Repeat</u> moving the 4 from hundreds to tens to ones. <u>Repeat</u> using the following starting numbers: 6, 23.

Now start with the \times and \div and ask where to move the digit card. Switch back and forth, starting with a move and asking for the \times and \div and starting with a \times and \div and asking for the move. Type a number on the calculator (one digit, a ten or a hundred) and place digit card on PVC. Multiply or divide by 10 or 100. Ask students to move the digit to match. Ask if the students can see a pattern (move to the left, \times 10; move to the right \div 10).

<u>Completeness</u>. Make sure all the following are done:

- (a) Use of more than one digit. Move more than one digit along the PV positions.
- (b) Extension to moves of two and three PV positions. Encourage students to study changes that move two places and then three places in both directions.
- (c) Reversing. Ensure that all activities reverse everything. That is, show or give a movement \rightarrow ask for the \times or \div ; give a \times or $\div \rightarrow$ ask for the movement.

Mental models. Ask students to imagine the cards and PVC in their mind – ask them to move finger with eyes shut along their imaginary PVC as questions are asked and to translate movement back and forth to \times or \div .

Mathematics

Practice. Worksheets – these should operate in both directions, giving the multiplication or division and asking for the movement, and giving the movement and asking for the \times and \div 10 or 100.

Reflection

Generalising. Students should be asked for the patterns in movements and relation to $\times 10$ and $\div 10$ and to write this down in their own words (generalisation). (**Note**: The pattern is better seen with more place-value positions. The materials can also be in virtual form and displayed with a data projector or smart board.) This should be extended to $\times 100$ and $\div 100$, and to $\times 1000$ and $\div 1000$. Then reversed and the pattern asked for moving to the left one, two and three PV positions and the same to the right.

3.4 Continuous vs discrete: Number-line model H-T-O

In this section, we take up the ideas in section 2.4 and use them in whole numbers up to four digits.

3.4.1 Introducing continuous vs discrete

A number line is a continuous line that is not naturally divided into discrete sections that can be counted. An interval of length one is used as a unit to divide the line into discrete parts. The placement of the number is based on this equal partitioning of the line. Since many measures are associated with lines it is important that all numbers are related to positions on lines as well as sets of objects.



The RAMR activities extend the ideas in section 2.4 to teach this to older students.

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Count discrete objects in the environment.

Hand. Use concrete materials such as blocks to "count" length or area of objects.

Mind. Discuss what things can be counted and what cannot. Point out that, normally, items have to be discrete (individual and separated) to be counted. Point out that the world is full of discrete things (chairs, people, animals, days, grains of sand, etc.) but that some things have had to be "changed" to be countable. Discuss length – it is not countable unless a unit of measure is used. Length is continuous (as is area, volume, mass, time, and so on) but units make it discrete. Ask why we want to turn the continuous into the discrete? [So we can apply number to it.]

Mathematics

Practice. Continue to engage in counting and measuring activities that demonstrate the continuous vs discrete big idea. Continue with worksheets that relate position to value on a line with two end numbers (e.g. place 929 on the line starting from 700 and getting to 1000).

Connections. A huge connection here is with length.

Reflection

Get students to generalise that discrete can be counted but continuous needs a way (e.g. a unit of measure) to change it into discrete for a number to be used on it. Discuss changes to numbers – count steps from one number to the next and that 0 is now the starting point – not nothing.

3.4.2 Rank

Regardless of the number of digits that make it up, each number is a single point on a number line. That is, 247 is a point close to and just below 250. This form of representation ranks numbers in terms of distance from zero (e.g. 239 is closer to 0 than 247, so 247 is larger) or the end points of the line. It is very important not to link the idea that the greater the number of digits in a number the larger the number is, even though it does hold true for whole numbers. When students learn about decimals and the extensions of the place-value system to represent numbers between whole numbers this causes conflict. The number 34.5 is not larger than 67 just because it has a greater number of digits. The value of the digits needs to be considered and, as discussed in comparing and ordering, it is the larger place values which need to be the starting point for such comparison and ordering activities. The RAMR activities below show some classroom activities that have been found useful.

Materials: Rope and pegs, pen and paper, number tracks, number lines, MAB and PVCs.



Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body

Use pacing, number tracks, rulers, and racetrack game boards to find the position along a line or a track of numbers by counting – notice the positions of numbers (larger numbers are further along the line or track).

Pick a distance at the school (e.g. length of a long building or a playground) – say it starts at 0 and ends at 100. Get students to walk to 50, 25, and other numbers. Discuss where each number will be. Repeat for other starting and ending numbers (e.g. 0 to 999, 50 to 150).

Put a start and end number on two students (say 0 on one student and 100 on the other) and put them in front of the class (students will see 0 on the left) and have the students hold a rope between them. Give other students numbers on paper between 0 and 100 and pegs and they have to peg on rope where they think this number would be. Get other students to help more accurately place the number. Discuss where numbers would be (e.g. in the middle, near an end). Repeat for other starting and ending numbers. Go both ways (i.e. reverse) – give number \rightarrow students place; give position \rightarrow students guess number.

Hand

Use rope and pegs. Get two students to hold a piece of rope at either end – hang 0 around the left person and 1000 around the right person. Give other students numbers on card/paper and pegs and get them to peg on rope. Get class to help place the numbers. Discuss how we could show that the placement is reasonable. Use examples like 487 (near half), 990 (near one end), and so on. Make sure you reverse the activity – peg cards with A, B, C, etc. on them onto line and get students to write down/guess what A, B, C, etc., are.

Repeat the above activity by putting numbers on a number line drawn on paper or on board (or virtually). Construct a number line from cm graph paper or straws – cut straws into 10 cm pieces, join together to make 100 cm, mark the start, finish and joins with 0, 10, 20, 30, and so on.

Use the number line to: (a) place numbers, and (b) state what number is in a particular place. Discuss the numbers in terms of ten – e.g. 56 is 5 tens (5 pieces 10 cm long) and then a little over half the next piece, while the position which is 7 ten pieces and a little more along is about 72. Reverse the activity – get students to find things with given lengths.



Mind

Have students imagine the line in their mind and then use the imagined line to place numbers.

Mathematics

Practice. Transfer to actual number line. Draw a number line as a copy of the number track. Discuss how numbers are at the end of the spaces not in the middle of the spaces. Discuss how to count along the numbers but that the count on or the count back is the number of jumps between numbers not the number of numbers. Extension ideas: Make sure students can work with number lines with all numbers marked, then fives and tens only marked, then only some tens marked, and finally no numbers marked. Reverse the activity – ask students to place numbers on lines; ask students to state what the numbers are that have been marked on lines.

Connections. Relate to a measuring tape mm-m – measure things and mark length on a line.

Reflection

Validation/Application. Have students construct their own measuring device to use as a number line. Use a variety of starting and finishing numbers.

Reversing. Go both ways – give number \rightarrow students place; give position \rightarrow students guess number, AND get students to construct number lines as well as interpret and work on number lines supplied to them.

Changing parameters. Do number lines still work for more than three digits?

3.4.3 Comparison/order

By placing on a number line, the largest number can be considered as the number further along the number line from zero – it can also be considered as the number further along the count.

However, the comparison of numbers needs to consider place value as it is the comparison of the highest place value of each number being compared that will provide the best comparison especially when comparing numbers with different numbers of digits. With whole numbers the number of digits will give an indication of the comparison and order as the number with more digits must have a digit in a higher valued place than the number with fewer digits. This is not a sound basis to teach students to make comparisons as it will not hold for decimals. It is better to compare the value of the highest place in the number, for example 345 compared to 67. The first number has three hundreds which is greater than zero hundreds in the second number.

Representation of numbers on a line provides an indication of size, so numbers can be compared and ordered in terms of position along the line. The placement on the number line also assists students to gain the understanding that higher place values are the starting point for comparing and ordering numbers (see figure below). Thus, comparison is aligned with rank.



The number 400 is a hundred more than 300, so 368 cannot be larger than anything over 400. This understanding should be related to place value so that the higher place values are the starting point for comparing and ordering numbers. This makes comparison align with rank.

Materials: Rope and pegs, pen and paper, number tracks, number lines, MAB and PVCs.

Reality

Where possible, find real-life contexts to embed the activities in; for example, comparing sport scores or heights, and so on.

Abstraction

Body. Use pacing, number tracks, rulers, and racetrack game boards to find the position along a line or a track of numbers by counting.

Hand. Repeat the activities in section 3.4.2 (e.g. with pegs and rope, or on a line or board or virtually) but, this time, use two numbers and state which is larger. Repeat this for three or more numbers – putting in order from smallest to largest or vice versa (remember to identify the "between" numbers).

Relate the results of number line to MAB/PVC represented numbers – develop the generalisation that the left-most place value is the most important in ordering.

Make a straw number line. Cut 10 cm lengths of straws (five each of two different colours). String straws alternately on string to make a "straw tape":



Compare final length with 1 m – how accurate is it?

Label the end of each straw as follows.



Using the straw number line. Find where the following would be on the straw tape: 30, 80, 28, 64. State the numbers that are where the teacher is pointing. (Teacher points to positions on one of the straw tapes.)

Crucial – *focus on teaching the importance of the hundreds position in determining the order of three-digit numbers.*

Mind. Imagine using the straw number line – what length would be greater? 56 or 73 or 29?

Vathematics

Formalise language and symbols. Ensure students know the correct words and symbols and can relate these to pictures.

Practice. Use the straw tape to work out the larger (smaller) of the following:

26 or 46	
----------	--

68 or 64

48 or 51

Place the following in order from largest to smallest: 54, 61, 16, 59, 65

42 or 39

Reality

Reflection

Abstractio

Maths

Complete a worksheet that will practise the ideas in the above material. Play games to become independent of material – the games below focus on getting students to see that the largest digit – in the hundreds – is the most important.

Three-digit "Chance number" games

Materials: digit cards, boards as below, digit cards to fit into board, card deck (0–9 or 1–9 only).



- (a) Chance number. Three cards are dealt one at a time, use first number to place a digit card on board (have to choose tens or ones), second number fills the other position. If make higher/lower number, score 1 point, 0 otherwise. Winner is largest score after five games. (Variation – when complete, can give up a number and take the value of a fourth dealt card.)
- (b) Chance order. Six cards are dealt, use the numbers to make the left-hand three-digit number smaller/larger than the right-hand number with digit cards on game board as required. Score 1 point if left-hand three-digit number is correctly larger (smaller) than right-hand three-digit number. Score 2 points if smaller three-digit number is largest possible. The winner is whoever has highest score after five games.
- (c) Chance order. Six cards are dealt one at a time, use first number to place a digit card on board (have to choose tens or ones in either the left-hand or right-hand three-digit number), continue making choices and placing digits on board before next card called. Score 1 if correct and 0 if not. The winner is whoever has highest score after five games.
- (d) Chance order. Six cards are dealt one at a time, use first number to place a digit card on board (have to choose tens or ones in either the left-hand or right-hand number), continue making choices and placing digits on board before next card called. Score 0 if not correct but score the value in the tens place of the smaller three-digit number if correct. The winner is whoever has highest score after five games.

Have students work in pairs to design their own game based on innovations of the models just played.

Connections. *Relate to measures*.

Reflection

Validation/Application. Apply comparison and order to the world of the students.

Flexibility. Find all situations where comparisons are used (money, height, mass, etc.)

Reversing. Give numbers \rightarrow put in order AND give one number and an order and have to find other number.

Generalising. This is the most crucial aspect – have to get students to generalise the importance of the largest *PV* position as the starting point of comparing.

Changing parameters. Extend above to four and more digits – is the largest PV position still the starting point?

3.4.4 Density

The concept of density relates to the amount of numbers between numbers in a counting system. This mathematical idea does not really relate to whole numbers as there are no whole numbers between consecutive whole numbers; for example, there are no other whole numbers between 34 and 35. However this idea is very powerful for decimals and is discussed further in the next chapter. The concept can be helpful when working with number sequences, such as multiples of ten, and identifying that between the multiples of ten there will be other numbers of the smaller place values. For example, between 100 and 200 there will be no other multiples of 10 like 20 and 30 and there will be multiples of 1 like 45, 46 and 47. There will be fewer of the larger place values and more of the smaller ones.

3.4.5 Rounding and estimation

Rounding and estimating are valuable skills that relate to students' sense of the size of numbers and their understanding of place value. Rounding and estimating are usually completed to a particular place value, for example, to round a number to the nearest hundred. Number lines enable values to be determined to which numbers under consideration are nearest. For example, in the number-line figure in section 3.4.3, it is obvious from the placement on the number line that 432 is nearer 400 than 500. So 432 rounded to the nearest hundred is 400. If similar placements are made on 0 to 100 number lines, students can come to understand that 32 is nearer 30 than 40, so 432 is 430 to the nearest 10. Placement of numbers on number lines also allows ease of estimation.

Materials: Rope and pegs, pen and paper, number tracks, number lines, MAB and PVCs.



Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction/Mathematics

Repeat the earlier rank and number-line activities but, this time, have other numbers (e.g. every 10 for 0–100 line; every 100 or 50 for 0–1000 line) already pegged or drawn on the line. Now when placing numbers, look at which 10 or 50 or 100 is nearest to the number.

Discuss what is the best approximation. Discuss what to do when halfway between two options.

Get students to construct their own number lines and place in points (such as every 10 or 20 or 50).

Reflection

Discuss what works best (too many points and it's too crowded; not enough points and it's too hard to read) - do 700–900 lines as well as 0–1000 lines. Check whether it matters if numbers were six digits and line went from 0 to 1 million.

3.5 Equivalence H-T-O

The big idea of equivalence focuses on the understanding that it is possible for a number to be the same and the numeral to be different. The concept of equivalence relates to the use of zeros in numbers. Our number system is unique in its use of zero. Sometimes adding a zero to a number makes the number different but other times it does not. Students need to be able to recognise when the inclusion of a zero changes a number and when it does not change a number. This concept is of particular importance with decimals but it can be investigated in relation to whole numbers. For example, changing 45 to 045 does not alter the value of the number but placing the zero in other places does, e.g. 405 or 450.

However, equivalence of whole numbers depends on the **purpose of the numbers**. If the purpose is to provide a measure of value, then the normal rules for zeros follow (e.g. 043 = 43). On the other hand, if the purpose is to identify, then the normal rules do not apply. Consider, for example, the numerical part of a car's registration. In this case, 043 is very different to 43 - in fact, 43 would be seen as a partial plate. Another example is the way computers order numbered documents. To ensure that 10 follows 9 we have to, normally, number the files 01, 02, 03, ... up to 09, 10, otherwise we will find 10 in between 1 and 2 in the list.

In this section, we attempt to integrate the above and build a more complex view of equivalence than simply the view that it occurs when zeros are put in front of numbers.

Reality

Look at the world of the students to find examples in their lives of where a different numeral is given for the same number. For example: How are dates entered on many forms? How are dates entered on credit cards? How would you write today's date? Are there any dates where you would write something different to normal numerals for numbers?



Abstraction

Hand. Use the numbers in A below as names for files. Save these files into a folder. Ask Word to order the files (click on "Refresh"). Repeat for numbers in B.

A:	7	3	8	2	4	6
B:	7	3	18	2	14	26

What happened? Is this a problem? What do we do to get B into numerical order? Is there a difference if B was changed to C?

C: 7 3 18 2 114 26

Discuss role of zero in numbers. Look at where zeros change and zeros don't change numbers. Reverse this – give students a number to start with and ask for the change or no change, give students a number which is an end and ask for the start if there was change/no change. Ask students for more than one answer.

Sort the following numbers into groups where the number is the same.

24 024 240 0024 204 2004 0204 00240

Mind. Imagine a number with no zeros - imagine putting in zeros - when/where does the zero make a difference?

Mathematics

Practice. Practise situations where zeros change and don't change numbers:

- give examples like 56→506, 45→045, 708→78 and so on; ask students to circle change and tick no change;
- give a set of examples based on 0 and three other digits; ask class to sort into groups where all numerals are the same number; and
- give a three-column table with starting number, change and no change as headings; fill in one of the columns for each row; ask students to make up the other rows.

Connections. Where else in mathematics do we have this situation where two different numerals give the same number/value?

- In measures?
- Time? (what is 5 past "oh five?") Dates? (what does year '95 or '04 mean?)
- Identification numbers? (Fred's student number was 00843 why is the 00 there?)

Reflection

Validation/Application. Ask students to explore the world and find everywhere that numbers are expressed in different numerals to the normal 6, 11, 25, 327, etc.

Flexibility. When numerals are different – why? What is the generalisation? Why is it done? [If numbers are to be ordered, and they go to three digits, then often there are 3 places to fill in and 4 has to be written 004, 57 has to be written 057 and 307 has to be written as 307.]

Reversing. Reverse everything. If 64 has to be written as 0064, then what number is 0206?

Generalising. Generalise the ways numerals can be changed without changing the number [add 0s at the start of a whole number] and change the number [0s in between digits or at end of digits of a whole number].

Changing parameters. What if we had six digits, does the rule change? Is this a change or not: 6781 \rightarrow 0060781? What number is a no change if it ends up as 0060781?

3.6 Extending to large numbers

The big idea here is that it is possible for mathematics to be learnt by a gestalt jump where new knowledge is seen as an extension of old without having all steps for learning the old repeated.

This is based on seeing PV as two parts: (a) a **macrostructure** of a sequence of larger numbers built around multiples of 1000 – ones, thousands, millions, billions, and so on; and (b) a **microstructure** within each of these PVs where there are hundreds, tens and ones. That is,, H-T-O millions, H-T-O thousands, H-T-O ones.

YDM recommends this as the best option for extending the five big ideas and their concepts and processes from three digits to nine digits, that is from H-T-O ones to H-T-O thousands and millions.

However, the alternative of repeating all the steps for part-whole, additive structure, multiplicative structure, continuous vs discrete/number line and equivalence is given in Appendix A.

3.6.1 Pattern of threes to read and write numbers

The ability to use place value to read numbers (symbol to language) and to write numbers (language to symbols) requires: (a) knowledge of the role of zeros; and (b) understanding that digits in numbers follow a strict pattern even if language does not. It also covers the ability to write numbers given out of order as place values (e.g. 4 ones 2 hundreds and 3 tens is 234). However, there are language difficulties; although the digits follow a strict pattern in their representation of numbers, this pattern is not followed by the language with respect to teens and zeros (e.g. "two hundred and two" is 202; "three hundred and seventeen" is 317).

Large numbers must be taught. There is still the need to use PVCs, numeral expanders and calculators in their teaching. There is still the need to put digits on PVCs and to use the kinaesthetic sense to drive home the pattern of threes. There is a special number expander for the pattern of threes:

	Millions			Thousands			Ones	
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones

A way of setting up stating a number like 346 786 254 is to cover, in turn, two of the threes 346, 786 and 254 and to read the three remaining digits as hundreds, tens and ones. Then apply millions to LHS three digits, thousands to middle three digits and ones to RHS three digits and read the number.

Obviously, we must ensure that students know the progression of pattern-of-three names:

... trillions \leftarrow billions \leftarrow millions \leftarrow thousands \leftarrow ones

... tera ← giga ← mega ← kilo ← ones

Once we have understanding of hundreds-tens-ones in terms of big ideas, concepts and processes, it is easy to extend this to large numbers by using the patterns of three (see below) and simply stating the hundreds, tens and ones as for three-digit numbers but adding the pattern-of-three name. For example, the number below is five hundred and forty-one trillions, eight hundred and forty billions, six hundred and thirty-four millions, five hundred and sixty-seven thousands, nine hundred and eighty-two (we do not say the ones).



It is important to make sure students understand multiplicative structure – that moving three places to the left is ×1000 and moving three places to the right is \div 1000. This is the basis of most metric conversions. Once students understand thousands, extend the hundreds-tens-ones understanding to large numbers by breaking the numbers and writing the numbers in groups of three numerals. Focus understanding of the multiplicative structure on the pattern of threes by moving three-digit numbers left and right to show what happens as you move left and right across the patterns of three (i.e. moving numbers between patterns of three is left ×1000 and right \div 1000).

Students find comparison and order difficult with large numbers. They need to align place values and look for the left-most place values. A common misconception is to believe that 52 567 is larger than 328 561 because the 5 is bigger than the 3 without checking if they are the same PV. Using a rule of "longer is bigger" is fraught with danger as this does not extend to decimals. Always use understandings that move easily between whole numbers and decimal numbers.

3.6.2 Extending positional understandings

This covers the first three big ideas and their concepts and processes: (a) notion of unit/part-whole, place value, and reading/writing; (b) counting, seriation and odometer; and (c) multiplicative structure and renaming. We have also put equivalence in this section.

Unit, place value, reading/writing. Once the macrostructure and microstructure of the pattern of threes is known, the activities in section 3.6.1 should build good place value and reading and writing of large numbers.

The calculator is an excellent method of reinforcing the large-number work:

- Give number as digits, students enter on calculator, and read out number AND read out number as it should be, students enter on calculator and say digits.
- Play the game "Wipe-out" (much better with more numbers wipe one digit at a time (e.g. 45 786 wipe the 7) or have more than one repeat of a digit and have to wipe both (e.g. 48 786 wipe both 8s). Can have difficulty with zeros (e.g. students will subtract 500 for example 45 786 wipe 5). If this is the case, play the game with digit cards on PVCs so students can see the PVs, or spend a lot of time making numbers like 5 000 on the PVCs with the digit cards so students get used to zeros.

Counting, seriation, odometer. Count tens and hundreds in three-digit numbers and reinforce patterns in counting for three digits. Move the H-T-O to the thousands and repeat – is there any difference? Could use bodies or PVCs, have 368 in the ones pattern of three and then move 368 up to the thousands pattern of three – any difference in counting?

Use calculators to reinforce:

- Enter, say, 34 672 and add 1000 keep pressing the = sign and calling out the thousands position note the odometer pattern remains. Repeat for, say, 459 302 and subtract 10 000, pressing equals and calling out the ten thousands position again note the odometer pattern.
- Practise counting patterns, e.g. 34 568 901, 34 578 901, 34 588 901, 34 598 901, _____, ____, ____,

Multiplicativity, renaming. Reinforce the $\times 10/100$ and $\div 10/100$ movements in three digits (use bodies or PVCs/digit cards) – move the three digits to thousands – add in the ones so six digits – is there any change in what happens? Go both ways: movement to operation AND operation to movement – introduce the $\times 1000$ and $\div 1000$ movements.

Note how important ×1000 and ÷1000 movements and operations are to metric conversions.

Equivalence. Recap finding for three digits that zeros only make a difference when put at the end or in the middle of digits in a whole number. Zeros at the start of a number make no difference. Apply this to larger numbers and see it is the same.

Reality

Discuss with students where you find large numbers – numbers with many digits. Examples of ideas are: How much does a person earn working in the mines? How much is their favourite singer worth? What does a new car or house cost? How much money does the richest person in Australia have? Bring in people to describe



situations which have large numbers or go on visits to places where large numbers are used (e.g. visit the council and find out how much they spend in a year). Find out if students themselves have situations where large numbers are used.

Note that large-number situations may have new names. Megabytes and gigabytes are an instance. What does it mean for your phone to have a memory of 16 gigabytes? How many "megs" on a memory stick? What is a "meg"? Some excellent large-number activities are virtual – one enables student to travel across the solar system as numbers show distance from earth; another shows the history of the earth with the number of years since the earth was formed shown on the screen.

Abstraction

Body. Use the bodies of students. Put one ones, ten ones, hundred ones, one thousands, ten thousands, and so on, as labels or bibs on students. Organise them into place-value order and into threes. Alternatively do this on a 6×10 mat – students can walk the place values as they say them.

Hand. <u>PV materials</u> – use fold-out PV charts, Montessori cards and number expanders to show PVs as a pattern of threes. As can be seen in the number expander, these materials focus on how numbers can be seen in terms of the microstructure. For example, the number expander shows that 462 381 759 806 is 462 billions, 381 millions, 759 thousands and 806 ones.

4	6	2	Billions	3	8	1	Millions	7	5	9	Thousands	8	0	6	Ones
---	---	---	----------	---	---	---	----------	---	---	---	-----------	---	---	---	------

Reverse this process – break up large numbers into threes (e.g. 348 651; 832 526 851) – name each of the groups of three – then name the digits in each group of three as if they were three-digit numbers, for example, 832 526 851:

832	526	
0.52	520	

8H 3T 2 Ones of millions 5H 2T 6 Ones of thousands 8H 5T 1 Ones of ones

851

"Eight hundred and thirty-two million, five hundred and twenty-six thousand, eight hundred and fifty-one"

<u>Growing the numbers</u>. An excellent way to show how H-T-O ones moves to large numbers is to teach one microstructure at a time, that is, teach H-T-O ones and then bring in H-T-O thousands and then H-T-O millions, and so on. One way to do this is to use materials where PV charts for ones, thousands and millions are separate but can be brought together. One such material to do this is the **Pattern-of-threes PVCs** below. There are three PVCs and digit cards – this material should be cut out and laminated. (Note: If bigger numbers are wanted a fourth PVC for billions can easily be made.)

The technique for teaching is to start with the ones PVC, then bring in the thousands PVC when ready and place it on the left, then do the same with the millions PVC. Activities are as follows.

1. Give students numbers to place in the pattern-of-threes PVC using the digit cards – get students to read the numbers, moving their left hands as they read across the digits and the ones, thousands, etc. It is important to **move the hand** to ensure that kinaesthetic activity is present to build the PV system as a mental image.

Repeat this with other numbers, with hand movements, to drive home the macrostructure and microstructure of large numbers.

2. Reverse the above and this time read out the number – ask students to make the number with the digit cards. State numerals and PV positions out of order and see if students can correctly put out the digit cards and then read the numbers (moving their hands).

Mind. Shut eyes and imagine large numbers and PV positions and say and write numbers in the mind.

Pattern-of-threes PVCs

HUNDREDS	TENS	ONES									
MILLIONS											

HUNDREDS	TENS	ONES										
	THOUSANDS											

HUNDREDS	TENS	ONES			
	ONES				

Digit cards

0	1	2	3	4	5	6
7	8	9	0	0	0	0
0	1	2	3	4	5	6
7	8	9	0	0	0	0

Mathematics

Practice/symbols. Continue the reading and writing of numbers using the pattern-of-threes PVCs/digit cards, followed by just reading and writing without PVCs. Use worksheets that relate PVCs, language and numbers. Particular activities are as follows.

- 1. <u>Unit-PV-reading-writing</u>. Once the macrostructure and microstructure of the pattern of threes is known, activities like those in Abstraction should build good skills in place value and reading and writing of large numbers, particularly if saying numbers, writing numbers, recording on calculators, and using materials are integrated. In particular, the calculator is an excellent method of reinforcing the large-number work:
 - give number as digits, students enter on calculator, and read out number AND read out number as it should be, students enter on calculator and say digits;
 - play the game Wipe-out wipe one digit at a time (e.g. 45 786 wipe the 7) or have more than one repeat of a digit and have to wipe both (e.g. 48 786 wipe both 8s).

Note: There can be difficulty with zeros in Wipe-out, e.g. students will subtract 500 from the number 45 786 when asked to wipe 5. If this is the case: (a) play the game with digit cards on PVCs so students can see the PVs; (b) spend a lot of time making numbers like 5 000 on the PVCs with the digit cards so students get used to zeros; and/or (c) relate 5 000 to the 5 in 45 786.

<u>Counting-seriation-odometer</u>. Count by tens and hundreds in three-digit numbers and reinforce patterns in counting for three digits. Move the H-T-O up to the thousands and repeat – is there any difference? Could use bodies (with bibs to make 368) on a mat and then move them from the ones to the thousands. Could use calculators and count in tens for 368 and count in ten thousands for 368 000.

Use calculators to reinforce. Enter, say, 34 672 and add $1\,000 - \text{keep}$ pressing the = sign and calling out the thousands position – note the odometer pattern remains. Repeat for, say, 459 302 and subtract 10 000, pressing equals and calling out the ten thousands position – again note the odometer pattern.

Practise counting patterns, e.g. 34 568 901, 34 578 901, 34 588 901, 34 598 901, 34 608 901, _____, ____, and so on.

3. <u>Multiplicativity-renaming</u>. Reinforce the ×10/100 and ÷10/100 movements in three digits (use bodies or PVCs/digit cards). Move the three digits to thousands, add in ones so six digits and again do ×10/100 and ÷10/100 movements. Is there any change in what happens? Remember to go both ways, that is, teacher says movement, students give operation AND teacher gives operation, students give movement. Introduce ×1000 and ÷1000 movements.

Note how important ×1000 *and* ÷1000 *movements and operations are to metric conversions.*

4. <u>Equivalence</u>. Recap finding for three digits that zeros only make a difference when put at the end or in the middle of digits in a whole number. Zeros at the start of a number make no difference. Apply this to larger numbers and see it is the same.

Connections. Large numbers have become commonplace with money and have always been a part of measurement (e.g. solar system) and time (e.g. age of the earth). The important thing is to make use of such examples at this point.

Reflection

Validation/Application. Organise the students to look at large numbers in their lives (e.g. population figures, odometer on cars, gigabytes, hits on YouTube, and so on) and how these affect the students. The idea is to get them to look at these numbers with new understandings.

Flexibility. Do a poster on large numbers – "where do we find numbers like this (e.g. 27 465 000)".

Reversing. Always ensure activities that relate representations of large numbers (e.g. PVC, language, symbols) are reversed, that is, go symbol to language AND language to symbol.

Generalising. Ask the students if they can see a pattern in how numbers get bigger and bigger (e.g. how do we understand a number that says how far it is to Saturn?).

Changing parameters. Discuss what words are used for very large numbers – thousands, millions, billions trillions – what is next? Encourage students to look up names of larger numbers on the Internet (based on Latin – quadrillions, quintillions, and so on). Use language and position, for example:

 trillions	billions	millions	thousands	ones
 terabytes	gigabytes	megabytes	kilobytes	bytes

Note: This RAMR cycle covers three big ideas and their concepts and processes: (a) notion of unit/part-whole, place value and reading/writing; (b) counting, seriation and odometer; (c) multiplicative structure and renaming; and (d) equivalence. If teachers find that this extension does not work and more teaching has to be done with large numbers to get understandings in these four areas, **Appendix A** at the end of this book has more detailed teaching instructions for teaching thousands and millions.

3.6.3 Extending number-line understandings

This covers rank, order and rounding (and approximation and estimation).

Use pegs and rope, or a drawing of a number line, to reinforce placement of numbers on a 0–1000 line and rules for order (biggest place-value position is the important position – the biggest digits there mean the biggest number).

Extend pegs and rope or drawing of number line to 0 to $1\,000\,000$ – think of this as 0 thousands to 1000 thousands – numbers are placed in the same way. Stress that the rule stays the same.

Play "Chance number" (see section 3.4.3) for four and five digits to reinforce that the largest PV position is the important one. Compare numbers by placing them in PVCs with digit cards – so students can see PV positions for comparison and use digits in larger PVs as means to determine order. **Do not teach that the number with the most digits is bigger – this does not extend to decimals. It is a consequence of the more general PV order rule.**

Reality

Try to find a way that a number line showing large numbers has meaning – timelines and distances may offer opportunities.



Abstraction

Body/Hand. Use pegs and rope, or a drawing of a number line, to reinforce placement of numbers on a 0–1000 line and rules for order (the biggest place-value position is the important position – the biggest digits there mean the biggest number).

Extend pegs and rope or drawing of number line to 0 to 1 000 000. Think of this as 0 thousands to 1000 thousands. In this way the 0 to 1 000 000 number line will be similar to the 0 to 1000 number line – the numbers are placed in the same way. Extend to 0 to 1 000 000 000 number line – this time, the number line can be thought of as a 0 to 1000 million number line – structurally the same as a 0 to 1 000 000 or a 0 to 1000 number line.

Get students to make the three number lines, on the floor with tape and numbers, or on paper: one for 0 to 1000, one for 0 to 1000000, and one for 0 to 10000000. Drawings of these lines below show their similarity, which seems easier to see if using language. Think of two numbers on the line – which is bigger?

Think of the rule for 0 to 1000 – the number with the more hundreds. So what is the rule for 0 to 1 000 000 000 – the number with the more hundred millions.



Mind. Shut eyes and imagine a line, say from 0 to a billion or 1000 million – think of this as similar to the 0 to 1000 line – now place numbers like 435 678 121 on it.

Mathematics

Practice/symbols. Aligning – compare large numbers by putting both on the same PVC and seeing which has the larger digit in the highest PV position, as below:

			НТО	ΗΤΟ		
A is 345 678			thousand	s ones		largest PV is 100 thousands
B is 96 723	A is 345 678 B is 96 723 → aligr		345	678	\rightarrow	A has a 3 in it, B has a 0
			96	723		A is the larger number

Worksheets – provide pairs of numbers and PVCs to put them in – align PV positions and determine larger/smaller.

Connections. The main connections here are to measurement (e.g. large distances, volumes, masses and so on) and to data and probability (e.g. hits on websites, Gross National Products, census data, chances of winning Lotto).

Reflection

Validation/Application. Check students can discuss large number lines in their world.

Flexibility/Reversing. Be flexible with numbers and reverse relationships (line and placement \rightarrow guess number, and number and line \rightarrow placement).

Generalising/Changing parameters. Look at PV as a number line as follows.

	 н	т	о	н	т	0	н	Т	0	Н	т	о
<u> </u>	E	BILLION	S	N	AILLION	S	THOUSANDS			ONES		

Where do we go from here? Does this help below?

Н	т	о	н	т	о	н	т	0	н	т	ο
 G	GIGABYTES		M	EGABYT	ËS	KILOBYTES			BYTES		

Motivating activity – both the activities below can involve a class in very large numbers as they explore possibilities.

- 1. <u>Chessboard and rice</u>. Look up the famous puzzle about wanting one grain of rice for the first square of a chessboard, two grains for the second, four for the third, doubling all the time until you get to the last square. How many grains in the last square? How many on the board in total? How much does this weigh? If you had 100 people each putting a grain on the board every second, how long would it take to finish all the squares?
- 2. <u>Tower of Hanoi</u> (look it up on Internet) how many moves for tower of height 2, tower of height 3? At a move per second, how long to move a 50 tower?

Note: There is a lot in the next section that can help extension from three to six and nine digits. However, YDM believes that acceleration through extension works for extending whole numbers from three digits to nine digits, and it should not be necessary to reteach all of sections 3.1 to 3.5. Nevertheless, if students are having difficulties, Appendix A has detailed number-line teaching information.

3.7 Directed number

There is a **mathematics joke**: A mathematician is a person who knows that if they send three people into an empty room and five walk out, they have to send in two more people to make it empty again. This is funny because we know that this is true for arithmetic (i.e. 0 + 3 - 5 + 2 = 0) but it makes no sense. However, if we change the story to temperature: A mathematician is a person who knows that if they raise the temperature in a room that is zero degrees Celsius by three degrees and then lower it by five degrees, they have to raise the temperature by two degrees to make the room zero again. Now the story makes sense. The moral of this is that, although theoretically there are negative numbers, the reality they come from is measures that go below zero where zero is not nothing but the point between positive and negative.

Thus to introduce the idea of directed numbers, we have to use measures, such as temperature (above and below zero), height (above and below sea level), buildings (floors above and below ground level) and money, e.g. assets and debts (liabilities). We have to get across the idea that some numbers **can be both positive and negative** – they can be something that is less than zero as well as more than zero.

3.7.1 Models for directed number

Directed numbers are numbers that are used to represent situations that involve both quantity and direction. The directions are opposites and are described as being either positive or negative. Directed numbers allow the whole-number system to be extended beyond zero in a negative direction. The concept of negative numbers makes most sense in relation to money and measures. A person can spend more money than they earn and the result will be a negative quantity or an amount of money will need to be earned to get the person back to zero. Temperatures can fall to below zero. Directed whole numbers are also referred to as integers.

When introducing directed numbers to students it is helpful to refer to them in terms of directed numbers being opposite to the whole numbers they know. It is important to make the distinction between negative numbers and the operation of subtraction. For example the number that is opposite 7 is negative 7 and is written as -7. A number line is an effective tool for demonstrating how these numbers are opposites. The numbers are the same distance or are the same number of jumps in each direction from zero.





Often a vertical number line is particularly effective for modelling directed numbers as it matches contexts where negative numbers occur in everyday life, e.g. thermometers, ground above and below sea level, and even the levels of a building above and below ground level.

3.7.2 Teaching directed numbers

We introduce negative numbers through two measures – money and temperature – in the following RAMR activity.

Reality

Discuss with students where do we have negatives? Try to find a local instance (e.g. diving – above and below water level). Also discuss local possibilities – students can discuss what happens when we spend more than we have in our credit card? Or if we borrow more than we can pay back?

Abstraction

Body. Get students to experience a local instance of negatives. If lacking local instances, set up experiences. Students can experience negative temperatures in a freezer. There are also negatives in buildings with basements – going down below ground level.

Hand. Start with money – look at a film star or someone famous – look at assets and debts – what they have and what they owe. An example is below:

The Courier Mail reported on 27 April 2009 the following about the net worth of Kylie Minogue: "Kylie Minogue is singing the recession blues, having lost a sizeable chunk of her wealth in the global financial crisis. The 2009 rich list of musos compiled by *The Sunday Times* revealed that the pop princess has seen her net worth drop by \$12.25 million from \$83.5 million. She also recently sold property in Melbourne worth \$1 million."

What is her net worth now? What does it mean to have this net worth?

What sort of assets could Kylie have to have this net worth? Make a list of possible assets. Kylie will also have debts. What kind of debts? Make a list of possible debts.

Is it possible for someone to have more debts than assets? Look at this example of Ron: Assets – car \$23,000, bank balance \$2,000. Debts – car loan \$15,000, owes uncle \$5,000, credit card debt \$13,000. What would be his net worth in thousands of dollars? What does this mean?

Move onto another measure – say a lift in a large building 28 floors up and 6 basements down. Start on ground level – go up 8 floors, down 5 floors, up 2 floors and down 7 floors. Where are we? What could we call that floor?

Develop a vertical line with positive and negative numbers on it as on right. This can be a large one on the floor with tape, a reasonably sized one on the board or small ones for each student. **The Maths Mat is an excellent material for this**.

Use this vertical number line to act out instances of negative: 0 can mean all even with assets and debts, the ground floor of a building, 0 degrees centigrade in temperature, present day in a time machine, and so on. Act out stories and end up with positive and negative numbers.

Mind. Picture a vertical line in the mind, shut eyes and act out stories in the mind.

Mathematics

Practice/symbols and language. Practise situations which end up in negative numbers – relate symbols to situation; e.g. you take \$500 on credit card to a sale, spend \$700 – what is your situation on the card? [-\$200].

6

5

4 3

2

0

-1

-2

-3

-4

-5

-6 -7

Abstraction

Maths

Reality

Reflection

Connections. Ensure negative numbers are related to appropriate measures and any other topic in mathematics. Is it possible to have negative fractions? Is it possible to have negative area (what about the area needed for a display and the area available – display could be bigger so not enough area so negative area)?

Reflection

Validation/Application. Apply understanding of negative numbers back into students' lives.

Flexibility. Do a poster on negative numbers and where they are in the world (Note: some sports have negative numbers – fullbacks in gridiron have rushing yards and on a bad day this is negative).

Reversing. Very important – provide students with a negative number and a context and they have to make up the story. This is reversing what we have so far done.

Generalising. This is students understanding when a vertical number line goes from 0 upwards or in both directions with negative numbers.

Changing parameters. Is it possible to need numbers to go horizontally positive and negative as well as vertically?

3.8 The whole number system

Most countries today use the decimal number system, called "decimal" in English because it has a base of 10 and 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). The word, *decimal*, comes from the Latin word, *decem*, which means 10. The whole numbers in the decimal number system do not stop but go on to infinity. Two of the big ideas, multiplicative structure and the pattern of threes, form the basis of being able to take this learning further to understand the whole number system.

Once students have been introduced to the concept of exponents, usually by exploring square numbers and using the index notation to represent squares they can see the connection between the exponents of 10 and the structure of the whole number system. A table like in the RAMR activity to follow will help show the connection between the place values, the value, the multiplicative structure and the exponential notation.

When the pattern-of-threes structure is included, numbers can be considered as just powers of 10 or a patternof-threes powers of 10. Thus the system can be considered as:

Н	Т	0	Н	Т	0	Н	Т	0	Н	Т	0
	Billions		1	Villions	S	Th	nousan	ds		Ones	
1011	1010	10 ⁹	10 ⁸	107	10 ⁶	10 ⁵	104	10 ³	10 ²	10 ¹	10 ⁰

OR

Н	Т	0	Н	Т	0	Н	Т	0	Н	Т	0
Billions Millions					Thousands				Ones		
10 ⁹			10 ⁶			10 ³			10 ⁰		
1011	1010	10 ⁹	10 ⁸	107	10 ⁶	10 ⁵	10 ⁴	10 ³	10 ²	10 ¹	10 ⁰

Reality

Reality Abstraction Reflection Maths

Get students to look at PVCs to millions – use pattern-of-threes PVC – can students see the pattern that enables this to go on forever? What would be the next pattern of three to add? The one after that? How could we think of it so it goes on forever?
Abstraction

Body. Set up pattern of threes on the mat. Students walk – H-T-O ones, then H-T-O thousands, then H-T-O millions – where do we walk to next?

Hand. Introduce powers for 10 – list the PV positions – put beside them how many 10s have to be multiplied – introduce the indice as a symbol to show number of 10s:

Tens	10	10	10 ¹
Hundred	100	10×10	10 ²
Thousand	1000	10×10×10	10 ³
Ten thousand	10000	10×10×10×10	10 ⁴
Hundred thousand	100000	10×10×10×10×10	10 ⁵
Million	1000000	10×10×10×10×10×10	10 ⁶
and so on			

Discuss what one would be – use patterns to show that it is one indice less that 10^1 so it is 10^0 .

Using the mat, make up a PVC that is just the PV positions and a second PVC that is pattern of threes – replace names with indices, what have we got? **Can we see the pattern?**

HB	TB	В	HM	ΤM	М	HTH	TTH	TH	Н	Т	0
1011	10 ¹⁰	10 ⁹	10 ⁸	10 ⁷	10 ⁶	10 ⁵	10 ⁴	10 ³	10 ²	10 ¹	10 ⁰

Н	Т	0	Н	Т	0	Н	Т	0	Н	Т	0		
E	BILLION	S	N	1ILLION	S	TH	OUSAN	DS	ONES				
	10 ⁹ 10 ⁶						10 ³		10 ⁰				

Mind. Think of place values in the mind going up by a factor of $\times 10$ – think of the indices increasing.

Mathematics

Practice/symbols and language. Get students to describe and draw the system and show how it continually gets bigger – get them to look up the names for bigger and bigger numbers.

Connections. Relate this system to metrics, for example:

Н	Т	0	Н	Т	0	Н	Т	0	Н	Т	0		
1	.000 km	IS		km			m		mm				
	10 ⁹ mm			10 ⁶ mm			10³mm		10 ⁰ mm				

Reflection

Validation/Application. Look in the world for whole-number systems that students use, for example:

Н	Т	0	Н	Т	0	Н	Т	0	Н	Т	0
GI	GIGABYTES MEGABYTES				KI	LOBYT	ES		BYTES		

Flexibility. Do a poster showing many examples of systems of numbers – on kilos, megas, gigas and so on.

Reversing. Give a number, say 64 gigabytes, and students have to find where to put this on a diagram of the system. Then reverse, give a position and students have to make up a situation. That is:

number situation \rightarrow system AND system \rightarrow number situation

Generalising. Can the students see the two forms of the system in their mind? Can they make up their own system? What happens when we move two places in multiplicative structure? We are at 10^6 , we multiply by 100, we are now at 10^8 . Can students generalise – 2 places is 10^2 , 3 places is 10^3 ?

Changing parameters. What if we changed base (e.g. seconds, minutes, hours)? What would the Mayan base 20 system look like? What would the Babylonian base 60 system look like?

4 Decimal Number Numeration

The sequence for decimal numbers includes work on (a) tenths and hundredths, (b) thousandths, and (c) number systems as shown on right. Decimal numbers are developed by combining common fraction knowledge and whole number knowledge. Thus, this chapter will include prerequisites (whole numbers and fractions) as well as focusing on sequencing, connections and big ideas. There will not be the same focus on providing complete RAMR activities – the chapter will focus on the most important parts of the cycle for the given topics.

Decimal numeration for tenths, hundredths and thousandths will use the same sequence as whole number numeration. However, the detail is not as necessary because decimal numbers follow on from whole numbers. The following sections will be covered:

- Connections to prerequisites
- Part-whole and place value to thousandths
- Additive and multiplicative structures to thousandths
- Continuous vs discrete and number line to thousandths
- Equivalence and extension to thousandths and millionths
- Directed numbers and the real number system



Work with decimals is a pivotal component of number. It leads on from whole numbers and common fractions; leads to measures (metrics), percent and rate; and covers both rational and irrational numbers.

4.1 Connections to prerequisites

This section will begin by looking briefly at the notion of unit, the connections to whole numbers and common fractions, renaming and reunitising, and describing a direct extension of whole numbers to decimal numbers.

4.1.1 Notion of unit

In whole numbers, the starting unit was grouped into powers of 10. In this section, the unit is both grouped and partitioned into powers of ten.



Reality

Look at things, in the world of the students, that are in boxes – e.g. one box set of DVDs (one season, 10 DVDs, 10 episodes per DVD), one DVD, one episode on DVD – discuss each position as unit. For example, if DVD is the unit, then box set is 10 DVDs (more than one) and episode is one part in 10 of DVD (less than one).

Abstraction

Body. A group of 10 people – one person – part of a group. Another 10 people – two groups or 20 people. Another four people – two groups and four parts or 24 parts. Two cakes cut into five parts – each could be two groups or 10 parts. It could also be one part depending on the guidelines of the "group".

Hand. Use counters to partition groups, make second groups, etc. -e.g. 24 counters could become three groups and each group could be four smaller groups - so what is the unit? And for each unit, what is one counter? Look

at eight counters – one counter could be 1 if the 8 counters is 8 and $\frac{1}{8}$ if the 8 counters are one group. What if the 8 counters are 100%, 400, $\frac{1}{2}$, and so on – what is one counter?

Mind. Picture 16 people – how many groups of 10 if there are 16 people? 28 people? 46 people? How many people if there is one group and eight people altogether? Three groups and four people altogether? Two groups and 16 people altogether?

4.1.2 Connection to whole numbers

It is important that students see whole numbers and decimals as the **same** in as many ways as possible. When students can see the connections, learning about decimals is much smoother. The following list of similarities and differences between whole numbers and decimal numbers is provided to assist teachers to see these connections as well. Each topic which was talked about in the Whole Number chapter is then discussed here in relation to decimals.

The following list of **similarities** summarises some of the big ideas as they relate to this chapter. Each point is discussed further in related sections of the chapter.

- PV positions for whole numbers and decimals are determined from the ones in whole numbers, the ones is the right-most position in decimal numbers, the ones is the place with a dot after it.
- In whole numbers, the PV positions move left from the ones following tens, hundreds, thousands, and so on. In decimal numbers, the PV positions move left and right symmetrically about the ones position as below:

hundreds tens **ones** tenths hundredths

- Larger numbers have larger digits in the largest PV positions it is only because of the placement of the ones position in whole numbers that more digits means larger numbers. This latter idea does not translate but the former idea is the same for decimal and whole numbers. Students need to understand that they need to place numbers to be compared into PV positions and then compare.
- Multiplying and dividing by 10 moves PV positions to the left and to the right not to be remembered in terms of adding or removing zeros or movement of decimal point as these are only consequences of the PV movement.

At the same time it is important to remember **differences**; decimal numbers have the following which whole numbers do not have:

- PV positions which are fractions rather than whole numbers;
- a different convention for determining the ones position look for the decimal point, not the rightmost digit;
- high density (no whole numbers between 24 and 25, while infinite decimal numbers between 24 and 25, 24.2 and 24.3, 24.37 and 24.38, and so on).

4.1.3 Connection to common fractions

Tenths as part of a whole (area model)

The steps for introducing part of a whole are: (a) identify the whole; (b) partition into equal parts (i.e. 10 equal parts); (c) name the parts – tenths (may have to acknowledge other fractions here but we are only working with tenths); (d) consider a certain number of parts (e.g. five parts); and (e) name the fraction (five tenths). RAMR activities are as follows.

Reality

Discuss what it is that makes a whole. Can we use the same criteria every time we are looking at breaking something into parts? It is important that it is understood that a whole is not necessarily ONE, for example:



On Monday I bring in one cake to share with 10 students – one cake is the whole that will be shared.

On Tuesday I bring in two cakes to share with 10 students – two cakes is the whole that will be shared.

On Wednesday I bring in half a cake to share with 10 students – half a cake is the whole that will be shared.

On Thursday I bring in one cake that is the size of the table to share with 10 students – one large cake is the whole.

Each day/situation is different but at each point we are identifying the whole as what is available to be shared.

Abstraction

Body. This can be done using the strip mat. Use a square card to cover the "parts" to indicate 1 tenth. Three parts covered with square card is 3 tenths. We are only using the language here without introducing the symbols.

Hand. Different shapes and quantities that are/can be broken into 10 equal pieces. Cover the appropriate pieces for different fractions, e.g. 3 tenths, 8 tenths.

Mind. I am running a race and I run 2 tenths of the race – am I closer to the start or the finish? etc. I cut a cake into 10 pieces and kept 4 tenths and I gave the rest to my friend. Who got more cake?

Mathematics

Introduce the symbols for one tenth $-\frac{1}{10}$ means one part out of 10; $\frac{5}{10}$ means five parts out of 10.

Shade these fractions – 3 tenths, 9 tenths, 14 tenths etc. on shapes – rectangles, squares, L shapes etc.

Shade 10 tenths – what is $\frac{10}{10}$, 20 tenths?

We are not teaching fractions as such here, just the idea that breaking a whole into 10 equal parts gives tenths. Two parts is two tenths, 20 tenths is two wholes etc. We are relying here that there is some basic knowledge of fractions to be able to label this as tenths. If not, you need to spend a little bit of time building up one half, one quarter, one twentieth etc., just a couple of fractions to get the idea.

Reflection

Reverse working from the whole-to-part and work part-to-whole (identifying the whole from the part).

Identify fraction (e.g. $^{2}/_{10}$); Ask to make whole.

Find one part (e.g. construct 1_{10}); Observe construction (e.g. to find 1_{10} when given 2_{10} means halving what has been identified as 2_{10}).

Find whole by putting together the necessary copies of the one part (e.g. make 10 copies to give 10 tenths or make 8 tenths to put together with 2 tenths).

Construct more than one differently shaped whole (e.g. L shape, P shape, steps etc.).

Hundredths as part of a whole (area model)

The steps for introducing hundredths as part of a whole are: (a) identify the whole; (b) partition into equal parts (i.e. 100 equal parts); (c) name the parts – hundredths; (d) consider a certain number of parts (e.g. five parts); and (e) name the fraction (five hundredths). RAMR activities are as follows.



Reality

Identify situations where a part or a whole is separated into further parts, for example:

One school separated into three parts (grades), these parts are separated into further parts (classes) – there are now three parts separated into three parts again – nine parts altogether.

I cut the cake into 10 pieces for my students. Each piece is one part – one tenth. Each of the 10 people must share their piece of cake with their brother – each part is separated into two parts – 10 parts each separated into two more parts – now 20 parts altogether.

Abstraction

Body. Again using the strip mat – take one of the one-tenth squares and break it into 10 pieces. "How many of these will fit into one square?" [10]. "How many do we need to cover each of the 10 squares?" [100]. "If we need 100 of these to cover the mat, what do you think we call this strip?" [One hundredth]. Using the hundredth strips, put out 7. "How much of the mat is covered?" [Seven hundredths]. Put out 18 on the mat covering 18 hundredths. "How much of the mat is covered?" [Eighteen hundredths].

Hand and Mind. Have students make the tenths and hundredths step by step, recognising the significance of each cut as they make it and labelling each part as they go. Begin with a whole – they can construct a square (easy if it is a multiple of 10 cm, e.g. 20×20 cm) – label this "one whole". Accurately cut this shape into 10 equal-sized pieces (strips) – label each of these "one tenth". Accurately cut these tenths into 10 equal pieces each (small squares) – label each of these "one hundredth".

This can be done with different shapes, not just squares. These pieces can be kept in envelopes for later activities whenever students need to use concrete materials.

Mathematics

Introduce the symbols for one hundredth $-\frac{1}{100}$ means one part out of 100; $\frac{5}{100}$ means five parts out of 100.

Use a hundreds grid to shade 3 tenths, shade 18 hundredths, shade 1 whole, 8 tenths and 4 hundredths etc.

Reflection

Reverse the activity of making one hundredth from a whole and give the students the "one hundredth" and they must construct the whole by constructing the tenth, labelling it and then constructing the whole. (One small box – they need 10 lots of 10 boxes; half a circle – they need 100 half circles.)

4.1.4 Renaming/reunitising

To be represented by a decimal, 43 hundredths has to be 4 tenths and 3 hundredths. This requires seeing 40 hundredths as the same as 4 tenths – this means seeing one whole as a hundred parts because one whole has 10 lots of 10 parts – this is called reunitising. RAMR activities are as follows.



Reality

Think of situations where you may need to reunitise. In the example of the school and classes above there may be situations where you need to walk about the whole school or the three grades or the nine classes.

A cake that is pre-cut in 10 pieces – the cake decorator will want to know that there are two cakes to decorate. The person serving the cake will want to know that it is 20 pieces.

Abstraction

Body. On the Maths Mat (6×10), using the elastics, show one whole as one row of 10. Surrounding each of the parts with elastic, count the squares to determine:

One whole row = ? tenths, Two whole rows could also be ? tenths.

One row and 8 tenths = ? tenths, Two rows and 3 tenths = ? tenths, Five rows and 8 tenths = ? tenths.

8 tenths = ? hundredths, 4 tenths and 2 hundredths = ? hundredths

One row, 2 tenths and 4 hundredths = ? tenths and 4 hundredths = ? hundredths, etc.

Hand. Using MAB or hundreds charts, begin with the initial number – 8 wholes and 2 tenths and 3 hundredths – how many tenths? How many hundredths?

Mind. Students to picture the body and hand activities in their heads – whether they pick the mat or the shading they answer the same type of questions but they only do it in their minds.

Mathematics

Activities that involve the formal recognition that there are 10 tenths in one whole and 100 hundredths in one whole and that allow students the flexibility to convert between the three units.

Students need to practise with multiple wholes, e.g. 4 wholes 3 tenths, to learn that the same rules apply for all numbers, not just between 0 and 1.

Reflection

Give two other ways to express 4.56. etc.

4.1.5 Whole numbers to decimal numbers

Use the multiplicative structure big idea that was developed during the Whole Number chapter to continue into the multiplicative structure of decimals. RAMR activities are as follows.

Reality Abstraction Reflection Maths

Abstraction

Body. Use number bibs or cards with students standing in front of labelled place values, to experience this concept.

Start with a number, e.g. 23 (two students in front of tens label and three students in front of ones label). At this point, only have the ones, tens, hundreds, thousands etc. labels displayed.

 \times 10, place values change to reflect this change (record the changing number at each multiple) – first change, tens students move to the left (their right if facing class) to the hundreds and ones students move to the tens etc.

Repeat three more times – ask students for ×10 rule (×10 means digits move left – **not** add a zero).

Now work backwards (e.g. starting with 23 000), ÷10 on calculator 4 times, changing PVC numbers and recording – again ask students for ÷10 rule (÷10 means digits move right – **not** remove a zero).

Then ask question, "What would happen if we \div 10 again?" Discuss possibilities, directing the discussion toward the idea that the ones digit will now be 10 times smaller therefore shift to the right. This position is called **tenths** – draw on the earlier experience of dividing a whole into 10 equal parts being one tenth.

We need a way to record the ones position – how do we record 2 ones and 3 tenths – it can't be 23 as this means something different – encourage students to understand that they need a new way of showing the ones. Let students use their own new way (e.g. hat on top of ones) to record the ones position.

Then ask question, "What would happen if we ÷10 again?" Discuss possibilities, directing the discussion toward the idea that the tenths digit will now be 10 times smaller therefore shift to the right. This position is called **hundredths** – draw on the earlier experience of physically dividing a tenth into 10 equal parts being one hundredth.

Hand. Students start with calculators, PVC/digit cards or multiplicative sliders on PVCs and pen/paper and can follow the same process as they did with Body but here they are working the calculator and recording all themselves. Students can use the concrete materials they had developed earlier to show that each next PV was a multiple of 10 of the previous one. Each time the "unit" was cut into 10, there were 10 times more units of the new PV.

Mind. Picture the number 54 in your head. If you multiply this number by 10, what number do you have? If you divide it by 10 and then 10 again, what number do you have? etc.

Mathematics

Spend a brief time with recording method chosen by students – then introduce decimal point. Stress that the point has no position – it only exists to show the ones. Always write decimal numbers with no position for point (i.e. 2.3 4 not 2 . 3 4 or 2.34 – this is difficult to show with typed numbers). It is useful to always circle the ones and the point to drive home that it is the ones about which decimals pivot (not the point), for example:



Discuss continually ÷10 and what this means for new decimal PV positions – stress symmetry around the ones, for example:



Ones

Ones

Tens Or

Ones tenths

Hundreds Thousands hundredths thousandths

Н	т	0	н	т	0	н	т	0	tenths	thousand ths	
Millions Thousands					s		Ones			Parts of Or	ne

Reflection

The reality of this concept could not necessarily be done at the start of this cycle as it would have revealed part of the structure that students were to build (around the decimal point).

Look at examples in a newspaper or on TV etc. where decimal numbers are used. What are they used for? What do they mean in each example?

Is it reasonable that any measure could be expressed with a decimal? A metre, the number of laps of the oval, the number of people? Often a whole number is sufficient – especially when counting.

4.2 Part-whole/Place value to thousandths

In this subsection, we look at the notion of a unit again, along with place value, but in terms of decimals. Then, we look at early place-value activities showing size and extend this to reading and writing.

4.2.1 Notion of unit and place value

The notion of unit is central to understanding numbers. A unit is grouped to make larger numbers and is partitioned to make smaller numbers. In our number system the groups for making larger numbers are based on multiples of ten and the partitions to make smaller numbers are into multiples of tenths.





The structure of whole numbers extends to decimal numbers in two ways: (a) by **adding decimal-fraction place values** (e.g. tenths, hundredths, thousandths, and so on) to the right of the ones; and (b) by **changing the convention for identifying the ones position** from the right-most digit to the digit with a dot on its right.

For whole and decimal numbers, it is particularly important to flexibly change unit while moving through place values and parts and wholes. For example, for 428.6 to be read and understood as 4 hundreds, 2 tens, 8 ones and 6 tenths, readers' perceptions of unit have to change from hundreds to tens, ones and tenths as their focus moves across the place values. This is because the 4 hundred does not refer to 400 single units which are ones, but to 4 units that are hundreds. Similarly, for mixed numbers, $3^{4}/_{7}$ has to be seen in terms of ones and sevenths.

The notion of unit is particularly important in decimals as the decimal point signifies the unit. However, it is possible for the ones position not to be the unit, for example, 243.6 is really considered to be 243.6 ones – it is

also 24.36 tens, 2.436 hundreds, 0.2436 thousands, 2436.0 tenths, and 24360.0 hundredths – it all depends on what is the unit. This flexibility is very important for metric conversions and percent – it is a part of renaming.

Recapping, the whole number place-value system extends to decimals by including fraction place-value positions to the right of the ones. To identify the position of the ones place, a decimal point is included after the ones and before the decimal component of the number. The decimal point is simply a marker to identify the ones so the place values can be determined for reading and writing numbers. The decimal point is similar to a full-stop used in writing text. The full-stop indicates that the reader has come to the end of an idea or the end of a sentence. A decimal point does a similar job for the reader of numbers. It indicates where the end of the whole part of the number is so methods using the pattern of threes to read and write numbers can be used. This can be extended to include decimals as these are the digits to the right of the decimal point.

4.2.2 Early activities showing size

This activity combines fractions with place value to introduce the first two decimal place-value positions.

Tenths

Cut bundling sticks into ten equal pieces. Then have three types of materials – single sticks, bundles of 10 sticks, and sticks cut into 10 equal pieces.

Place material on a tens-ones-tenths PVC. Have students also writing numbers on small versions of the PVC, and placing numbers on calculators, and writing with pen/paper.

Relate stories \leftrightarrow materials \leftrightarrow pictures \leftrightarrow language \leftrightarrow symbols for tenths. Go both ways: stories and materials \rightarrow symbols AND symbols \rightarrow stories and materials.

Hundredths

Obtain 10×10 grids, ones-tenths-hundredths PVC, small pictures of PVC, Unifix, calculators and pen/paper.

Relate stories \leftrightarrow materials \leftrightarrow pictures \leftrightarrow language \leftarrow symbols for tenths and hundredths – recap 50 hundredths = 5 tenths and 34 hundredths is 3 tenths and 4 hundredths. Go both ways: stories and materials \rightarrow symbols AND symbols \rightarrow stories and materials.

An effective way is to use 10×10 grids to direct students to shade 3 wholes, 4 rows and 6 little squares – transfer this to a O-t-h PVC using coloured counters for ones, tenths and hundredths – translate this to a small PVC using digits, and write as a decimal (or put on a calculator).

Then reverse this process by giving the decimal and asking for small PVC, large PVC, and shading to be completed. Move forwards and backwards like this. Emphasise the change from shading 40 squares to seeing this as 4 in the tenths place and vice versa.

4.2.3 Extension to reading and writing

In this activity, it is necessary to:

• Use digit cards on PVCs – focus on the position.



- Give students place-value cards and get to line up in correct place-value positions. Have other students stand in front of them with digits to make numbers. Go both ways: numbers → students in position AND students in position → numbers.
- Extend the ideas from earlier to include thousandths by extending tenths/hundredths by seeing the pattern that relates PVC to language and symbol go both ways.
- Relate back to whole numbers.

You could also relate tenths, hundredths and thousandths to Hundreds, Tens, Ones of thousandths to continue the pattern of three as below in the example RAMR activity.

Place-Value Chart

н	т	0	Н	т	0	Н	т	0	tenths	hundredths	thousandths
Millions Thousands				S		Ones			Parts of Or	ne	

Abstraction

Body. Use the "body" place cards for students to "be" the digit in different place values. Have the class read out the name of the number they see. *Remember that this exercise is more for the students being the numbers and locating their positions etc. than it is for the remainder of the class, so change the students so they all have a turn in both roles.

Hand. Place PV cards on mat and use this as a basis to discuss what is happening. Practise reading and writing decimal numbers to thousandths – play games such as "Wipe-out". Use digit cards to represent numbers on PVCs to help students see how many zeros in wiping 6 from 1.268.

Mind. *Play "Wipe-out" without the calculator:*

- Say the numbers "one and forty-nine hundredths" and students must picture the number in their heads.
- Say a number that has 8 hundredths, 9 tenths and 4 ones and students must only write down the number (4.98).

As in the RAMR activity for section 4.1.5, discuss continually \div 10 and what this means for new decimal placevalue positions, again stressing the symmetry around the ones.

Mathematics

Allow students opportunities to read and write the numbers as well as write the numbers for the word descriptions. The place values do not need to be given in order. In fact they should be given out of order to really determine if students understand place value.

Reflection

The Activity Sheet, "Colour in Decimats" (from Australian Primary Maths Classroom magazine, 2010) could be used here to extend the reading and writing of decimals as it changes the unit and parameters that you are working with throughout the activity. This activity can be modified as a game with different outcomes.

4.3 Additive and multiplicative structures

Here we look at the counting, seriation, odometer, and renaming that goes with these two big ideas.

4.3.1 Additive structure

The additive structure of place value relates to the ability to count in each place value. This ability extends to decimal place values. We can count on and back in tenths, hundredths and thousandths etc. places and each count is in units of the place value. It is possible to determine one more or less in each place value for decimals in the same way it was determined for whole numbers (e.g. one tenth more than 5.3 or one hundredth less than 4.19, and so on). Each place-value position counts like the ones. For example if you are counting on in

tenths each count is a tenth, that is, 0.1, 0.2, 0.3. 0.4... You have counted 4 tenths and arrived at 0.4. This is closely related to counting and odometer and is also the basis for patterning in decimals as it was with whole numbers.

Students need to develop an understanding of the odometer principle in whole-part situations. It includes whole and decimal numbers, where the grouping is 10, and common fractions and mixed numbers, where the grouping may be any whole number. It is particularly important for whole and decimal numbers where each PV counts forward 7, 8, 9 and then goes to 0 with the PV on the left increasing by 1; and each PV counts backwards 2, 1, 0 and then goes to 9 with the PV on the left decreasing by 1. It should also be noted that odometer, along with place value, rank, and multiplicative structure, is one of the four meanings of whole and decimal number numeration.

Seriation is the ability to determine one more or less in each place value (e.g. one tenth more than 4.2, one hundredth less than 5.56, and so on). This ability is related to counting and odometer as 10 more is one count forward in the tens place and 10 less is one count backward in the tens position. It also is the basis for patterning (e.g. What are the next three numbers? 2.76, 2.86, 2.96, ___, ___, is solved by one tenth more, one tenth more, and so on, using the odometer pattern when 9 is reached).

To **teach**, recap role of counting and seriation in whole-number PV positions – ensure students can work over the bridging numbers, i.e. if counting in tenths 2.8, 2.9 to 3.0, NOT 2.91 or 2.10 etc.

Use a variety of activities to get students to count in decimal place-value positions:

- Place digit cards on PVC or use an abacus. Pick a PV position and change numbers in this position as count. Discuss what happens after get to 9 in a PV position.
- Repeat counting back discuss what happens when get to 0 in a PV position?
- Use calculators to count in decimal PV positions (e.g. enter 3.258, + .01 and keep pressing = to count forwards in hundredths position subtract to count backward).
- Extend counting to show examples of odometer pattern for counting on and counting back ask students to identify pattern.
- Look at examples of seriation looking at making one decimal PV positions larger and smaller.
- Practise counting patterns e.g. 4.7, 4.8, 4.9, ____, ___; 4.66, 4.68, ____, ___.
- Discuss how odometer is a recurring pattern that affects whole numbers, decimals and metrics (base 10) and time, angle, common fractions for other bases (e.g. 2⁴/₆, 2⁵/₆, 3⁰/₆, 3¹/₆, and so on).

4.3.2 Multiplicative structure

As with whole numbers multiplicative structure is both a principle and a concept that relates to decimals. The decimal place-value positions relate to each other multiplicatively, that is, one position to the left is 10× larger (multiplication by 10) and one place to the right is 10× smaller (division by 10) just as they do for whole numbers. It is important that students understand that it is the change in numbers in relation to the place-value positions. These changes to numbers should not be remembered as the adding or removal of zeros. This understanding is important when they work with decimals and when converting measures, for example, converting cm to m and vice versa. The addition or removal of zeros are consequences of the place-value movement, not the focus. The figure below shows the multiplicative structure of the place-value system. Moving the number 3 two places to the left changes the number from 0.03 to 0.3 then to 3. The number gets larger as places are moved to the left. Moving 3 to the right changes it from 3 to 0.3 and then to 0.03. The number gets smaller as the places are moved right.

Ones	Tenths	Hundredths	Ones	Tenths	Hundredths
0.	0	3	3.		
0.	3		0.	3	
3.			0.	0	3
K	\sim \sim	/			
×1	L0 ×	10	-	-10	÷10

Multiplicative structure of the place-value system

Students need to develop flexibility to recognise that numbers can be renamed in relation to their place values for decimals as they do with whole numbers. There are two "types" of renaming – simple ones that rename all of a PV position, for example, 3.62 is 3 ones, 6 tenths and 2 hundredths or 36 tenths and 2 hundredths; and more complex ones which rename only part of each place value, for example, 3.62 is 2 ones 14 tenths 22 hundredths. Students who understand this flexibility of numbers based on sound knowledge of the structure of numbers related to their place value will be able to utilise this knowledge in computations where they separate numbers flexibly to make computations more manageable.

The PV positions form a symmetry around the ones place – tens on one side and tenths on the other; hundreds on one side and hundredths on the other, and so on.

To **teach**, use techniques as follows:

- Use students as place-value positions and have another student moving up and down in front of them

 discuss what operation changes numbers as students move? Reverse this ask what ×10 does for
 the position of the number? Do the same for division.
- Place digits on a PVC going from thousands to thousandths, place on calculator and record, ×10 and ÷10 and note how digits move. Do this as doing body work above. Go both ways:

$\texttt{Operation} \rightarrow \texttt{move} ~ \texttt{AND} ~ \texttt{move} \rightarrow \texttt{operation}$

- Repeat this for ×100, ÷100, ×1000, and ÷1000. Ask students to state rules for all these activities.
- Record all the changes (e.g. 2.54 × 10 is 25.4) and discuss what moving digits a number of PV positions means for the positioning of the ones position (as determined by the decimal point – never give any rules in terms of decimal points) – see below:

$$2(5)$$
 $3 6 \times 10 = 2 5 (3.6)$ – the ones position moves to right as digits move to left

- Practise examples such as 7.23 × 10 = _____; 26.5 ÷ 100 = _____. Discuss how the ×10/÷10 structure applies in metrics and, in other bases, to time, angle and so on.
- Place digits on PVC from thousands to thousandths, place an object on PVC to act as decimal point discuss role of decimal point.
- Place digit cards to make a number (e.g. 256.78); discuss this as 25 tens, 6 ones and 78 hundredths move decimal point to tens discuss how this makes 25.678 tens from 256.78 ones. Set up examples where decimal point is moved and discuss what the number is (e.g. 34.6 becomes 0.346 hundreds, 0.457 becomes 45.7 hundredths discuss how this is also 45.7 %). Relate to metrics.
- Discuss how flexibility of decimal point means renaming and vice versa. Relate to \$4.65 being 46.5 ten cent pieces.

4.4 Continuous vs discrete: Number-line model

As described earlier, number can be applied to both discrete and continuous attributes, and this is essential for all number types. Thus whole numbers, decimal numbers, common fractions, rates and ratios must be understood in terms of discrete and continuous – that is, seen in relation to separate objects and seen in relation to continuous entities. In practice this means that number should be represented in terms of sets, number lines and area.

Whole numbers can be used to describe discrete items, e.g. fingers, people, chairs or continuous measures like lengths or volumes that have been separated into countable units. When considering decimals it is more difficult to think of the smaller partitions of a unit as discrete parts. To consider tenths you need a concept of what has been partitioned into 10 equal parts. Continuous models like number lines allow this to be visualised and understood more easily than discrete models. Once a line has been broken into 10 discrete parts that can be labelled as tenths we can work with these as decimals.

4.4.1 Density

Concept

The concept of density relates to the amount of numbers between numbers in a counting system. This mathematical idea does not really relate to whole numbers as there are no whole numbers between consecutive whole numbers, e.g. there are no other whole numbers between 34 and 35. However this idea is very powerful for decimals. Between any two consecutive decimals there are other numbers. For example, between 0.4 and 0.5 there are many other numbers. Because the structure of our number system allows us to represent infinitely small numbers, there are in fact an uncountably infinite number of numbers between any two given numbers. For example, numbers 2.65, 2.67, 2.648, 2.6780201, and so on, are between 2.6 and 2.7. This means that decimal numbers are extremely dense and whole numbers are not.

Using a number line can help highlight the concept of density. The figure below attempts to show how a whole number line can be expanded by taking the section between 0 and 1 and considering the tenths that lie between 0 and 1. Then the section from 0 to 0.1 is taken further to show the hundredths between these numbers. This process can continue to infinitely small numbers.



Density of decimal numbers

Teaching density



Reality

Discuss how looking at decimals as a number line (or length) changes discrete to continuous – that there are an infinite number of numbers between 7 and 8 and there are an infinite number between 7.4 and 7.5 etc.

Abstraction

Body. Two students stand on either side of the classroom. They are the ends of the number line, say 11 and 14 and they write their number on a card that they hold at their point. Then the next student picks a number between these and stands on the number line, say 12, writes a card and holds it in place. The next student must choose a number between 11 and 12, say 11.6. The next student chooses a number between 11 and 12, say 11.6. The next student chooses a number between 11 and 12. Say 11.6. The next student chooses a number between 11 and 12. Say 11.6. The next student chooses a number between 11 and 12. Say 11.6. The next student chooses a number between 11 and 11.6, say 11.54. The next between 11.5 and 11.54 and so on up to 6 or more decimal places.

* Students may need some reminding of the multiplicative nature of the number system, that each new "smaller" PV position is a division of 10 from the previous position.

Hand. Students can make the place cards that begin with two consecutive whole numbers and are broken down until there are 6 decimal place values. They should test these on their peers.

Mind. Students should be able to answer the activities in their minds.

Mathematics

Students should be formally recording numbers that are in the ten thousandths, hundred thousandths. These names are worth familiarising students with.

Reflection

Choose a number 2.5468 and students reverse the concept and choose the previous two "place value" numbers that it would have been between, i.e. 2.5468 is between 2.546 and 2.547.

Choose a number, e.g. 5.2145, and choose two other numbers that could have been chosen, i.e. 5.2145 is between 5.214 and 5.215 – other numbers could have been 5.2148 or 5.2143.

4.4.2 Rank and order

Concepts

The concept of **rank** as it applies to whole numbers holds for decimals as well. Regardless of the number of digits that make up a number, each number is a single point on a number line. That is, 2.47 is a point close to and just before 2.5. This form of representation ranks numbers in terms of distance from zero (e.g. 23.9 is closer to 0 than 24.7, so 24.7 is larger). With whole numbers, the greater the number of digits in a number, the larger the number is but with decimals this does not hold true. The number 34.5 is not larger than 67 just because it has a greater number of digits. The value of the digits needs to be considered as it is the larger place values which need to be the starting point for such comparisons and ordering activities: 34.5 has 3 tens where 67 has 6 tens and therefore 67 is larger.

Students need to know that the number names and symbols have an **order** that relates to their relative quantity for decimals as well as for whole numbers. As described above in relation to rank, the number of digits in a number is not a basis to compare decimals. Decimals are best compared using a number line. To

understand decimal numbers on a number line, students need to understand the concept of density as described above in section 4.4.1. Representation of number on a line provides an indication of size, so numbers can be compared and ordered in terms of position along the line. The placement on the number line also assists students gain the understanding that higher place values are the starting point for comparing and ordering numbers (see figure below). Thus, **comparison and order** are aligned with rank.



Comparing numbers on a number line

The number 0.4 is one tenth more than 0.3, so 0.368 cannot be larger than anything over 0.4.

Teaching rank and order

Ranking and ordering decimal numbers is an important big idea that will transfer to the measurement units. Students must have an appropriate method to do this.

Reality

Relate examples of ordering decimal numbers to the examples that students uncovered during previous activities (e.g. 5 023 and 5 203, or 1 metre 45 cm and 1.5 metres). The actual unit or measure of the examples need not be emphasised as it is just the numbers we will be examining.

Abstraction

Body. Use rope, pegs and pieces of paper to place decimal numbers on number lines.

- Begin with basic tenths one tenth, two tenths, three tenths,, ten tenths (one) to give students a reference to be able to place numbers on the number line. Add the numbers 0.1, 0.2, 0.3, ..., 1 to the number names on the number line.
- On an empty rope number line place your beginning and end numbers, i.e. 0 and 2 then have students peg numbers on the number line first tenths, then tenths and hundredths. Allow students time, or encourage them, to self-correct ask "do all the numbers appear to be in the correct places", "are there any changes that anybody would like to make"?
- Using masking tape on the floor the same activity can be completed using digit cards and word names. The tape on the floor makes it easier to extend to thousandths.
- Compare decimal numbers by their PV positions and identify larger number by digit in largest PV position.
- Show that decimal numbers with more digits can be smaller, e.g. 3.45 is smaller than 4.2.

Mind. By picturing the position on the number line of numbers, or the numbers' distance from zero, students can tell you which number is larger – 7.8 or 7.9; 5.678 or 5.7?

Mathematics

Complete activities where students must order and rank and compare three or more decimal numbers. They should be able to do this without necessarily using a number line every time.

Reflection

Challenge students to rank numbers in the thousands or millions, where there is more information than is necessarily important to the decimal ranking, e.g. rank 154 639.3268 and 154 639.6587, etc.

Reality

Reflection

Abstractior

Maths

4.4.3 Rounding and estimating

Concept

Rounding and estimating are valuable skills that relate to students' sense of the size of numbers and their understanding of place value of whole numbers and of decimals. Rounding and estimating are usually completed to a particular place value, e.g. to round a number to the nearest hundredth. Number lines enable values to be determined to which numbers under consideration are nearest. For example, in the number line figure on the previous page, it is obvious from the placement on the number line that 0.432 is nearer 0.4 than 0.5. So 0.432 rounded to the nearest tenth is 0.4.

Teaching rounding and estimation

The ability to estimate with decimals is important when exploring money and measurement.



Reality

Discuss the importance of being able to estimate and where it may be used. For example, if a measured distance from the airstrip to the school is 4.678 km and a visitor needed a rough idea how far to go, you would give an estimate of "about 5 km" and not 4 km as 5 km is more accurate.

Abstraction

Body. Students place numbers on a number line, e.g. start at 3 and put place cards down in tenths until 5. Examine each of the tenths values and decide which whole number they are closest to. Discuss:

- The rule about the midpoint (5) being rounded up.
- That if the number 5.678 is being rounded to the nearest whole number then the hundredths and thousandths are insignificant. If the number is being rounded to the nearest tenth then the thousandth place value is insignificant.
- Place digit cards on the number lines and decide what they should be rounded to.
- Have students "be" the digit decimal numbers using the digit bibs, i.e. make the number 4.621, rounding to the nearest tenth the digits that are not considered (thousandths) should sit down on the spot. The number is then rounded.

Hand. In pairs or singly, students use a number line at their desks, e.g. between 1.7 and 1.9 and have number cards (e.g. 1.75, 1.84, 1.8995) that they place on the number line then slide to the correctly rounded tenths position. Have a number of packs ready so students can swap when they are finished.

Mind. Ask students to mentally determine the appropriate rounding for decimal numbers.

Mathematics

- Place decimal numbers on number lines so that can determine rounding to various decimal PVs (e.g. 2.457 is between 2.4 and 2.5 and is closer to 2.5 than 2.4 so 2.457 is 2.5 (rounded to the nearest tenth).
- Practise these skills use the game "Target" (e.g. 17 × ? = 500.6 so use calculator to determine by trial and error what the ? is).

Reflection

Reverse the questions – "what are two numbers that if rounded to tenths could be rounded to 5.6?"

4.5 Equivalence and extension to millionths

This covers the last two ideas, equivalence and extension to millionths, by extending the pattern of threes to decimals.

4.5.1 Equivalence

The big idea of equivalence focuses on the understanding that it is possible for a number to be the same and the numeral to be different. The concept of equivalence is particularly important when working with decimals and is the cause of many misunderstandings with decimals. Sometimes adding a zero to a number makes the number different but other times it does not. Students need to be able to recognise when the inclusion of a zero changes a number and when it does not change a number. For example changing 4.5 to 4.50 does not alter the value of the number but placing the zero in other places does, e.g. 4.05 or 40.5.

To **teach**, recap equivalence from whole numbers. Note that it is not extensive and relates to placement of zeros. For example, which changes the number: 3 to 30 or 3 to 03?

Extend this to decimals by looking at adding zeros into decimal numbers – which of these change the value of the number?

- 2.46 → 2.460
- 2.46 → 2.406
- 2.46 → 20.46 or 2.046
- 2.46 → 02.46

Do many of these and see if students can build a **pattern** – where can 0s be added without changing the value of the decimal number?

4.5.2 Pattern of threes to read and write decimals

Students need to be able to extend the place-value structure to include decimals. A PVC that extends from millions to thousandths can be used to show how the patterns that exist within the whole number system are extended into the decimal number system but with some differences.

Similar to the whole-number place values, there is a pattern of threes but it is somewhat difficult because the symmetry of place-value names, the pattern of threes and the conventions for saying numbers do not match as coherently as for whole numbers. The pattern of threes can be seen when decimal numbers exist in even threes of decimal places but is obscured for other places because it relies on tenths being a hundred one thousandths, hundredths being ten one thousandths, hundred millionths being ten thousandths, and so on. This pattern is shown below. (*Note:* This can be confusing for students if PV names are not clearly understood – this is not a pattern for early development of PV understanding.)



The language for decimal place values is based on common fraction language. Thus, e.g. 0.35 is "thirty-five hundredths". In language terms, therefore, the name of the decimal PV part of a decimal number depends on the smallest place value (e.g. 45.685 is "forty-five and six hundred and eighty-five thousandths"). This leads to difficulties, for example, does "two hundred and forty-five thousandths" represent 200.045 or 0.245?).

There is a difficulty with language $\leftarrow \rightarrow$ symbol in decimal numbers. For example, 0.35 is 35 hundredths as a fraction but in terms of place values it is 3 tenths and 5 hundredths. Thus, students have to understand that 30 hundredths is 3 tenths and that 3 tenths is 30 hundredths. Similarly, for 0.254, students have to understand that 200 thousandths is 2 tenths and 50 thousandths is 5 hundredths.

To assist students to understand the concepts of place value as they relate to decimals early work with decimal numbers should be read using the place values rather than reading each digit:

- 1.37 is one and thirty-seven hundredths, rather than one point three seven.
- 9.534 is nine and five hundred and thirty-four thousandths, rather than nine point five three four.

You could also relate tenths, hundredths and thousandths to Hundreds, Tens, Ones of thousandths to continue the pattern of three as below. There needs to be consideration of the overlapping use of the ones as being part of the Hundreds, Tens and Ones of Ones as well as being part of the Ones, Tenths and Hundredths group of three for the Parts of One.

н	Т	0	н	Т	0	н	т	0	tenths	hundredths	thousandths
Millions Thousands					Ones			Parts of C	ne		

It can be seen that ${}^{10}/_{1000} = {}^{1}/_{100}$ so that ten thousandths is one hundredth; similarly, ${}^{100}/_{1000} = {}^{1}/_{10}$ so that a hundred thousandths is one tenth. This would enable the change from above PV chart to the below PV chart, and build the pattern to go to smaller and smaller parts.

н	т	0	Н	Т	0	н	Т	0	Н	Т	0	Н	Т	0
	Millions		Tł	housand	ds		Ones		T h	ousand	ths	N	1illionth	ns

This would mean that 3.4673 could be read as "three and four thousand, six hundred and seventy-three ten thousandths" but also as "three and four hundred and sixty-seven thousandths and three hundred millionths".

4.6 Directed numbers and the real number system

This section looks at directed numbers and real numbers and, therefore, also rationals and irrationals.

4.6.1 Decimal directed numbers

As with whole numbers, directed numbers can be used to represent situations that involve both quantity and direction involving decimals. The directions are opposites and are described as being either positive or negative. Directed numbers allow the decimal number system to be extended beyond zero in a negative direction. An example of the use of negative decimals can be with temperature where the temperature drops a whole and decimal degrees below zero.



As with whole numbers, directed numbers are opposite to the corresponding positive decimal. It is important to make the distinction between negative numbers and the operation of subtraction. For example the number that is opposite 0.7 is negative 0.7 and is written as -0.7. A number line is an effective tool for demonstrating how these numbers are opposites. The numbers are the same distance or are the same number of jumps in each direction from zero.



- terminating decimals (e.g. $\frac{3}{4} = 0.75$), or
- infinite decimals with a repeating part (e.g. ²/₇ = 0.285714285714285714285714285714 ... which is repeating 285714).

Show how terminating decimals and repeating part decimals become common fractions, for example:

- $0.46 = \frac{46}{100}$ so fraction is $\frac{23}{50}$;
- 0.46464646 ... let f be fraction, therefore 100f f = 46.464646 ... 0.464646 ..., thus 99f = 46, and so fraction = ⁴⁶/₉₉. (These decimals are all common fractions and are called rationals the other ones are called irrationals.)

Discuss decimals that are infinite but with no repeating part (examples of these are pi, square root of 2, and so on). These are not fractions and are called irrationals – infinite decimals that do not repeat.

The argument of why square root of 2 is not a common fraction was important in history and relates to whole numbers having a religious role and, therefore, the importance of all numbers being in some way whole numbers or at least having two whole numbers in their symbol.

Argument: say the square root of 2 was p/q – then squaring both sides gives 2 = p^2/q^2 – but this is impossible – p^2/q^2 can only be a fraction, an odd number or, if even, a multiple of 4 (can you work out why?).

Discuss how rational and irrational numbers come together to form the real number system.

Discuss how the rationals (which are all common fractions) are infinite in number and countable (can be put in an order and counted) **and** the irrational are also infinite in number but uncountable (because they are not common fractions).

4.6.3 The real number system (decimals)

We have seen so far that decimal numbers are an extension of whole numbers – so the system should be as well. Some properties that appeared evident earlier are that, in terms of decimals, the decimal number system is: (a) symmetrical about ones; (b) bi-directional in relationship, in that the place-value positions work left and right from the ones and ×10 will move the place value one place to the left and $\div10$ will move the place value one place to the right; (c) exponential in structure and this structure extends to decimals; and (d) continuous across the decimal point.

The crucial factor in teaching/developing the real number system is patterns – look for things that hold true for all numbers. What is evident is that the system is generalisations. Some activities to do this are as follows.

1. Pattern of powers of 10

Look at millions to thousandths as a PVC – expand out to millions to millionths – discuss the PV positions in terms of $\times 10$ and $\div 10$, as below (O is one or ones, T is ten, H is hundred, Th is thousand, M is million, t is tenths, h is hundredths, th is thousandths, and m is millionths):

нм	TM	ОМ	HTh	TTh	OTh	Н	Т	0	t	h	th	Tth	Hth	m

OR in pattern of threes:

HM	TM	ОМ	HTh	TTh	OTh	Н	Т	0	Hth	Tth	Oth	Hm	Tm	Om

Discuss the pattern in terms of indices or powers of 10 as below:

10 ⁸	107	10 ⁶	10 ⁵	10 ⁴	10 ³	10 ²	10 ¹	10 ⁰	10-1	10-2	10-3	10-4	10-5	10-6

OR in pattern of threes:

H10 ⁶	T10 ⁶	O10 ⁶	H10 ³	T10 ³	010 ³	H10 ⁰	T10 ⁰	O10 ⁰	H10 ⁻³	T10 ⁻³	O10 ⁻³	H10 ⁻⁶	T10 ⁻⁶	010 ⁻⁶

Discuss how the pattern on left and right of the Ones goes:

 $\ldots 10^5 \quad 10^4 \quad 10^3 \quad 10^2 \quad 10^1 \quad 10^0 \quad 10^{\text{-1}} \quad 10^{\text{-2}} \quad 10^{\text{-3}} \quad 10^{\text{-4}} \quad 10^{\text{-5}} \ldots$

OR

HTO 10⁶ HTO 10³ HTO 10⁰ HTO 10⁻³ HTO 10⁻⁶

2. Counting patterns

Trials and investigations will show that all place values, no matter how large or small, count the same way, for example:

37, 38, 39, 40, 41, ...
6.18462**37**14, 6.18462**38**14, 6.18462**39**14, 6.18462**40**14, 6.18462**41**14, ...

It is important to get students to construct as well as follow these patterns, and to be aware of problems when there are many 9s, for example:

3 4 9 9 9 9 9 9 9 8 2 3 4 9 9 9 9 9 9 9 9 2 3 5 0 0 0 0 0 0 0 2

3. Multiplicative structure patterns

We can investigate the $\times 10 \pm 10$ bi-directional relationship in PV positions. We can translate the changes from $\times 10$, etc., to indices, for example:

PV 10^7 will be changed to 10^{-2} by division by 10^9

4. Equivalence patterns

Adding a 0 before the non-zero digits if first non-zero digits are in whole number PVs makes no change as does adding a zero after the non-zero digits when the last non-zero digit is in the fraction PVs, for example:

00004294 = 4294, 3.6871000 = 3.6871 243.415 ≠ 2430.415 ≠ 243.0415 and so on 0.035 ≠ 0.0035 and 35 ≠ 3500

5. Density patterns

There is always an infinite (uncountably infinite) number of numbers between any two numbers, for example, between 2.468 and 2.469.

6. Any base patterns

A number system on any base (say 12) could be built as per this positional system – it would simply have powers of 12 not 10.

5 Common Fraction Numeration

This chapter deals with fractions other than decimals – common fractions and mixed numbers, equivalent fractions, percent, rate and ratio. From a part-whole perspective, common fractions and mixed numbers include the major meanings – part of a whole, part of a set, fractions as division, and fractions as operator. From a continuous vs discrete perspective, the fifth meaning is fractions as a single point on a number line. For common fractions/mixed numbers, this chapter also follows the whole/decimal numbers sequence:

Part-whole: PV Additive/multiplicative structures Continuous vs discrete / Number line

Equivalence



In addition to the whole/decimal numbers sequence, common fractions and mixed numbers includes a section on equivalent fractions. Similar to the remainder of this book, all activities will be classified by the RAMR framework components but only the most crucial components will be detailed. Also, because of the similarity of topics and their sequential nature, new topics can often be taught by extending the teaching activities of the previous topic. This means that sometimes a teaching sequence only needs activities from the mathematics and reflection components.

Fraction concepts

There are five concepts or meanings of fraction:

- a fraction as part of a whole dividing a whole into equal parts (e.g. ³/₄ is 3 parts out of 4 parts equalling 1 whole), an approach that easily leads to common fraction notation;
- a fraction as part of a set dividing a group into equal parts (e.g. ³/₄ of 8 is dividing 8 as a whole into 4 equal parts and taking 3 of the parts, gives 6);
- a fraction as a quantity/position on a number line a fraction is a single point on the line (e.g. ³/₄ is a single point halfway between ¹/₂ and 1), an approach which is good for ordering fractions;
- a fraction as division fractions are given as numerator divided by denominator (e.g. 3 cakes shared among 4 people is ³/₄ of a cake to each person, so ³/₄ is equivalent to 3÷4); and
- a fraction as a multiplier/operator fraction ³/₄ is that which multiplies by 3 and divides by 4 (i.e. acts as ×3 ÷4).

It is useful to see fractions in terms of similarities and connections with percent, rate and ratio. They can all be forms of multiplicative comparison; for example, Jack has \$24 and Bill has ¾ of this, June has 30% of this, and Ann's money is in ratio 3:4 with Jack's money. Meanwhile, Alan changes \$24 to 16 litres of fuel at \$1.50/litre.

As well, Whole/Part charts and renaming show the similarity between mixed numbers and tens and ones – both can be separated into components. Furthermore, measuring fixed amounts with units is similar to partitioning a whole into equal parts, so there are similarities between measurement and fractions.

However, the great similarity or connection is that fractions are division. This enables many difficult ideas to be made easy. For example, dividing money between a lot of people means each gets little – this means that measuring with a large unit gives a small number, and fractions with large denominators are small.

It is also useful to remember differences. Fractions have a different language and symbol structure than decimals (see the previous chapter). Further, there is an important difference between fraction and ratio – fraction is part-to-whole while ratio is part-to-part. That is, ratio 2:3 is fraction $\frac{2}{5}$.

The first thing in fractions is to develop meaning. The recommendation is that this is first taught with materials and language only – that notation be left until after the decimal notation is known.

Sequence

To develop a rich meaning for fractions, it is essential to cover all the meanings from real-world situations and later mathematics. This means introducing the following concepts in the following order:

FRACTION AS PART OF A WHOLE (part of a cake, part of a liquorice strip)

FRACTION AS PART OF A GROUP (part of a class)

FRACTION AS A NUMBER OR QUANTITY (¾ is halfway between ½ and 1)

FRACTION AS DIVISION (¾ is 3 divided by 4 – puts fractions on calculators)

FRACTION AS OPERATOR (¾ is multiplication by 3 and division by 4)

5.1 Part-whole

The basis of fractions is partitioning wholes into parts. The identification of the whole is a vitally important aspect of working with fractions and lack of attention to what the whole is or was can lead to misunderstandings about fractions. Students need a wide range of experiences splitting both collections of discrete objects and continuous wholes into equal parts. Collections can be partitioned into fractions by sharing or equal dealing out. Continuous quantities can be partitioned into fractions by cutting, folding, pouring or weighing. It is also valuable for students to work in reverse and to take a number of equal-size parts and use them to re-form wholes. This skill of being able to see parts as a whole is called *unitising*. Unitising is essential for part of a group because the set of objects has to first be unitised (seen as one whole) before it can be partitioned.

5.1.1 Partitioning and materials

As well as partitioning and unitising, it is crucial that parts being partitioned to and unitised from are equal – in area as well as length. However the parts do not need to be the same shape. It is the quantity or size that needs to be the focus. When fractions of different wholes are to be compared the original whole needs to be the same. In the figure below the two squares are the same whole so the halves of each shape will be the same. However, as the two triangles are not the same size, halves of these two shapes are not the same size, even though they are the same shape.



Comparing fractions of different wholes

Fractions involve partitioning wholes into parts. When this is done, it is important, for fractions, that **the whole is maintained as the unit** and that **the part does not become the whole or the unit**. This is part of the wholepart principle. Thus, for a fraction such as ¾ to be understood, ¾ must be seen as **three parts out of four parts as one whole**. Three parts out of four parts, without seeing the parts as the whole, is the ratio 3:1.

Examples of materials

Some examples of materials that are useful for partitioning, and directions for folding paper to make different fractions, are given below. These folds are part of the technical knowledge needed by teachers to successfully work with students in classrooms on practical work on fraction as part of a whole.

1. Real-world materials. Everyday materials that can be cut into fractions using area or length models (e.g. cakes, pies, pizzas, liquorice, and so on).

2. Paper rectangles.

Halves	-	fold in half					
Quarters	; —	fold in half and half again					
Eighths	-	fold in half three times					
Thirds	-	fold so that fold goes halfway back along rectangle					
Fifths	-	roll rectangle 2½ times, and flatten where rolls end					
Sixths	-	roll 3 times for sixths, 3½ times for sevenths and so on					
Paper strips.							
Halves [.] F	old in ł						

3.



Models and concepts

There are three models that lead to two concepts for the part-whole big idea.

- 1. Area model. Examples of real-life and classroom area models for representing fractions include cakes, pizzas, blocks of chocolate, paper folding, circles and rectangles (fraction discs), diagrams of 2D shapes. This leads to fraction as part of a whole.
- 2. Length model. Examples of real-life and classroom set models for representing fractions include continuous items such as liquorice strips, paper strips, fraction mats. This also leads to fraction as part of a whole.
- 3. Set model. Examples of real-life and classroom set models for representing fractions include discrete items such as Unifix cubes, counters, logic attribute blocks, students, chairs, etc. This leads to fraction as part of a set or group.

5.1.2 Teaching fraction as part of a whole and part of a group

Part of a whole

This is the first concept to be taught and relates to area and length models. The process is: (a) Identify the whole, (b) partition into equal parts (say 4), (c) name parts (by the number of parts – four parts means fourths), (d) take a number of these parts (say 3), and (e) name the fraction (3 fourths or $\frac{3}{4}$). The RAMR activities below show how this concept is taught.



Materials: As above in 5.1.1.

Reality

Using local culture and environment. Look for things in local environment that use fractions (e.g. half a glass of water, halfway home). Try to find unique things.

Existing knowledge. Check that students can count, know numbers, can partition in equal parts and know basic shapes.

Kinaesthetic activities. Find things in the environment that students can find a fraction of or cut into fractions (e.g. their bodies, fruit). For length model, go outside and walk alongside the building – stop half way, one quarter of the way and so on.

Abstraction

Sequences. A common sequence is as follows.

<u>*Play.*</u> Ask the students to fold the paper rectangle, square and circle to make halves in as many ways as they can, and then to make fourths/quarters in as many ways as they can.

Materials (paper rectangles).

Halves. Hold up a paper rectangle: Say: "This is a whole." Ask the student to say what it is. Say: "Fold the whole to show halves." [Check and discuss how students did this – some may have folded down and some may have folded across.] "How many equal parts have you made?" [2] "I wonder why we don't call them 'twoths'?" Say: "Can you fold it from one corner to the opposite corner to show halves?" [No] "Cut the rectangles from corner to opposite corner. Now put one on top of the other. Are the two parts equal?" [Yes] "So you can't fold a rectangle to show halves but you can cut it to show halves." Repeat this with a paper square. The students should be able to fold along each diagonal.

Fourths/quarters. Take a new piece of paper. Hold it up and say, "This is a whole. Now show me fourths/quarters. How many parts have you got? Is each part the same size? So one whole equals how many fourths/quarters?" Ask the students to say how they folded their paper to show fourths.

Thirds/sixths. Take a new piece of paper. Hold it up and say, "This is a whole." Ask the students to fold it to show thirds. Say: "How many parts have you got? I wonder why we don't call them 'threeths'? Are the parts the same size? Can you make this into sixths or will you need a new piece of paper?"

Repeat for fourths and eighths, and for fifths and then tenths.

Materials and language.

Paper rectangles

- Identify whole (hold up your coloured rectangle, say "this is one whole", hold up the white rectangle, say "this is also one whole");
- Partition into equal parts (fold the white rectangle into four equal parts, fold in half and fold again);
- Name the equal-sized parts (ask "how many parts has the whole been divided into?", count the parts "one, two, three, four", state "each part is a fourth");
- Determine number of parts (shade three of the parts, or cut out three of the parts, place this against the coloured whole);
- Associate fraction name (ask "how many parts are shaded one, two, three", state "three out of four parts of one whole, so fraction is three quarters").

Paper circles

- Identify whole (hold up your coloured circle, say "this is one whole", hold up the white circle, say "this is also one whole");
- Partition into equal parts (fold the white circle into six equal parts, fold in half and fold "half" line into centre);
- Name the equal-sized parts (ask "how many parts has the whole been divided into?", count the parts "one, two, three, four, five, six", state "each part is a sixth");
- Determine number of parts (shade two of the parts, or cut out two of the parts, place this against the coloured whole);
- Associate fraction name (ask "how many parts are shaded one, two", state "two out of six parts of one whole, so fraction is two sixths").

Mental models. Get students to shut eyes and then imagine shapes cut into fractions.

Mathematics

Appropriation. Repeat the Abstraction activities but go another step, write out the fraction symbol. First be informal (e.g. "three fourths"), and then formal (e.g. %). Ensure students have appropriated the correct symbol.

Practice. Use a variety of practice activities: (a) worksheets with four columns (rectangle/strip shaded, circle shaded, language, symbol) and fill in only one column (different for each example) – students fill in other columns. Play games like Mix-and-match cards and Bingo.

Connections. Relate fractions to division and measuring with units.

Reflection

Validation. Search for examples of fractions in the local environment.

Application. Apply knowledge to solve simple examples – Andie ate $\frac{3}{4}$ of the cake – show what he ate.

Flexibility. Look for all the places/objects that fractions could be used. In older students ask for anything that is *%*. For example, 75%, 0.75, 75 cm, 45 minutes 270 degrees, and so on.

Generalising. Try to get across the generalisation that if we take a whole and break into q equal parts and shade p of them that the fraction is p qths or p/q.

Reversing. One of the very important aspects of teaching the fraction meaning is to reverse the process – to ensure that teaching covers **all** the following:

- WHOLE \rightarrow PART (give students a paper square, say it is one whole, and ask to fold to get $\frac{3}{4}$; give students 12 Unifix, say it is one whole, and ask to construct $\frac{2}{3}$;
- PART \rightarrow WHOLE (give students a paper square, say it is $\frac{3}{4}$, and ask to make one whole; give students 12 Unifix, say this is $\frac{2}{3}$, and ask to make one whole); and
- WHOLE/PART \rightarrow WHOLE/PART (give students a paper square, say it is $1^{1}/_{4}$, and ask to construct $1^{1}/_{2}$; give students 20 Unifix, say this is $1^{2}/_{3}$, and ask to make $2^{1}/_{2}$).

<u>Note</u>: It is crucial to ensure that students maintain the whole throughout. When a paper rectangle is folded into four, some students see four wholes not one whole. Thus, we spend time at the start stressing what the whole is and keep a coloured whole to compare the part with. Similarly, for Unifix, we spend time at the start ensuring students see the Unifix as one whole group. Other methods to do this are running a finger around the whole while saying "this is one whole" or putting the Unifix on a coloured piece of paper or drawing a circle around the Unifix. The idea is to act out the formation of the whole, so that the kinaesthetic sense is in action as well as sight, hearing and touch.

The two notions that underlie the teaching of fraction are unitising, making a whole out of parts (even if only in the mind), and partitioning, making parts out of a whole.

Part of a group

This is the second concept to be taught and relates to area and length models. As for part of a whole, the process follows the same 5 steps: (a) Identify the whole (much more difficult when start with a set), (b) partition into equal parts or groups (say 4), (c) name parts (by the number of parts – four parts means fourths – not the number in each part), (d) take a number of these parts (say 3), and (e) name the fraction (3 fourths or $\frac{3}{4}$). The RAMR activities below show how this concept is taught.



Materials: Unifix cubes/counters.

Abstraction

Sequences. Follow sequence below.

- Identify whole Take 12 Unifix. Cover them with hands and say "this is one whole".
- Partition into equal parts Partition the Unifix into three parts do this by sharing among three (start with an area model already divided into thirds and gradually increase complexity to an area model that is not divided).
- Name the equal-sized parts ask "how many has the whole been divided into?" Count the parts "one, two, three", state "each part is a third".
- Determine the number of parts choose two of the groups.
- Associate fraction name ask how many groups chosen "one", "two", say the name [two thirds].
- Provide numeral $[^{2}/_{3}]$ note there are 8 Unifix in $^{2}/_{3}$ of 12.
- Reverse this process cover 12 counters, call the 12 Unifix $^{2}/_{3}$ and ask students to make a whole.
- Become more complex cover 20 counters, call the 20 counters 1¼ ask students to make a ½.



Reading and writing fractions

The language for fractions comes from the number of equal parts. We recommend that to get the pattern, names be called "twoths", "threeths", "fourths" and "fiveths" until the pattern of "fred" equal parts being "fredths" is seen. Once this language is gained, full fraction language and symbols can be seen to emerge from the parts being considered, that is, three out of four equal parts as one whole is "three fourths", then "3 fourths", and "3 line 4" or ¾. It is useful to remember that ¾ is a single symbol – its parts are not separate.

The naming of the fractions is similar to the naming for the order of numbers (e.g. thirds, fifths, sixths, twenty-firsts) except for the early ones (second – half, fourth – quarter). This is another place where language and numeral do not follow the same pattern (e.g. four – quarter, seven – seventh, eleven – eleventh, twenty-one – twenty-first). The names go as follows:

2	TWO	HALF (TWOTH)				
3	THREE	THIRD (THREETH)				
4	FOUR	FOURTH or QUARTER				
5	FIVE	FIFTH				
6	SIX	SIXTH				
7	SEVEN	SEVENTH				
and so on						

For mixed numbers, the language is almost place value. It is given in terms of wholes and parts, that is, $4^{2}/_{7}$. A Whole/Part chart can be used to show the mixed numbers. After this, it is important to maintain counting, that is, 4^{4} , 1, 1 $\frac{1}{5}$, 1 $\frac{2}{5}$, and so on.

5.1.3 Mixed numbers

This is the way of describing fractions larger than 1 as wholes and parts (mixed numbers). Mixed numbers such as $3\frac{1}{3}$ can be placed on Whole/Part charts with 3 in the wholes and $\frac{1}{3}$ in the parts, thus they relate/connect to place value – placed in a Whole/Part chart mixed numbers can be seen to have two positions – the whole and the parts. If the parts are sixths, for example, then there are 6 parts to each whole, like there are 10 ones to each ten on a Tens/Ones chart.



To teach the concept, use activities like those in the RAMR framework described below.

Materials: Fraction materials cut up into wholes and parts (e.g. fraction discs), Whole/Part chart.

Abstraction

Sequences. Mixed numbers – the idea is to use a Whole/Part chart to introduce mixed numbers – wholes are put on the left and parts on the right.

- Materials Place materials for, say, 2 and $\frac{3}{4}$ on the Whole/Part chart.
- Language Ask how many wholes? [Two] Cover wholes, ask how many parts left over (loose parts)? [Three quarters].

Mathematics

Appropriation. Mixed numbers – Symbols – Ask how many wholes? [Two] Write the number 2 on the left. Cover wholes, ask how many parts left over (loose parts)? [Three quarters] Write this fraction on the right. $[2^{3}/_{4}]$

Practice. Formation of improper fractions through renaming of wholes to parts can be practised with shading and trading games.

<u>Sandwich relish</u>. Rules: Each player has a game board. A suitable dice is thrown in turn by each player (here dice will have six of $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$ and $\frac{7}{8}$ on the six sides) and the same number of parts are shaded on the board as the number on the dice. Player states how many wholes and parts they have shaded in total (i.e. gives the fraction shaded across the board as a mixed number, e.g. "two and three-eighths"). If stated incorrectly, the player misses a turn. The first player with all four wholes shaded wins.



<u>Card games</u>. Use the fraction discs, Whole/Part charts and cards with appropriate fraction names (e.g. 3 tenths, 2 eighths) for the following games. "Lose 5 ones" (an adaptation of the whole-number game "Lose 5 tens"): Students start with 5 wholes on the Whole/Part chart, take turns in drawing a card and taking away that number of tenths, eighths, etc. Students have three turns each, compare numbers when finished and winner is the one who has the smallest number. "Win 5 ones": The opposite to the game above – start with zero and build to 5 ones.

5.2 Additive and multiplicative structures

In this subsection, we put the two big ideas associated with additivity and multiplicativity together.

5.2.1 Additive structure (counting, seriation, odometer)

Additive structure was described in relation to whole number and decimals as part of their place value. Fractions do not have place value as whole numbers and decimals do. However the denominator effectively names a fraction and can be used for counting as the place values of whole and decimal numbers can be. For example, it is possible to count in whatever the denominator of a fraction is, e.g. 1/3, 2/3, 3/3, 4/3 etc. Each new fraction is one more part of the same equal parts, that is, *a unit fraction more or less.* For example 1/6 more than $3^2/_6$ is $3^3/_6$. This leads to counting and odometer $- 3^3/_6$, $3^4/_6$, $3^5/_6$, $4^0/_6$, $4^1/_6$, and so on.

	WHOLES	PARTS
Counting activities		
Set up a Whole/Part chart as on right. Obtain some physical materials in the form of		
wholes and parts – say circle wholes and circles cut into sixths. These then can be		
placed on the Whole/Part chart to make mixed numbers – e.g. three wholes and		

• Show materials \rightarrow Say and write fraction AND Say and write fraction \rightarrow Put out materials.

four sixths forms 3⁴/₆. So mixed numbers can be made. The activity should go as

- When saying fraction 3⁴/₆, put out materials, place hand over wholes place and say "three wholes" and then move hand to RHS over parts place and say "four sixths".
- Count out wholes and sixths from a starting point, say 3²/₆, adding a sixth piece of a circle each time, and saying 3 and 2 sixths, 3 and 3 sixths, 3 and 4 sixths, and so on past 3 and 5 sixths, 4 and 0 sixths and so on.
- Count backwards removing sixths as go.

follows:

• Place race games – for example, for sixths, spin a 1-2-3-4 spinner, in turn, and add the number of sixths that is shown, rename as necessary to have a mixed number (no improper fractions allowed), first player to six wholes wins. (*Note:* Can reverse game and start with six wholes and remove sixths.)

Odometer activities

The odometer pattern is a generalisation and so teaching needs a lot of examples as follows.

- Repeat the counting activities above for many different types of fractions (e.g. fifths, eighths, thirds, and so on).
- Write out the resulting counting sequences for each type of fraction.
- Look through these sequences for a pattern for each sequence (e.g. for sixths, 5 is crucial after 5 parts the count goes to the next whole, at 0 parts the count goes to the previous whole and 5 parts).
- Look through these sequences for a pattern for when the counting forwards reaches the next whole number and the counting backwards reaches the previous whole number when does this occur?
 [Counting forwards go to the next whole number (increases by 1) and zero parts after you count to where the number of parts is one less than fraction denominator; counting backwards after you count down to zero parts, go to the previous whole number (decreases by 1) and the number of parts that is one less than fraction denominator.]
- Get students to write their generalisations in their own way.
- Get students to extend this work where else do we see different odometer numbers (other than 9 which is the odometer number for whole and decimal numbers)? [What about weeks and days, hours and minutes, metres and centimetres?]

Note: Get students to notice odometer pattern in each fraction type before looking for common pattern across all pattern types.

5.2.2 Multiplicative structure and renaming

The basis of fractions is partitioning wholes in parts. It is also the reverse of this which is being able to see parts as a whole (which is called unitising). Unitising is essential for part of a group because the set of objects has to first be unitised (seen as one whole) before it can be partitioned.

As well as partitioning and unitising, it is crucial that parts being partitioned to and unitised from are equal – in area as well as length. This need for equality of parts is related to the multiplicative structure of fractions. There is a multiplicative relationship between the number of equal parts and the whole. One of the concepts of a fraction is that a fraction is a division e.g. $\frac{1}{2}$ is 1÷4 (one whole divided into 4 equal parts). $4 \times \frac{1}{2} = 1$.

Repeated dividing of wholes into smaller parts, e.g. repeated halving, is also multiplicative and each time the whole is halved the size of the parts is proportionally smaller and the denominator is proportionally larger. For example $\frac{1}{2}$ of $\frac{1}{2}$ is $\frac{1}{2}$, $\frac{1}{8}$ of $\frac{1}{8}$ is $\frac{1}{16}$ and so on. This multiplicative relationship between fractions is the basis for equivalence understandings and operating with fractions.

Renaming/Improper fractions

Renaming is the ability to change mixed numbers to improper fractions and vice versa by renaming (e.g. $3 \frac{1}{3} = \frac{9}{3} + \frac{1}{3} = \frac{10}{3}$, while $\frac{14}{5}$ is $2 \frac{4}{5}$ because $\frac{10}{5}$ is 2). This ability is based on the multiplicative structure of fractions and their related whole. Once again we have the similarity to whole numbers, e.g. $3 \frac{1}{6}$ can be seen as $2 \frac{7}{6} = 1 \frac{13}{6} = \frac{19}{6}$.

Improper fractions are the results of renaming – they are a fraction notation where the numerator is larger than the denominator – they occur when the wholes in the mixed numbers are renamed as parts (e.g. $3^{1}/_{3}$ is a mixed number composed of three wholes and one third; 3 wholes is 9 thirds so, after renaming wholes as thirds, $3^{1}/_{3}$ is improper fraction ${}^{10}/_{3}$).

The best way to teach improper fractions is how renaming was taught in whole numbers.



• Reverse the process – go from a pile of parts to wholes and parts.

Division and operator meanings

Division comes from a multiplicative structure viewpoint, e.g. sixths are one whole partitioned/divided into 6 equal parts. Thus, the same result comes from dividing a whole equally among six people – each will get a sixth. This leads to a collection of activities, where two cakes are shared among three people – each gets $\frac{2}{3}$, four pizzas among five people – each gets $\frac{4}{5}$, and so on. In this way division is connected to fraction to show the mathematical equivalence between division and fraction. It leads to the meaning that **fractions are division**.

These dividing activities are great fun and require problem solving. Put out, say, three circles of paper as cakes, choose four students, and set the task to divide the cakes equally among the four students. This will be done differently depending on how students go about doing it – for example, they might divide each cake among the four people, giving three single quarters; or they might give a half to each person, leaving two halves left which are divided into quarters so that each person gets a half and a quarter. *Note:* The use of plates, table and settings, can be a great way to set up the sharing, as shown on right.



Reality

Reflection

Abstraction

Maths

Operator or multiplier is also a part-whole and multiplicative approach. For example, if we look at eight objects, we can consider the eight objects as one unit (as we do for part of a set) and then a single object is 1/8 of the whole unit. Alternatively, the eight objects can be seen as eight which makes a single object the unit. (Note, we can also consider the 8 as 100% which makes the unit 12.5%.) If we think of the 8 as eight, we can multiply by 3 and divide by 4 and get $8\times3\div4 = 6$. If we now think of the 8 as one, we can divide into 4 parts and take 3 of them, and we see that $\frac{3}{4} = 6$ also. In this way, we arrive at the meaning **fraction as operator**, **i.e. fractions** ($\frac{3}{4}$) **are that which multiplies by numerator (3) and divides by denominator (4)**. This allows cancellation of fractions in operation situations.

Teaching fraction as division and fraction as an operator

Materials: Division – Cakes, pizzas, liquorice, paper circles, strips and rectangles; Operator/Multiplier – Unifix cubes.

Abstraction

Sequences. Follow sequence below.

- <u>Division</u>. Cut paper strips. Say they are liquorice. Share 3 strips among 5 people. This will take time but students will see that the only way to do this is to cut the strips into 5 equal pieces and distribute each of the 5 people will get ³/₅. This means that 3 shared among 5 is ³/₅ and fractions are division.
- Repeat this for 3 cakes shared among 4 people.
- <u>Operator/Multiplier</u>. Take 8 Unifix. Treat each Unifix as the unit. Multiply the 8 by 3 and divide by 4 get answer 6. Take the 8 Unifix but this time treat the 8 as the unit. Find ¾. The 8 as a whole is broken into 4 equal parts (2 Unifix in each quarter). Consider ¾ this is 6 Unifix. This means that ¾ is ×3 and ÷4.

<u>Note</u>: When developing symbols, the best idea is to move as follows:

TEACHER RECORDS STUDENTS' MODELLING

It is also useful to relate models to language and symbols in the six directions from the Rathmell triangle:

model \rightarrow language, language \rightarrow model, model \rightarrow symbol, symbol \rightarrow model, language \rightarrow symbol, symbol \rightarrow language.

The meaning "fraction as division" is used with the symbols to provide a way to put fractions on the calculator. For example, $\frac{3}{4}$ is 3 divided by 4 and is, therefore, 0.75. This allows fractions and decimals to be converted to each other (in both directions):

* ⁷/₈ is 8 <u>7.000</u> which is 0.875
 * 3.47 is ³⁴⁷/₁₀₀ or 3 and ⁴⁷/₁₀₀

The meaning of fraction as operator or multiplier means that $\frac{3}{4}$ can be considered as $\times 3 \div 4$.

Thus ${}^{3}\!/_{4}$ of 20 is 20 × 3 = 60 ÷ 4 = 15; and ${}^{3}\!/_{4}$ of ${}^{2}\!/_{3}$ is found by cancelling the 3s, gives ${}^{1}\!/_{4}$ × 2, and cancelling 2s, gives ${}^{1}\!/_{2}$.

5.3 Continuous vs discrete/Number-line model

As described before, whole numbers and decimals can be applied to both discrete and continuous attributes. Thus whole numbers, decimal numbers, common fractions, rates and ratios must be understood in terms of discrete and continuous – that is, seen in relation to separate objects and seen in relation to continuous entities. In practice this means that number should be represented in terms of **sets**, **number lines** and **area**.

It also means that this leads to the meaning of **fraction as quantity** or a single point on a line.

Whole numbers can be used to describe discrete items, e.g. fingers, people, chairs, or continuous measures like lengths or volumes that have been separated into countable units. Fractions provide a means for considering the smaller partitions of a unit as discrete parts. Discrete models (area models such as pizzas, 2D shape, etc.) allow us to divide a whole into equal parts and count the discrete parts. Fractions can also be represented using continuous models like a piece of ribbon or rope. The use of number lines allows fractions to be visualised and counted. The placement along a number line allows us to investigate other aspects of fractions.

5.3.1 Teaching fraction as quantity (a point on number line)

Materials: Number lines, liquorice, strips of paper.

Reality



- Choose a distance walk it call it one whole. Ask students to estimate and walk half/quarter etc. Walk whole way counting off quarters and end at ⁴/₄.
- Mark off parts of body feet ⁰/₄, knees ¹/₄, hips/waist ²/₄, chest/shoulders ³/₄, head ⁴/₄ get students to point as teacher says fraction get students to say as you point at own body (reversing).

Abstraction

Body. Count a distance (fence panels, concrete squares or a number of steps ...) – mark halves and quarters – walk whole way counting off quarters and end at 4 quarters.

Hand. Draw a line – make strips of paper the same length as the line – fold the strips to make fractions of the line $(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{3}, and so on)$. It is crucial to mark in lines for halves, quarters, eighths, sixteenths and thirty-secondths,

because these are used in American engines/cars, etc. This exemplifies the notion of density as discussed for decimals; e.g. What is the wrench between $\frac{5}{8}$ and $\frac{3}{4}$?

Sequences. Follow sequence below.

- Identify whole draw a line and mark 0 and 1 at ends.
 Move fingers along it say "this is one whole."
- Partition into equal parts partition the line into four equal lengths – do this by cutting a strip of paper to the length of the number line and folding it in four equal parts, then using the folds to mark lines onto the number line.
- Name the equal-sized parts ask "how many has the whole been divided into?" Count the parts, state e.g. "each part is a fourth".
- Determine the number of parts choose three of the lengths.
- Associate fraction name ask how many lengths chosen – "three", say the name [three fourths].



1

1

• Provide symbol $[\frac{3}{4}]$ – mark $\frac{3}{4}$ on number line.

5.3.2 Rank

Rank is the concept that the further along the number line a number is, the larger it is. The focus is on the distance of numbers from the end of a number line. Fractions are numbers that each have a place on a number line and so rank applies to fractions. The complexity with rank and whole numbers and decimals is that it is not the number of numerals in a number that determines the size of a number but the relative distance along the line. This requires understanding of density and place value. With fractions there is a different complexity. The ability to place fractions on a number line requires comparison of the fractions. If the denominators of the fractions are different, equivalent versions of the fractions need to be considered to allow for the placement along a number line.

So the size of a fraction in comparison to another cannot be determined by considering the denominator or numerator in isolation. If the denominators are the same the numerators can be compared and the fractions can be ranked. If the denominators are different, equivalent versions of the fractions need to be found by converting them to the same denominator. Therefore "the larger the denominator, the smaller the fraction" does not necessarily apply as it will depend on the numerator as well. For example, $\frac{9}{12}$ is larger than $\frac{2}{3}$ even though twelfths are smaller parts than thirds – it is the number of parts that need to be considered as well.

To **teach rank**, use the techniques of whole and decimal numbers:

- Get a rope and pegs or put a line on the ground place starting and ending numbers at ends of line have students holding the rope or placed at ends of the line could be 0 and 1 for proper fractions, could be 0 and 5 for improper fractions or mixed numbers.
- Give students fraction numbers to peg on line or place on line on floor discuss the placements in terms of order or guesses and in relation to start and end numbers.

5.3.3 Density

The concept of density relates to the number of numbers that exist between two given numbers. Fractions, like decimals, are dense. There are always fractions (in fact, a countably infinite number of fractions) between any two given numbers. For example, numbers $2^{1}/_{3}$, $2^{2}/_{3}$ are between 2 and 3; $2^{1}/_{24}$, $2^{1}/_{12}$, $2^{1}/_{6}$ are between 2 and $2^{1}/_{3}$ and so on. This means that fractions are dense and whole numbers are not – in theory, for fractions it is possible to continue partitioning between whole numbers so that the next whole number is never actually reached.

• Show how fractions can be put between any two fractions, and that there are many fractions that could go in between any other two fractions.

0			1
	1/3	⁹ / ₁₅	
1/3			⁹ / ₁₅
	$\frac{13}{30}$ $\frac{1}{2}$ 6	1/120	. 10

• Expand previous number lines as below showing how the equivalent fractions grow.

0																1
⁰ /2								¹ / ₂								² / ₂
⁰⁄₄				¹ / ₄				² / ₄				3/4				4/4
⁰∕ଃ		¹ /8		² /8		³ /8		4/8		⁵ /8		⁶ /8		⁷ /8		⁸ /8
⁰∕16	¹ / ₁₆	² / ₁₆	³ / ₁₆	4/16	5/16	⁶ / ₁₆	⁷ / ₁₆	⁸ / ₁₆	⁹ / ₁₆	¹⁰ / ₁₆	¹¹ / ₁₆	¹² / ₁₆	¹³ / ₁₆	¹⁴ / ₁₆	¹⁵ / ₁₆	¹⁶ / ₁₆
ممط																

and so on.

5.3.4 Comparing and ordering

The ability to compare and order fractions is an important aspect of learning about fractions and is a complex activity. For fractions and mixed numbers, there are four ways to compare and order them:

- If the fractions have the same denominator the larger numerator is bigger (e.g. $\frac{4}{5} > \frac{2}{5}$).
- If the fractions have the same numerator the larger denominator is smaller (e.g. $\frac{3}{7} < \frac{3}{5}$).
- If the fractions can be translated to a common denominator the larger equivalent fraction is bigger (e.g. ²/₃ = ⁸/₁₂, ³/₄ = ⁹/₁₂, ⁹/₁₂ > ⁸/₁₂ means ³/₄ > ²/₃) see subsection 5.4 on equivalence to see how this is done.
- If the fractions are mixed numbers the one with **more wholes** is larger (e.g. 4 is bigger than 3, therefore $4\frac{1}{2} > 3\frac{1}{2}$).

Benchmarking is another strategy that can assist with the comparing and ordering of fractions. Here, the fractions are related to benchmarks (e.g. whole numbers, halves, quarters) to determine size. For example, $\frac{4}{11}$ is less than $\frac{1}{2}$ and $\frac{4}{7}$ is greater than $\frac{1}{2}$, thus $\frac{4}{11}$ is less than $\frac{4}{7}$.

There is a progression of difficulty with these different types of comparisons. When students are introduced to the idea of comparing fractions it makes sense to follow this sequence. For example, the following pairs of fractions should be compared in the following order:

L	²/9	4/9	(like denominators)
	²/3	²/5	(like numerators)
	3/4	5/6	(near 1)
	³ /7	⁵ /9	(benchmarking around a $^{1}/_{2}$)
¥	⁵ /8	7/11	(equivalence)

The list of techniques for comparison provides a repertoire of strategies that teachers should know. They should be used with students in diagnostic situations (i.e. used with students for whom they appear to be suitable), **not necessarily taught to every student regardless of need**. Appropriate and inappropriate rules for the various strategies are as follows. (See Post, T., & Cramer, K. (1987). Children's strategies in ordering rational numbers. *Arithmetic Teacher*, October.)

Same denominators, different numerators (e.g. $\frac{4}{9}$ and $\frac{2}{9}$):

• Appropriate strategy: Compare the numerators as though they were whole numbers.

• *Inappropriate strategy:* Confusing the numerator with the size factor of the denominator, then using the fraction rule "the larger the denominator, the smaller the fraction" so that 4 ninths is smaller than 2 ninths.

Same numerators, different denominators (e.g. $\frac{2}{3}$, $\frac{2}{5}$):

- Appropriate strategy: Using a half as a benchmark so that 2 thirds is more than a half but 2 fifths is less than a half and therefore 2 thirds > 2 fifths.
- Inappropriate strategy: Whole-number comparison because 5 > 3, 2 fifths is larger than 2 thirds.

All different (e.g. ³/₄, ⁵/₆):

- Appropriate strategy: Use 1 as a benchmark so that 3 quarters has 1 quarter left which is larger than the 1 sixth left from 5 sixths.
- *Inappropriate strategy:* Finding the differences, that is, 3 is 1 less than 4 and 5 is 1 less than 6 so the fractions are equivalent.

All different (e.g. ³/₇, ⁵/₉):

- Appropriate strategy: Using a half as a benchmark.
- *Inappropriate strategy:* Finding the differences (in which case these fractions would be equivalent) or using the fraction rule.

From this activity, teachers should understand the importance of providing the students with plenty of approximation activities and the importance of having their students explain their answers because correct answers can be based on inappropriate cognitive strategies.

5.3.5 Rounding and estimating

This is the ability to round a fraction to the nearest whole number, or 10 or $\frac{1}{2}$. For example, $2\frac{3}{8}$ rounded to nearest whole number is 2. The ability to round and estimate fractions is related to the idea of benchmarking. For this, students need to recognise the relationship between the denominator, numerator and related whole.

To **teach rounding and estimating**, set up number lines marked at the "roundings", place in fractions and see which rounding is closest. For example round $3^{3}/_{5}$ to the nearest half, the following steps hold:

- (a) construct a number line 0 to 5 and mark in the halves $-\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5;
- (b) place on the number line the example $3^{3}/_{5}$ this will be between $3^{1}/_{2}$ and 4; and
- (c) note that $3^{3}/_{5}$ is closer to $3^{1}/_{2}$ than to $4 \text{so } 3^{3}/_{5}$ rounded to the nearest $1/_{2}$ is $3^{1}/_{2}$.

5.4 Equivalence and applications

The concept of equivalence is that it is possible for a number to be the same but the numeral or way it is represented could be different. Because of their nature, more than one fraction name applies to each fraction. For example, $\frac{2}{3}$ of a cake is the same size as $\frac{4}{6}$ of the cake. This makes equivalent fractions an example of the equivalence principle (equivalence is equality by adding 0 or, in this case, multiplying by 1, so $\frac{2}{3} \times 1 = \frac{2}{3}$; $\frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$; $\frac{2}{3} \times \frac{3}{3} = \frac{6}{9}$; and so on). There are any number of other fractions that have the same value as, or are equivalent to, any fractions. As a fraction is a whole (or a number of wholes) partitioned into smaller parts the continued partitioning into smaller parts will result in equivalent fractions that look different but represent the same value.

When dealing with equivalence, students need to be able to partition (divide wholes into parts) and unitise (see parts as a whole). If $\frac{1}{3}$ of a whole is compared with $\frac{2}{6}$, then students need to understand $\frac{2}{6}$ as one group of two out of three groups of two seen as one whole. This ability to see groups of groups as a whole is called
reunitising (as it requires unitising for the group of two to be seen as a unit and then again when the three groups of two is the whole).



5.4.1 Sequence and material for teaching equivalent fractions

Equivalence is where one fraction has two (or more) symbolic representations, e.g. $^{2}/_{3} = ^{4}/_{6}$. The sequence for teaching equivalence is as follows:

REAL-WORLD SITUATIONS OF EQUIVALENCE (e.g. showing that $\frac{1}{2} = \frac{2}{4}$ for a cake)

3D EQUIVALENCE SITUATIONS (showing $\frac{2}{3} = \frac{4}{6}$ with fraction discs, etc.)

DEVELOPING SEQUENCES OF EQUIVALENT FRACTIONS (using paper folding or discs and strips to find a sequence of fractions)

MORE SEQUENCE WORK WITH 2D (using fraction mats to write down sequences)

LOOKING FOR A PATTERN (using existing sequences and fraction sticks to find the pattern that determines equivalence)

PRACTISING THE PATTERN

There is a variety of materials and some interesting techniques to achieve this sequence.

- 5. **Real world**. Cakes, pizzas, liquorice, chocolate bars, and so on.
- 6. **Paper folding.** If an A4 sheet is folded longways and half shaded then it forms a foundation for equivalence. If a set of these A4 sheets is folded:

in half shortways – it shows $\frac{1}{2} = \frac{2}{4}$ in thirds shortways – it shows $\frac{1}{2} = \frac{3}{6}$ in fourths shortways – it shows $\frac{1}{2} = \frac{4}{8}$ in fifths shortways – it shows $\frac{1}{2} = \frac{5}{10}$ and so on ...

The basis of the folding is that $^{2}/_{4}$ has to be seen as "1 lot of 2" in "2 lots of 2" (as a whole) to enable $^{2}/_{4}$ to be seen as the same as $^{1}/_{2}$. This is an example of reunitising and is difficult for students to do. However, when $^{1}/_{2}$ is folded the other way into three, both the 1 part and the whole of 2 parts are broken into three – showing $^{3}/_{6} = ^{1}/_{2}$. This provides the starting point for the equivalence pattern.

- 7. Fraction discs and strips. To compare $\frac{2}{3}$ with $\frac{4}{6}$, the disc material for each is taken out, one material is placed on a whole and the other *on top* of the first material to see whether they cover the same area. For fraction strips, the two materials are compared by placing the material beside each other and looking to see if they have the same length.
- 8. Bases and overlays. One technique for equivalence is to have a square, say, with half of the square shaded (this is a "base"). Then a number of plastic see-through overlays are made of the square cut into halves, thirds, quarters, fifths, etc. all the same way. Then if the overlays are put on top of the base, a sequence of equivalent fractions is determined by how the overlay cuts up the half.
- 9. **Fraction mats**. These are a mat with length examples of ones, halves, thirds, and so on that can be visually compared. An example is provided later.
- 10. Fraction sticks. These are strips of paper that show equivalence. Examples are given later.

If a sequence of equivalent fractions is written down, then many patterns can be seen in the sequence, say, $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, and so on. The obvious sequence is that the "tops" increase by 2 while the "bottoms" increase by 3. However, the **equivalence pattern** required is: the numerator and denominator both change by the same multiplier or divisor – this relates to multiplication by 1 which equals $\frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5}$, and so on. A consequence of this is that equivalent fractions are fractions that can be cancelled down to the same starting fraction.

5.4.2 Activities for teaching equivalent fractions

The sequence of materials, real world, paper folding, discs and strips, fraction mats and fraction sticks appears to offer some opportunity in teaching equivalent fractions. A sequence of activities is provided below using the RAMR framework.

Materials: See after activities.

Reality

Kinaesthetic activities

<u>Real-world material and meaning</u>. A round cake can be used to show simple equivalence. Cut the cake into two halves. Cut one half into two quarters. Show that $^{2}/_{4}$ is equal to the $^{1}/_{2}$ by comparing the amount of cake in the half and the two quarters. The following could be used to show the equivalences: (a) 12×8 chocolate block – show $\frac{1}{3} = \frac{2}{6}$; (b) a long piece of liquorice – show $\frac{3}{4} = \frac{6}{8}$.

Sequences

Abstraction

Physical material and meaning

- Area (fraction discs). Fraction circles can be used to compare the following, by placing material on top of each other, and determine which pair is equivalent: (a) $\frac{1}{2}$ and $\frac{2}{4}$, (b) $\frac{3}{4}$ and $\frac{2}{3}$, (c) $\frac{4}{6}$ and $\frac{2}{3}$.
- Length (fraction strips). Fraction strips can be used to compare the following, by placing material beside each other, and determine which pair is equivalent: (a) $\frac{1}{3}$ and $\frac{2}{6}$, (b) $\frac{3}{4}$ and $\frac{4}{5}$, (c) $\frac{1}{2}$ and $\frac{3}{6}$.
- Set (Unifix). Use sets of Unifix to compare the following, by partitioning two sets of Unifix material and comparing the result, and determine which pair is equivalent: (a) 12 Unifix – $\frac{1}{3}$ and $\frac{2}{6}$, (b) 20 Unifix – $\frac{3}{5}$ and $\frac{6}{10}$, (c) 28 Unifix – $\frac{3}{4}$ and $\frac{5}{7}$.
- Base boards and fraction pieces. Make a set of base boards as on right and a set of fraction pieces that fit into the base boards – halves, thirds, quarters, fifths and sixths. When looking for an equivalent fraction, first find the appropriate base board and then fill it in with fraction pieces. Then the focus is on the parts – e.g. the third base board down on right fits into the first!
- Overlays and base boards. Make a set of base boards as on right and then a set of plastic overlays with the *fraction shapes turned 90 degrees*. Shade fractions on the base board and then use overlays on top of these fractions to show that $\frac{2}{3}$ is equivalent to $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$ and so on. Place and remove overlay to see relation (can be done really well with computer).

Physical material and sequences of equivalent fractions

Fold two rectangles of paper longways. Shade one half of each. Take one of the shadings and fold it the other way - into half, thirds, fourths, etc. In this way, we can develop a sequence of equivalent fractions equal to $\frac{1}{2}$: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \dots$ and so on. Use the same method to develop a sequence of fractions equivalent to $^{2}/_{3}$.













Pictorial material and sequences of equivalent fractions (fraction mat)

Use a ruler and a fraction mat (see below) to do the following: (a) find if the following fractions are equivalent: (i) ³/₄ and ⁹/₁₂, (ii) ²/₅ and ⁴/₉, (iii) ⁶/₈ and ⁹/₁₂; and (b) develop a sequence of fractions equivalent to ¹/₃.

Patterning material and pattern for equivalent fractions (fraction sticks)

- Make a set of fraction sticks (see next page). Pull out the 2 and 5 stick and make $\frac{2}{5}$. Note that all equivalent fractions are shown. Look for a pattern that shows they are equivalent. Note that students tend to see this pattern additively $-\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \dots$ is seen as adding 2 on the "top" and adding 3 on the "bottom"; students need to see this pattern multiplicatively $-\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \dots$ needs to be seen as multiplying "top" and "bottom" by the same amount $-\frac{2}{3} \times \frac{2}{2}, \frac{2}{3} \times \frac{3}{3}$, and so on.
- Repeat this activity for other fractions until students realise that two fractions are equivalent if **they are both a "multiply top and bottom by same amount" of the same fraction** that is, they "cancel down" to **the same fraction**.

Note: this is an equivalence of expression example – multiplying by $^{2}/_{2}$ and $^{3}/_{3}$ and so on is the same as multiplying by 1 so the fraction is not changed.

Fraction mat

	1 whole																
1 half $\frac{1}{2}$									1 half $\frac{1}{2}$								
	1 th	ird -	$\frac{1}{3}$					1 thi	rd $\frac{1}{3}$ 1 third $\frac{1}{3}$								
1 fo	ourth	$\frac{1}{4}$			1 f	ourth	$1\frac{1}{4}$	-		1 fourth $\frac{1}{4}$				1 fourth $\frac{1}{4}$			
1 fift	$h \frac{1}{5}$		1	L fift	h -	1 5		1 fift	th $\frac{1}{5}$ 1 f			fift	$h \frac{1}{5}$		-	1 fif	th $\frac{1}{5}$
$\frac{1}{\frac{1}{6}}$	th		1 si $\frac{1}{6}$	xth		1	$\frac{1}{6}$	th	$\begin{array}{c c} 1 \text{ sixth} & 1 \text{ s} \\ \frac{1}{6} & \frac{1}{6} \end{array}$			1 si <u>1</u>	sixth 1 6		1 sixth $\frac{1}{6}$		
$\frac{1}{\frac{1}{7}}$:h	$ \begin{array}{c} 1\\ \text{seventh}\\ \frac{1}{7} \end{array} $		se	1 eventh sev $\frac{1}{7}$		seve 1 7	l enth 7	:	1 seven <u>1</u> 7	ith	s	1 ever <u>1</u> 7	nth	S	$ \frac{1}{\frac{1}{7}} $	
1 eighth <u>1</u> 8		1 eight <u>1</u> 8	th	eig	1 ght <u>1</u> 8	th eighth $\frac{1}{\frac{1}{8}}$		eig	1 hth 1 8	ı e	1 igh <u>1</u> 8	th	ei	1 ghth <u>1</u> 8		1 eighth <u>1</u> 8	
1 ninth <u>1</u> 9	ni	1 inth <u>1</u> 9	r	1 nintł <u>1</u> 9	ı	1 nint <u>1</u> 9	th	1 nir 1 ç	l hth	ni	1 inth <u>1</u> 9	n	1 inth <u>1</u> 9	n	1 nint <u>1</u> 9	'n	1 ninth <u>1</u> 9
1tenth $\frac{1}{10}$	1 ten <u>1</u> 1(ith 0	$\frac{1}{10}$	l 1th 0	te	1 enth <u>1</u> 10	te	1 enth <u>1</u> 10	1 ten <u>1</u> 1(th)	1 tent <u>1</u> 10	:h	$\frac{1}{1}$	th L 0	1 ter 1 1	1 nth L0	1tenth $\frac{1}{10}$

Fraction sticks

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

The fraction sticks are an interesting material. A set is like below:

A fraction is made by putting two rows together, for example, $\frac{2}{5}$:

2	4	6	8	10	12	14	16	18	20
5	10	15	20	25	30	35	40	45	50

This can be used to show the pattern for equivalent fractions. Two of these can be used to compare, add or subtract fractions (e.g. $\frac{3}{7}$ and $\frac{2}{5}$) by aligning the "common denominator":

		3	6	9	12	15	18	21	24	27	30
		7	14	21	28	35	42	49	56	63	70
2	4	6	8	10	12	14	16	18	20		
5	10	15	20	25	30	35	40	45	50		

5.4.3 Applications of equivalence – comparison and order

There are many applications of equivalence. For example, comparison, addition and subtraction for unlike denominator fractions – construct a set of fraction sticks: (a) Make $\frac{2}{5}$ out of the 2 and the 5 stick. Look for the patterns that determine if two fractions are equivalent. (b) Make $\frac{2}{5}$ and $\frac{3}{7}$, find the common denominator and compare the fractions. Which is bigger? What is their sum if we add the equivalent common denominator fractions? (c) Which is bigger of $\frac{3}{4}$ and $\frac{7}{9}$? Subtract the equivalent common denominator fractions for $\frac{3}{4}$ from $\frac{7}{9}$.

In this subsection, we will look at comparison and order – operations are for the next book. First, let us recap – comparison is working out which fraction is larger and there are four ways to keep in mind – like denominators, unlike denominators but like numerators, benchmarking, and unlike denominators using equivalent fractions – see section 5.3.4.

The way that enables all fractions to be compared (and ranked) is equivalence. This suits any example, say comparing $\frac{3}{7}$ and $\frac{2}{5}$ (as in fraction sticks example above). Here, the fractions are converted to the same denominator using equivalence and then compared. This is done by moving the 3 and 7 sticks and forming equivalent fractions and the 2 and 5 sticks and forming equivalent fractions. If we look along the denominators of the two equivalent fraction double sticks, we see that $\frac{3}{7}$ and $\frac{2}{5}$ have a common denominator in one of their equivalent fractions – 35ths. We see that $\frac{3}{7} = \frac{15}{35}$ and $\frac{2}{5} = \frac{14}{35}$. Thus, $\frac{3}{7}$ is slightly larger than $\frac{2}{5}$.

However, here the sticks do all the work. We must use the pattern in the sticks to learn the process so we can do it without the sticks. If we do many examples, we will see that the common denominator is a multiple of the two denominators (or a factor of this) which is 35. Then we use the equivalence:

 $\frac{3}{7} = \frac{2}{35}$ equivalence is multiply by same on top and bottom so as 7 × 5 = 35, then top (the 3) must also be multiplied by 5 so $\frac{3}{7} = \frac{3}{7} \times \frac{5}{5} = \frac{15}{35}$

Similarly $^{2}/_{5}$ = $^{2}/_{35}$ means × $^{7}/_{7}$ and so becomes $^{2}/_{5}$ × $^{7}/_{7}$ = $^{14}/_{35}$

In algebra terms, this means that the equivalent fractions for a/b and c/d are ad/bd and cb/db or bc/bd, as common denominator is bd.

5.5 Directed number, rationals and the rational number system

This subsection briefly discusses directed numbers, the concept of rationals and what this means for the rational number system when working in fractions.

5.5.1 Directed number

Directed number applies to common fractions as it applies to whole and decimal numbers. The vertical number line is still the most useful for representing directed fractions. There is the positives (above zero) and the negatives (below zero) and these can be fractions as the figure on the right shows.

Once again, measures are the best to introduce this idea – however, not many measures are common fractions and not decimals. In fact time and angle are the only ones that could provide some variety – weeks and days are sevenths, hours and minutes are sixtieths, and angle is 360ths. With some creativity, hours could be quarters and we could have a countdown – as they say in Houston: 4, 3, 2, 1, 0, -1, -2, and so on. We could do this with ¼ hours.

5.5.2 Rationals and irrationals

From common fractions emerges the concept of **rationals** or rational number, a number that can be represented by a fraction which is whole number "over" whole number.



The Pythagoreans believed that whole numbers were the way the creator created the world. They were not worried about common fractions because they were made from whole numbers and were therefore rational (logical). However, they ran up against a problem with square root of 2. If it is assumed that the square root of 2 is a common fraction p/q fully cancelled down, then $2 = p^2/q^2$. This means that p^2 had to be even which means that p had to be even, but this means that q had to be even so that $\frac{p \times p}{q \times q} = 2$, which means that p and q are not cancelled down. Thus, the square root of 2 is not a common fraction, which means it is something else (and ungodly), therefore it is **irrational**. Many more irrationals were then found such as pi.

It is possible to show that all fractions, when divided out to make a decimal, are a **finite decimal** (end after a number of decimal place-value digits – e.g. $\frac{3}{8} = 0.375$) or a **repeating decimal** ($\frac{2}{7} = 0.285714285714$ repeating). Therefore, all the decimals which are **not finite and do not repeat** are irrationals. This means that there are an infinite number of rationals and an infinite number of irrationals.

The positive rationals can be written down in an order that shows that all will be included by advancing the denominator a counting number at a time.



These rationals can be counted in the order shown by the arrows, so they are **countably infinite**. It has been show that the irrationals are **uncountably infinite**. Both rationals and irrationals are always dense; there is an infinite number of them between any two numbers. Of course, being uncountable, irrationals are denser.

5.5.3 Rational number system

The rational number system is straightforward after the real number system. It is the real number system made up of only decimals that are finite or repeating.

6 Percent, Rate and Ratio

Percent, rate and ratio are **similar and yet different**. They can all be considered as **multiplicative comparison** but they have very **different notation**. Percent is based on decimals and common fractions, rate is largely based on decimals (and division), while ratio is based or related strongly to common fractions. They could all be expressed in decimal forms – e.g. 7.5% could be represented as 0.075, 5:2 could be represented as 2.5 (dividing first term by second term), and rate is normally a decimal.

However, we do have the three different notations and we have to take into account that they **cause differences.** For example, percent and rate can be directly compared, particularly in their decimal form, while ratio, being part-to-part, requires change to fraction or to equivalent ratio for comparison (just like fractions). Obviously, a higher cement mixture would be formed from sand:cement of 5:3 than 5:2. However, for two different ratio without a common number, say 5:2 and 7:3, we need to obtain an equivalent ratio where one number is in common. For example, 5:2 and 7:3 have a common first term of 35 giving 35:14 and 35:15, or a common second term of 6 giving 15:6 and 14:6. In both cases, the second ratio 7:3 gives the stronger cement mixture.



The sequence we will follow for this chapter is as on right. And we follow the same structure of big ideas as for the previous three topics (whole numbers, decimal numbers, and common fractions):

Part-whole/Notion of a unit Additive and multiplicative structure Continuous vs discrete/Number line Equivalence (proportion) Common models and meanings (extension)

6.1 Part-whole

Percent, rate and ratio all relate to part-whole but in different ways – <u>percent</u> are **hundredth**s so they partition a whole into 100 equal parts, <u>rate</u> is a decimal formed by **division** which is also part-whole, while <u>ratio</u> is based on part-whole but really comes from comparing the **part to the other part**.

6.1.1 Early "per", rate and ratio

One of the bases of the YuMi Deadly Maths program is that all mathematics should be pre-empted in an earlier grade. So, we should be doing work to lead into percent, rate and ratio in the early years.

Percent. Percent is a fraction out of 100 and a decimal that is hundredths. It just uses a new word "per" instead of fraction or decimal words. So can we bring this word in earlier – by relating $\frac{2}{5}$ to "2 per 5". Similar to the fraction where we have $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ and one, we have 1 perfive, 2 perfive, 3 perfive, 4 perfive and 5 perfive which is 1 as 100 percent is. We could do "per10" and "per20".

We could explore things like "per20". The number of shaded circles would then be "eight per20" (as each circle represents "four per20"). This could lead to "per100" which can be renamed as "percent" because 100 is centum in Latin.



Rate. Rate can be introduced early because it is used throughout early mathematics without being recognised. For example, 3 bags of lollies with 4 lollies in each bag is really 3 × a rate because 4 lollies/bag is a rate. In fact,

some mathematics educators argue that all simple multiplication is either number \times rate (e.g. 3 bags \times 4 lollies/bag = 12 lollies) or a rate \times rate (e.g. 3 bags/total \times 4 lollies/bag = 12 lollies/total).

Ratio. Ratio is related to fraction. In the example on right, the fraction is $\frac{2}{5}$. However, there are three components to the diagram (part – 2 counters; other part – 3 counters, and total – 5 counters). The fraction shaded in is 2 parts out of 5 parts as



one whole = $\frac{2}{5}$. The ratio is part to other part and is 2 to 3, or 2:3. Thus, why not introduce ratio when introducing fraction. From the start, consider a fraction as part-whole but also say there is a third component (the unshaded circle – the other part). This will mean three things – fraction $\frac{2}{5}$ (part-whole), fraction $\frac{3}{5}$ (other part-whole) and ratio 2:3 (part-part).

6.1.2 Percent (%) as a fraction and a decimal

This is a special type of fraction which is always parts per hundred or hundredths (e.g. 42% = 42 hundredths = 0.42). Percent as per-cent means per hundred, that is, a common fraction. Thus 37% really means 37 hundredths or 37 parts per hundred. The best way to think of this using a 10×10 grid as below:



This means that 100% is 100 hundredths, one whole, and that percentages over 100 are greater than one, for example:



In terms of decimal factors, this means that percent are when the decimal points moves to the hundredth (making this position the one).



Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
			3	6	7•	4

3.674 = 367.4%

6.1.3 Conversions and estimation

Once percent is known, it is important to be able to convert between fractions, decimals and percent. For an example activity, construct a number line and label one end 0 and the other end 1. Make up fractions, decimal and percent examples on paper (include examples that are the same) and give these randomly to students. Students place the numbers on the line, discussing where they go, and why they are the same thing. Extend on this by discussing rules for changing from one notation to the other two. This builds flexibility in percent.

It is also important to gain an understanding that allows percentage to be estimated in terms of length between zero and 100. One early activity by which to do this is as follows.

- 1. Teacher obtains a 1L measuring cylinder (or tall thin jug that you can calibrate) and coloured water.
- Teacher pours coloured water into cylinder students guess percentage full, and record guesses and accuracy (how far out); after each guess, give answer so that students can record errors. Begin with multiples of 10% (100 mL) in cylinder, then multiples of 5% (50 mL) and finally examples with multiples of 1% (10 mL).
- 3. Students record and graph errors and find averages for five trials. Students can also consider if it is possible and appropriate to find error in terms of percentage of amount being estimated (and to average this).

There is a simple game to learn what percent (%) means. It involves two players (A and B) and one calculator (or two calculators exchanged).

- Player A takes the calculator and chooses a number between 9 and 100, hides it from player B, then presses [number] [÷] [number] [%] without B seeing what is being pressed. Player A does not press the [=] key. The calculator should read 100%.
- Player A gives the calculator to player B who has to guess the number. B does not press clear! B presses [guess] [%]. And continues this until they guess correctly. B should get a % to help with the next guess (below 100% is too low, above 100% is too high). B keeps using the [%] button to make guesses until they guess the number (calculator will show 100%).
- 3. Player B's score is the number of guesses it takes to reach the correct number. Now player B sets up the calculator for player A who now has to guess and scores the number of guesses. Five trials each and the winner is the lowest score.

6.2 Additive and multiplicative structure

There is very little that is additive in percent, rate and ratio. Multiplicative is the structure which connects percent, rate and ratio.

6.2.1 Additive structure

The only part that is additive is when percent is a decimal and can be considered as hundredths – for example, 37.5% is 0.375 – then it has the additive and counting properties of decimal numbers.

6.2.2 Multiplicative structure

Along with rate and ratio, percent can be considered as multiplicative comparison. We can see this by looking at the three comparison possibilities; for example, two numbers 4 and 20 can be compared in three ways:

- (a) by number 20 is bigger than 4
- (b) by addition 20 is 16 more than 4
- (c) by multiplication 20 is 5 times as much as 4

Multiplicative comparison is percent, rate and ratio – 75 percent is a comparison by saying \$60 is 75% of \$80; rate \$1.40 per litre is a comparison by saying 10 litres is \$14 because \$1.4/L; and ratio is a comparison because sand to cement is 5:2.

Comparison of rate and ratio

Ratio differs from rate in that ratio is comparison between like attributes. That is, 15L to 6L is ratio while 15L for \$6 is rate. Look at the table below. The shaded areas are ratio.



Thus, rate and ratio are different in three ways:

- Ratio compares the same attributes (for example, sand and cement in mass are 5:2 that is 5 tonnes of sand is used with 2 tonnes of cement); rate compares different attributes (for example, the cost of petrol is \$1.20 per litre which compares money to volume).
- 2. Ratio uses rotation similar to fractions in that there are 2 whole numbers (but part-to-part, not part-to-whole for example, sand to cement is 5:2; rate uses a single number, for example, \$1.20 per litre). Rate could really be considered as a ratio with 1 as the second number (for example, \$ to litres is \$1.20:1 litre), it uses the single number and is like a fraction.
- Ratio problems are worked out by using proportion or equivalent ratio (for example, if sand to cement is 5:2 and we have 8 tonnes of cement, then we need 20 tonnes of sand as 20:8 is the same as 5:2); rate problems use multiplication (for example, if petrol is \$1.20 per litre then 4 litres is \$1.20 × 4 = \$4.80).

Multiplicative comparison

Multiplicative comparison is best understood as **change** – there is a start, a multiplier, and an end. This can be represented with arrows. Also, there is inverse – the end can change to the start and this uses division.

- Whole number. I have 4× as much as you do you have \$3, I have 4×\$3 = \$12.
- *Fraction*. Jack has $\frac{3}{4}$ of what Jill has; Jill has \$16 and Jack has $\frac{3}{4} \times $16 = 12 .
- *Percent.* Sue pays 40% of cost; cost was \$80, Sue pays 0.4 × \$80 = \$32.
- *Rate.* Pears are \$6.50 a kg; 5 kg of pears is 5 × \$6.50 = \$32.50.
- *Ratio.* Sand to cement is 3:2; the truck brings 6 tonne of sand; this means 2 × 2 = 4 tonne of cement.

The **Function Machine** is excellent in building this idea – see YDM Algebra book.

6.3 Continuous vs discrete/Number-line model

Percent and rate can be placed on a line and rank, order and estimation worked out because both percent and rate are decimals and decimals can be placed on a number line. Ratio cannot be placed on a line because it is part-to-part.

Now lines can be contracted and extended and can show relationships (see YDM Algebra book). This means that lines can show multiplicative relationship, for example, length of A is 3 × length of B.



This leads to many activities where length can be modified:

6.3.1 Stretching and squishing

Computers can be used for ratio and proportion because shapes can be easily changed by pulling on the edges or boundary indicators of the shape.

Example: Computer madness

Use PowerPoint to draw 2 identical people:

- Change A so that height A:B is 2:1
- Change B so that width A:B is 2:3
- Change A so height A:B is 1:3 and width A:B is 1:2
- Play with the two people, stretching and squishing. Can you make size A:B as 2:1?

Construct two identical candles X and Y on PowerPoint:

Change X and Y so that

- height X:Y is 1:4
- height X:Y is 3:4
- width X:Y is 9:1
- height X:Y is 3:2 and width X:Y is 2:1.

6.3.2 Developing the double number line

Two lines can be stretched and squished differently on a computer. This can also be done with elastic.

Construct two identical rulers P and Q on PowerPoint:







Stretch P so P:Q is 2:3 – that is, the 2 of P aligns with the 3 of Q.



Then, proportions can be seen, for example – "If P = 6, what is Q?" [Answer 9 as is evident from the diagram.] Other ratios can also be seen in the diagram – "What other ratios are evident?".

2:3 = ____ = ___ = ___ ?

From this we can see that 2:3, 4:6 and 14:21 are all in the same ratio. Then we can **explore multiples** below and **notice patterns** – that multiples are the same if ratios are in proportion.



Note: We can reverse and work out proportions by keeping multiples the same.



We can transfer this to both sides of a line as follows, then the change on both sides is the same.



This can be reversed to find answers to multiplicative comparison problems! That is, find ? = 8 and ?? = 24.



This is the double number line method.

6.4 Equivalent ratio or proportion

6.4.1 Developing proportion

There are three ways of dealing with proportion.

Cut squares into 3 parts, then 2:3 = 6:9





2. **Combining objects**. This method shows the reverse direction of the relationships shown above. For example, 4:6 could be shown with blocks, and the blocks could be put into pairs to show that 4:6 is 2:3 as below.



3. Proportion sticks. These are like fraction sticks but vertical not horizontal.

Difficulties with equivalent ratios or proportion can be helped by proportion sticks as on right.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

For 5:2, put the 5 stick beside the 2 stick (as on left); this shows that 5:2 = 10:4 = 15:6 and so on. They act like horizontal fraction sticks and can be used to find equivalent ratios, the rule for equivalent ratios or proportion (i.e. equivalent ratios cancel down to same proportion) or to compare proportions.

Similar to equivalent fraction, proportion should be taught through three steps:

- (a) first show that proportion is possible that, e.g. 2:3 = 4:6;
- (b) then show that proportion divides ratios into sequences such as 5:2 = 10:4 = 15:6 = 20:8 = ... and so on; and
- (c) finally show that the pattern relating ratios in the sequences is that the starting numbers are multiplied by 2 (5:2 = $5 \times 2:2 \times 2 = 10:4$), by 3 (5:2 = $5 \times 3:2 \times 3 = 15:6$), by 4 (5:2 = $5 \times 4:2 \times 4 = 20:8$), and so on.

The rule for proportion is that two ratios are in proportion if they cancel down to the same starting ratio, e.g. 60:24 is **in proportion** to 125:50 because the first = 5:2 if you divide both numbers by 12 and the second = 5:2 if you divide both numbers by 25.

6.4.2 Conversions

This is the ability to convert between the different forms of fraction – common fraction, decimal, percent and ratio. For example, the following are equivalent – the common fraction $\frac{2}{5}$ or $\frac{40}{100}$, decimal number 0.4, percent 40%, and ratio 2:3 or 40:60.

5	2
10	4
15	6
20	8
25	10
30	12
35	14
40	16
45	18
50	20

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6.5 Common models/meanings for percent, rate and ratio

There are 3 types of models – **set/area**, **double number line**, and **change**. These emerge from percent being a multiplier or an operator (like fraction). Thus percent, rate and ratio can be seen as mathematically similar.

6.5.1 Percent

Similar to rate and ratio, percent is a form of multiplicative comparison. The three types of models are as follows.



6.5.2 Rate

Rate is also represented by the three types of models – area/set, double number line, and change. However, like fraction, rate is conceptually something that multiplies – acts like a multiplicative operator or multiplier. Multiplication can be taught as rate by number (for example, 3 lollies per bag \times 4 bags = 12 lollies). Note that the attribute for the number is the same as the second attribute for the rate and the answer is the first attribute of the rate.

Area/set/size. This is only useful for simple rates like "\$5 per kg". If each \$ is represented by an object and the kg as one grouping of these objects, then 6 kg is 6 of these groupings. In this example this would be 6 lots of \$5 = \$30. Similarly, if we have \$15 then we have 3 groupings of \$5 so we can buy 3 kg.





Change (multiplier). Rate can be considered as a multiplication change from second to first attribute, for example, 4 L changes to \$6 by change \$1.20/litre.

(a) 4L is \$?
4 L
$$\xrightarrow{\times 1.20}$$
 \$?
? = 4 × \$1.20 = \$4.80
(b) \$6 is ?L
?L $\xrightarrow{\times 1.20}$ \$6.00
? = 6 / 1.20 = 5L

Note: End unknown is multiplication, while start unknown and change or multiplier unknown is division.

6.5.3 Ratio

Again there are the three models, as follows.

Area/set/size. The first of these is to look at ratio in terms of area or set. For example, if cordial to water is 1:2 then the ratio can be considered in terms of the diagram on the right.

This has two consequences: (a) it enables us to work out answers to problems; for example, if there is 8 L of cordial, it means that each square is 8 and thus we need 16 L of water; and (b) it involves a total as well as the parts – if there are 8 L of cordial and 16 L of water, then this means 24 L of mixture. This means that ratio normally differs from fraction in that fraction is part-to-whole while ratio is part-to-part.

Double number line. The second model is the **double number line**. This is a good model for problem solving:



In this model, the top of the horizontal line is one thing (here sand for the ratio sand:cement is 5:2) while the bottom of the line is the other (here cement). A vertical line is put in to show ratio (the 5:2). Then if we want how much sand for 6 tonnes of cement, the 6 is the bottom of another vertical line and the top of this vertical line is the answer. This unknown is worked out by ensuring that the change between the bottom numbers is the same as the change between the top numbers (which makes, as in the diagram, the answer = $5 \times 3 = 15$ t of sand).

Change. The third model is ratio as **change** – if sand to cement is 5:2 then the amount of cement is $\frac{2}{5}$ or 0.4 times the amount of sand – this leads to the following picture of ratio as change:

 $\begin{array}{ccc} & \times \ ^2/_5 & \mbox{OR} & \times \ 0.4 \\ \mbox{Sand} & \black \rightarrow & \mbox{cement} & \mbox{Sand} & \black \rightarrow & \mbox{cement} \end{array}$

cordial

water

7.1 Content framework for number

The teaching framework organises the content for number into a framework of four topics. Each of these topics is partitioned into sub-topics. Each sub-topic is described and any concepts or strategies used in the teaching framework are listed. They are also related to big ideas. Topics and sub-topics are chosen so as to represent ideas that recur across all year levels. The resulting framework is given in Table 1. Sequences based on this framework are provided in section 7.2 up to Year 7.

ТОРІС	SUB-TOPIC	DESCRIPTION AND CONCEPTS/STRATEGIES/WAYS	BIG IDEAS
	Attributes	Attribute recognition; sorting and classifying	
D	Counting	Notion of unit, counting (e.g. rote, rational, robust), 1:1 and 1:many correspondence; subitising	Part-whole Continuous vs discrete
Pre-number	Reading and writing numbers	Language, materials, models, symbols Payne and Rathmell triangle	Part-whole
	Logical connectives	Venn diagrams, logical connectives ('and', 'or' 'not' and 'ifthen')	Boolean logic
	Place value/ Counting/ Mutiplicative structure	HTO structure, reading and writing, pattern of threes Counting, seriation and odometer Multiplicative relationships (moving PV left, ×10; moving PV right, ÷10), renaming	Part-whole Odometer Additive structure Multiplicative structure
Whole number numeration	Number line	Quantity, rank (each number a single point on number line), comparison and order, density Rounding and estimating	Continuous vs discrete
	Position of zero	Representing one number in many ways, use of zero	Equivalence
	Directed number	Negative numbers; relation between negatives and positives	
	Systems	Extending to whole number system; continuous models	
	Place value/ Counting/ Mutiplicative structure	Tenths, hundredths, thousandths; HTO pattern of threes Seriation, odometer Multiplicative relationships (moving PV left, ×10; moving PV right, ÷10), renaming, flexibility	Part-whole Odometer Additive structure Multiplicative structure
Decimal number numeration	Continuous vs discrete	Rank (each number a single point on number line); Density; Rounding and estimating	Continuous vs discrete
	Position of zeros	Representing decimals in many ways, use of zero	Equivalence
	Directed number	Negative numbers; Relation between negatives and positives	
	Real numbers	Extending Real number system; Continuous models	
	Fraction concepts	Part of a whole/set, number line, division, operator	Part-whole, Mult struct, Continuous vs discrete
Fractions	Mixed numbers	Wholes and parts, improper fractions, representations using many models (area, set, number line), counting, order	Part-whole, Mult struct, Add struct, Odometer
	Equivalent fractions	Equal fractions, ordering fractions	Equivalence
	Early, rate, ratio and use of "per"	Representation using many models	Part-whole, Mult struct, Continuous vs discrete
Percent, rate	Multiplicative comparison	Multiplication as change	Change vs relationship Mult struct, Inverse
and ratio	Meanings and models	Fraction and decimal meanings, area, set, double number line and change models	Continuous vs discrete Inverse
	Equivalence and conversions	Proportion	Equivalence

Table 1. Framework for teaching number

7.2 Yearly teaching frameworks for number

SUB-TOPIC	YEAR PP	YEAR P
Counting	Sorting/Correspondence – experience sorting objects by attribute, putting sets into 1-1 correspondence, experience 1:1 and 1:many correspondence. Rote – forwards to 10. Rational – experience making and counting collections to 10, match 1-1 to compare collections. Symbol recognition – experience identifying numerals of personal significance (e.g. age); experience real world, language, set/line models, up to 10 (e.g. storytelling, forming sets of objects, acting out story on an un- numbered number track).	 Sorting/Correspondence – reinforce sorting objects by attribute, experience; experience 1:1 and 1:many correspondence with number names. Rote – rote counting forwards/backwards to 20, experience ordinal numbers to 20. Rational –rationally counting to 20, identifying 1-1 with objects to 20, with and without counting (subitising); count robustly (recognising can start and end count where wish and can count groups and objects in groups). Symbol recognition – introduce identifying digits and relate to objects to 20, introduce number names as words and relate to symbols and collections of objects to 20; reinforce connection between real world, language, set/line models, symbols, up to 20 (e.g. counting on a numbered number track). Odometer – experience what happens to ones in symbolic counting across the ten.
Whole number and decimal concepts	<i>Place value</i> – experience early group activities (put objects into groups, say how many groups and ones leftover) at subitisation level (up to 5 in a group).	<i>Place value</i> –undertake early grouping activities with material and on a groups/ones chart (groups on left of ones), stating groups and leftover ones; relate to tens and one in teens. <i>Rank</i> – experience number on a number track or ladder to 20.
Whole number and decimal processes	<i>Reading/Writing</i> – experience drawing numbers of personal significance. <i>Comparing/Ordering</i> – compare groups of objects to determine more or less.	Reading/Writing – numbers to 20 (using tens and ones). Comparing/Ordering – numbers to 20 (comparing collections or using a number track, ladder or line); experience terms first and second to indicate order.
Fraction concepts	Experience partitioning whole and forming groups (equal and unequal).	Use everyday language for fractions (e.g. slice, piece) and experience partitioning objects into equal parts (e.g. cutting a cake).

SUB-TOPIC	YEAR 1, SEMESTER 1	YEAR 1, SEMESTER 2
Counting	<i>Sorting/Correspondence</i> – sort objects of different attribute (e.g. same colour but different shape); experiencing 1:1 and 1:many [notion of unit big idea].	Sorting/Correspondence – sort objects of different attributes; reinforce 1:many correspondence (particularly 1:10).
	<i>Rote</i> – experience forwards/backwards to 20 and 40 by 1s and 2s, strategies to count collections (song and rhyme), introduce ordinal numbers to 40.	<i>Rote</i> – forwards/backwards to 100 by 1s, 2s, 5s and 10s, reinforce ordinal numbers to 100; discuss different things that could be counted (e.g. objects,
	Rational – make and count collections to 50, count out a subset from a set up	groups, steps, position along number line).
	to 50.	<i>Rational</i> – to 100, counting out subset (last number names tell how many).
	Symbol recognition – reinforce digit recognition to 40 (real world, set, line, language, symbol).	<i>Symbol recognition</i> – introduce symbols to 100.
Whole number	<i>Place value</i> – real world objects up to 40, language, set/line models; symbols	<i>Place</i> value – to 100 [look at notion of unit for one and tens].
and decimal	to 40, experience number names to 100.	<i>Rank</i> – to 100.
concepts	<i>Odometer</i> – to 40.	<i>Odometer</i> – to 100 [discuss odometer big idea].
	<i>Rank</i> – to 40; discuss difference in number on number line to number as	<i>Rank</i> – to 100 [continue discussion continuous vs discrete big idea].
	applied to a collection [continuous vs discrete big idea].	<i>Multiplicative structure</i> – experience to 100.
Whole number	Reading/Writing – numbers to 40.	<i>Reading/Writing</i> – numbers to 100.
and decimal	Seriation – experience 1 more/less, 10 more/less to 40.	Seriation – 1 more/less, 10 more/less to 100.
processes	<i>Comparing/Ordering</i> – up to 40.	<i>Comparing/Ordering</i> – up to 100.
Fraction concepts	<i>Common fractions</i> – experience partitioning and unitising, in particular, making haves and rejoining to make wholes (area model).	<i>Common fractions</i> –reinforce "whole" and "half" as "part of a whole" and part of a set" experience half of a half (quarters, fourths); connect real world, models, language and symbols.

SUB-TOPIC	YEAR 2, SEMESTER 1	YEAR 2, SEMESTER 2
Counting	Rote –forwards/backwards to 130 in 1s, 2s, 5s, and 10s. Rational –to 130.	<i>Sorting/Correspondence</i> – reinforce 1:100 as well as 1:10 and 1:1 correspondence.
	Odometer – experience to 130 (looking particularly at crossing the 100).	<i>Rote</i> – forwards/backwards to 1000 in 2s and 5s; counting halves, thirds and quarters.
		<i>Rational</i> – reinforce to 130, experience to 1000 (including collections of 10 and 50).
		Odometer – experience counting forwards/ backwards past multiples of 10 and 100 (10, 20, 30 and so on), discuss outcomes (patterns of ones, teens, and decades), understand to 130, experience to 1000; looking at odometer process as count halves, thirds and quarters.
Whole number	<i>Place value</i> – reinforce teens and zeros to 100, introduce symbol, language	Place value – connect symbol (teens and 10s), language (number names to 100) models to 1000
concepts	(number hames) with materials and PVCs to 130. Rank = to 130 (placing numbers on number line)	Rank - reinforce to 1000
	Multiplicative structure – to 100.	Multiplicative structure – experience to 1000.
Whole number	Reading/Writing – to 130.	Reading/Writing – to 1000.
and decimal processes	Seriation – to 130.	Seriation – experience to 1000 (1, 10 and 100 more, 1, 10 and 100 less).
P	Comparing/Ordening – to 130.	Comparing/Ordering = 10,1000. Renaming = experience to 1000.
	Rounding/Estimating – to 100.	Rounding/Estimating – experience to 1000.
Fraction concepts	<i>Common fractions</i> – experience equal parts, halves, quarters, thirds.	<i>Common fractions</i> – create equal parts; identify halves, quarters and thirds (real world examples, area model, language, materials, symbols). <i>Mixed numbers</i> – wholes with halves, thirds, and quarters on whole-part chart.
Fraction processes	<i>Reading/Writing</i> – halves, thirds and quarters.	<i>Reading/Writing</i> – common fractions and mixed numbers for halves, thirds and quarters.
		<i>Comparing/Ordering</i> – halves, thirds, and quarters when denominators the same.

SUB-TOPIC	YEAR 3, SEMESTER 1	YEAR 3, SEMESTER 2
Counting	Rote – reinforce forwards/backwards to 1000 in 1s, 2s, 5s, 10s and 100. Rational – reinforce to 1000. Odometer – reinforce to 1000.	Rational – extend beyond 1000 to 10 000; count wholes and parts for fractions. Odometer –extend beyond 1000 to 10 000; experience with wholes and fractions.
Whole number and decimal concepts	Place value – reinforce to 1000 (real world, symbol, language, materials and PVC). Rank – reinforce number-line model to 1000. Multiplicative structure – reinforce to 1000 (e.g. 8→800 is ×100 and 700→7 is \div 100).	 Place value – use pattern of threes (e.g. O, T, H ones → O, T, H thousands) to introduce place value to 10 000; connect real world, symbol, language, model. Rank – introduce line model to 10 000. Multiplicative structure – introduce for ones to 100 thousand, focus on ×1000 and ÷1000.
Whole number and decimal processes	Reading/Writing – reinforce to 1000. Seriation – reinforce to 1000. Comparing/Ordering – reinforce to 1000. Renaming – reinforce to 1000. Rounding/Estimating – reinforce to 1000.	Reading/Writing – introduce for 5-digits (to 10 000). Seriation – connect to odometer to understand one more or less for any PV position up to 100 thousands. Comparing/Ordering – experience to 10 000. Renaming – experience to 10 000. Rounding/Estimating – introduce for 5 digits.
Fraction concepts	Common fractions – experience part of a whole, part of a set and number line for a variety of fractions (connecting real world situations, area models, language). Mixed numbers – experience wholes with a variety of fraction [discuss application of notion of a unit big idea].	Common fractions – reinforce part of a whole, part of a set and number line understanding for a variety of fractions. Mixed numbers – reinforce understanding of wholes with fractions using a whole-part chart; relate to PV for whole and decimal numbers; experience improper fractions.
Fraction processes	<i>Reading/Writing</i> – experience a variety of common fractions and mixed numbers. <i>Comparing/Ordering</i> – experience comparing/ordering fractions and mixed numbers with materials and number lines.	Reading/Writing – common fractions and mixed numbers. Comparing/Ordering – fractions with common denominators; unit fractions. Renaming–renaming wholes to parts to get improper fractions.

SUB-TOPIC	YEAR 4, SEMESTER 1	YEAR 4, SEMESTER 2
Counting	Rote, Rational – reinforce forwards/backwards to 10 000. Odometer – reinforce to 10 000; reinforce counting wholes and fractions.	Rational – extend beyond 10 000 to 100 000; extend to 10ths and 100ths. Odometer – extend beyond 10 000 to 100 000 (develop general rule for counting forward/back through any PV position, e.g. 388, 389, 390, 391; 4362, 4361, 4360, 4359).
Whole number and decimal concepts	Place value – reinforce to 10 000; connect real world, symbol, language, model; introduce symbol, language, set model and real world to 10ths. Rank – number-line model to 10 000 (focusing on quantities of 100); introduce 10ths (number-line model), relate to reality, language and symbols. Multiplicative structure – to 10 000 (e.g. 7 →7000 is ×1000 and 3000→3 is ÷1000); experience place value pattern built on ×10 / ÷10.	Place value – symbol, language, set model and real world to 100ths. Rank – representing 100ths on number-line model and relating to real world, language and symbols. Multiplicative structure – introduce place value pattern built on multiplication/division of 10s; extend to 10ths (e.g. 10ths→1s is ×10 and 1s→10ths is ÷10).
Whole number and decimal processes	Reading/Writing – reinforce to 10 000 (connections words ← → symbols. Seriation – reinforce to 10 000. Comparing/Ordering – reinforce to 10 000. Renaming – reinforce to 10 000; introduce to 10ths and 100ths. Rounding/Estimating – reinforce to 10 000.	 Reading/Writing – introduce 6 digits (to 100 000); introduce 10ths, 100ths. Seriation – connect to odometer (one more or less for any PV up to 100 000). Comparing/Ordering – experience to 100 000. Renaming – experience to 100 000; reinforce to 10ths and 100ths. Rounding/Estimating – introduce for 6 digits.
Fraction concepts	Common fractions – introduce part of a whole/part of a set for all fractions (real world, area and length models, language, symbols); reverse activities to find whole from part; introduce 10ths and 100ths as a fractions. Mixed numbers – reinforce that have to 'unitise' a set of objects so that it is seen as one whole.	<i>Common fractions</i> – reinforce part of a whole/part of a set; experience fractions on a number line and fractions as division; reinforce 10ths and 100ths as common fractions; link common fractions to decimal fractions for 10ths and 100ths. <i>Mixed numbers</i> – reinforce improper fractions.
Fraction processes	Reading/Writing – introduce common fractions and mixed numbers for families of fractions (halves, quarters, eighths or thirds, sixths). Comparing/Ordering – fractions with common denominators; unit fractions; introduce equivalent fractions. Renaming – renaming wholes to parts to get improper fractions and reverse.	Reading/Writing – reinforce common fractions and mixed numbers for families of fractions (halves, quarters, eighths or thirds, sixths). Comparing/Ordering – fractions with common denominators; unit fractions. Renaming – renaming wholes to parts to get improper fractions and reverse.

SUB-TOPIC	YEAR 5, SEMESTER 1	YEAR 5, SEMESTER 2
Counting	<i>Rote, Rational, Odometer</i> – reinforce to 100 000, 10ths, 100ths.	<i>Rote, Rational, Odometer</i> – extend to 1000ths and beyond; extend to millions and beyond.
Whole number and decimal concepts	<i>Place value, Rank, Multiplicative structure</i> – symbol, language, set model, number-line model and real world to 100 000 (emphasising patterns of three).	<i>Place value, Rank, Multiplicative structure</i> – symbol, language, set model and real world to 1000ths (emphasise, e.g. 4 10ths = 40 100ths) and beyond.
Whole number and decimal processes	Reading/Writing – reinforce 6 digits (to 100 000). Seriation – reinforce odometer to understand one more or less for any PV position up to 100 thousands, 100ths, 10ths. Comparing/Ordering – reinforce to 100 000, 100ths, 10ths. Renaming – reinforce to 100 000, 10ths, 100ths. Rounding/Estimating – reinforce for 6 digits.	Reading/Writing, Seriation, Comparing/Ordering, Renaming, Rounding – from 1000ths up to millions and beyond each way.
Fraction concepts	<i>Common fractions</i> – reinforce part of a whole, part of a set; introduce number-line model and relate to other models, real world, language, symbols; introduce fractions as division.	<i>Common fractions</i> – reinforce part of whole/set, number line and division meanings of fractions; link common fractions to decimals for 1000ths and beyond. <i>Mixed/Improper fractions</i> – reinforce through counting.
Fraction processes	<i>Reading/Writing</i> – introduce common fractions and mixed numbers for families of fractions (halves, quarters, eighths or thirds, sixths). <i>Comparing/Ordering</i> – fractions with common denominators; unit fractions. <i>Renaming</i> – renaming wholes to parts to get improper fractions and reverse.	Counting – like denominator unit fractions $(1/4, 2/4, 3/4)$. Comparing/Ordering – fractions with like denominator. Renaming – improper fractions \rightarrow mixed numbers through counting with materials $(1/4, 2/4, 3/4, 4/4, 5/4 = 1 1/4)$.

SUB-TOPIC	YEAR 6, SEMESTER 1	YEAR 6, SEMESTER 2
Counting	<i>Rational, Odometer – g</i> eneralise counting/odometer principles to any whole number PV position.	<i>Rational/Odometer</i> – generalise counting/odometer principles to any PV position (including decimal PVs); reinforce prime, composite, square and triangular numbers; reinforce positive and negative numbers.
Whole number and decimal concepts	<i>Place value, Rank, Multiplicative structure</i> – symbol, language, set and number-line model and real world to any whole number (emphasise that the number system is structured around pattern of threes, adjacent positions relating by × or ÷ 10, and decimal point can be placed anywhere); introduce prime, composite, square and triangular numbers; introduce positive and negative numbers.	<i>Place value, Rank, Multiplicative structure</i> – symbol, language, set and number-line model and real world to any number (emphasise that the number system is structured around symmetry of PV position around 1s); reinforce prime, composite, square and triangular numbers; reinforce positive and negative numbers.
Whole number and decimal processes	<i>Reading/Writing, Seriation, Comparing/Ordering, Renaming, Rounding</i> – to any whole number.	<i>Reading/Writing, Seriation, Comparing/Ordering, Renaming, Rounding</i> – to any number.
Fraction concepts	Common fraction – reinforce meanings part of whole/ set, number line and division. Mixed/Improper – experience mixed numbers/improper fractions. Equivalent fractions – experience simple equivalences with materials. Percent – introduce notion of percent as part of a whole (broken into 100 equal parts) and as 100ths on PV chart; reverse to find whole from %.	<i>Mixed/Improper</i> – formally introduce mixed numbers/improper fractions (area and number-line model). <i>Equivalent fractions</i> – introduce notion of equivalences (e.g. $\frac{4}{6} = \frac{2}{3}$), develop sequences of equivalent fractions ($\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$ and so on). <i>Percent</i> – reinforce notion of percent as part of a whole (broken into 100 equal parts) and as 100ths on PV chart; reverse to find whole from %.
Fraction processes	Counting/Renaming – like denominator improper fractions and mixed numbers. Comparing/Ordering – fractions with like numerator. Conversion – fraction to decimals and vice versa (emphasise relationship, e.g. $^{24}/_{100} = 0.24$).	Renaming – improper ← → mixed numbers. Comparing/Ordering – fractions using models (e.g. number line). Conversion – fraction, decimals and percent.

SUB-TOPIC	YEAR 7, SEMESTER 1	YEAR 7, SEMESTER 2
Counting	Odometer – continue to reinforce this principle.	Odometer – apply principle to counting of fractions.
Whole number and decimal concepts	<i>Place value, Rank, Multiplicative structure</i> – reinforce structure of the number system (whole number and decimal PVs); define prime, composite numbers.	<i>Place value, Rank, Multiplicative structure</i> – reinforce structure of the number system (whole number and decimal PVs).
Whole number and decimal processes	Reading/Writing, Seriation, Comparing/Ordering, Renaming, Rounding – reinforce these for the number system (whole number and decimal PVs); compare prime, composite numbers; introduce prime factors and factor trees; investigate square-root notation.	Reading/Writing, Seriation, Comparing/Ordering, Renaming, Rounding – reinforce these for the number system (whole number and decimal PVs); reinforce prime, composite numbers, prime factors and square-root notation.
Fraction concepts	Mixed/Improper – reinforce mixed numbers/improper fractions. Equivalent fractions – reinforce equivalence fractions in many models, use lists to explore rule/pattern for equivalence. Percent – reinforce notion of percent (relate RW situations to set, area, number-line models, language and symbols); introduce ratio and rate (link to fractions and percent).	Mixed/Improper – reinforce mixed numbers/improper fractions. Equivalent fractions – extend equivalence to mixed numbers and improper fractions, reinforce notion of common denominator. Percent – reinforce notion of percent; introduce notions of discount, profit and loss; reinforce ratio and rate (link to fractions and percent).
Fraction processes	<i>Comparing/Ordering</i> – use equivalence to change fractions to common denominators and to determine order. <i>Conversion</i> – reinforce conversion between fraction to decimals and percent.	<i>Comparing/Ordering</i> – use common denominator to order any fraction. <i>Conversion</i> – reinforce conversion between fraction to decimals and percent.

Large numbers is an extension of the hundreds-tens-ones understanding. The same principles can be applied as those for 0 – 999. Large numbers are patterns of threes.

A1 Notion of unit, PV and reading and writing for large numbers

A1.1 Notion of unit

Big idea

Changing the perception of a unit as thousands and millions up to trillions – the 2 in 423 is 2 tens where tens is the unit, or 20 ones – must be applied to the understanding that the 2 in 423 000 is 2 ten thousands where ten thousand is the unit, or 20 thousands or 20 thousand ones.



Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Think of some way to get the body to experience a million. For example, act like the Roman army. They counted each time their left foot was placed down – they called this a passus. Walk 100 passuses around an oval – how far did you get? Think of a thousand passus, a million passus. Is it possible to count a long march in millions of passuses – what about 1000 passuses? [Note: 1000 passuses was called by the Romans a mile passus or a mile – the old British unit of length.]

Think of other ways to consider a million – how many plane trips across Australia – how long will this take?

Hand. Find ways to construct a million. For example, construct a metre cube with a kit or with metre rulers. Place a MAB cube inside it – 1000 cubes will fit in the cubic metre. Place a MAB unit inside the cubic metre – 1 000 000 units will fit in the cubic metre.

Mind. Ask students to shut eyes and imagine a million in some way. For example, imagine \$10 notes. The \$10 notes are packed into packets of 1000. Think of how many these would be? Pack the packets into a box that is 10 packets wide (10 shorter sides of the \$10 bills) and 10 packets long (10 longer sides of the \$10 bills). How high will this be?

Find other ways to imagine a million; for example, consider the MAB unit inside the MAB cube and/or the MAB cube inside the cubic metre.

Mathematics

Practice. Find ways to consider a million. For example, use the Internet to spend \$1 million dollars without buying anything over \$100 000. Was it more difficult than you thought?

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Connections. Take a large number with the macrostructure showing – for example, 47 million. Discuss what we would do if putting this number into boxes labelled millions, thousands and ones. Would we write "47" in the millions box or would we write "47 000 000"? Discuss why we write "47" – try to elicit it is because millions are our unit in this case.

Reflection

Applications. Apply the notion of considering very large groupings as single units back to real world. Where in the world do we have groups and ones interchanging depending on how we think of them? Consider particularly how to involve kilo, mega, giga, and tera as they refer to bytes. For example, what does a 135 gigabyte drive mean? Or in everyday language, what does a 135 gig drive mean? Or, how many megs is that memory stick?

Flexibility. Brainstorm examples of large numbers being used as units. For example, what is a millionaire, a billionaire? What does a \$21 trillion stimulus package mean? How many dollars?

Reversing. Go both ways – for example, give \$3 trillion and ask for dollars, and give \$3 000 000 and ask for how we can say this with less length in the numeral?

Generalising. Relate the making of large groups of numbers into units such as billions or gigs to other situations and try to get students to see that making new "units" is a generalisation that we always do when numbers get large. For example: (a) time – look at "centuries", "millennia", and names given to periods of earth's history such as "Jurassic"; (b) speed – look at "mach", and "speed of light" and names from science fiction such as that used in Star Trek for speed; and (c) distance – look at "light years" and names from science fiction. There must be other names – for example "par secs" – what is this? Use the Internet.

Changing parameters. What if we looked at very small instead of very large? There are special names for very small things (e.g. "pico", "nano") – look at these too – extend the generalisation to making units out of very small as well as very large.

A1.2 Place-value concept

Big idea

Large numbers are an extension of place value through a pattern of threes to large numbers. Place value is a visual image.

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Get six and then nine students to each take a digit from 1 to 9 and arrange themselves right to left to form a large number. Ask them to split into two (6 students) and then three groups (9 students) of three. Start from the right, discussing the place value of each student – get students to note the sequence from right:

hundred million, ten million, one million - hundred thousand, ten thousand, one thousand - hundred, ten, one

Discuss what each group of three could be called – thousands-ones for 6 students, and millions-thousands-ones for 9 students. Place a sign in front of each group with the macrostructure. Remind students that within the group of three there is the microstructure of hundreds-tens-ones (could give each student a hundred, ten or one along with their number). Do activities where you point to a position and students say the PV (e.g. 3 ten thousands) and the reverse, where you give a PV name (e.g. hundred million) and students point to it.



Millions	Millions	Millions	Thousands	Thousands	Thousands	Ones	Ones	Ones
Н	Т	0	Н	Т	0	Н	Т	0

Organise students to stand in order with PV cards. Get another student (or students) to move in front of PV cards with a digit card and ask remaining students to state PV position. Also reverse – state the PV position and get the remaining students to direct student with digit card to move to this position.

Hand. Look at 6 and 9-digit numbers and do the following (which is described for 9 digit numbers).

<u>Step 1</u>: Consider a nine-digit number such as 356 872 913, break digits into threes, i.e. 356 / 872 / 913, introduce and/or recognise that right-hand three digits are ones, middle three digits are thousands and left-hand three digits are millions. Cover the 872 913 with hand and say the remaining three digits as a three-digit number (i.e. three hundred and fifty-six) then add the millions – repeat covering 356 and the 913 and saying eight hundred and seventy-two thousands – repeat covering 356 872 and saying nine hundred and thirteen ones (remind that convention in ordinary speech is not to say the ones). Check students can see the PVs – teacher gives digit \rightarrow students give PV, and (reverse) teacher gives PV \rightarrow students give digit. Repeat this for other numbers from 5 to 9 digits.

<u>Step 2</u>: Construct and place the following cards on the wall, get a student to place a digit, say 4, in front of PV cards and to place that card in front of different PV cards.





Get the students to say the place value. For example, where the 4 is above is the ten-thousands PV position, it is "forty thousand". Ensure the activity is undertaken all ways – teacher places digit in front of PV card and asks students to say the PV, and teacher gives PV and students place digit.

<u>Step 3</u>: Provide a copy of the PVCs and digit cards on the next page. These PVCs have been designed to represent the macro and micro PV structure of large numbers. Organise the students to place their PVCs side by side (ones on right, thousands in middle and millions on left). Use the digit cards to represent numbers from 4 to 9 digits on the PVCs (choosing cards from a deck without K, Q, J or Joker, and with the Ace being a 1 and the 10 being a zero at random is a way to determine what the digits will be in each place). When the digits are placed, ask students to find ones, thousands or millions groups of three on PV charts and reverse by pointing to groups of three and asking students to give name. Repeat this for digits – teacher says digit \rightarrow students give PV, teacher gives PV \rightarrow students say digit. It is important to place digits on PVCs and to use the kinaesthetic sense to embed an understanding of the patterns of three.

Pattern-of-threes PVCs

TENS	ONES					
MILLIONS						
	TENS MILLIONS					

HUNDREDS	TENS	ONES					
THOUSANDS							

HUNDREDS	TENS	ONES				
ONES						

Digit cards

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

<u>Step 4</u>. In the situations in Steps 2 and 3, provide digits and place values out of order and construct the number. Start with macrostructure, for example, what number is 876 thousands, 5 millions, 234 ones? [5 876 234] Then do both structures together, for example, what number would be constructed from 4 ten thousands, 3 tens, 5 one millions, 7 hundred thousands, 8 hundreds, 4 ones and 0 one thousands? [5 740 834] Repeat this many times.

Reverse this – *get students to give all the different ways that 5 876 234 could be written.*

Mind. Shut eyes and imagine PVCs to help embed patterns from kinaesthetic experience to the mind. Assist this by, when placing material on PVC, always putting left hand on digits as the PV is said. Finally, write large numbers, look at them, break them into threes in the mind, assign macro-PV structure and say the PVs of some of the digits. Reversing, say PV positions and assemble this to a number in groups of three. Imagine numbers as being in a structure:

нто	НТО	н т о
millions	thousands	ones

Mathematics

Practice. Use charts and worksheets with charts to practise chart \rightarrow PVs and PVs \rightarrow chart. Then use worksheets to practise numerals \rightarrow PVs and PVs \rightarrow numerals. Also practise giving PVs out of order and determining numerals' positions. Reinforce with the game "Wipe-out" – see below.

Connections. Connect to metrics. The macro/microstructure in the PV charts above works excellently for km, m and mm; kL, L and mL; tonne, kg and g, and so on. For example, for length, the ones can be replaced with millimetres, the thousands with metres, and the millions with kilometres. More complex relationships exist for area and solid volume. These connections are an excellent way to build understanding of metric conversions.

Connect to indices – ones are 10° , tens are 10^{1} , hundreds are 10^{2} , thousands are to 10^{3} and millions are 10^{6} . Thus the PVs of large numbers (millions) are H-T-O 10^{6} H-T-O 10^{3} H-T-O 10° .

Connect to other names – bytes, kilobytes, and megabytes.

Game: Wipe-out (place value)

Materials: Calculator, worksheet (if wanted). Number of players: 2.

Directions: One student calls out a number, e.g. 673, 56 782, 24 875. Other students put in calculator then first student calls out a digit. Other students have to change number on calculator with a single subtraction, e.g. for "wipe the 7", answers would be 603, 56 082, 24 805. Do examples with two digit positions to wipe (e.g. 347 642 – wipe out both 4s). Can be done with a worksheet as below.

Number	Digit	Subtraction	Result
63 284	8	-80	63 204
7 452 892	5	-50 000	7 402 892

Reflection

Application. Use internet, papers and magazines to find examples of large numbers and get students to determine PVs of their digits.

Flexibility. Discuss and list different ways to get, for example, 7 536 500 (6 thousands, 5 hundreds, 7 millions, 3 ten-thousands, and 5 hundred-thousands; 75 365 hundreds; 7 536-and-a-half thousands; approximately 7½ millions; and so on). Extend applications and look for other examples of large numbers.

Reversing. This has been done throughout the cycle but check at this point that students can go from numbers to PV positions, and PV positions (even out of order) to numbers.

Generalising. Look at the pattern-of-threes structure (the macro and microstructure together as below.

нто	НТО	ΗΤΟ
millions	thousands	ones

Discuss how this could be extended – what is to the left of millions? Discuss other names, e.g. kilo, mega, giga, tera, and so on. Look at the numbers in terms of indices – how could this be extended?

Changing parameters. What happens if the base is not 10? For example, days, hours, minutes and seconds – how do these relate to "place value"? For example:



A1.3 Reading and writing process

Big idea

Building number as a visual image.

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Same as PV activities for smaller numbers, get six and then nine students to each take a digit from 1 to 9 and arrange themselves right to left to form a large number, splitting into groups of three. Again discuss what each group of three could be called – thousands-ones for 6 students, and millions-thousands-ones for 9 students. Place a sign in front of each group with the macrostructure. Remind students that within the group of three there is the microstructure of hundreds-tens-ones (could give each student a H, T or O along with their number). Get rest of class to read out number, pointing to the position as they say the digit – each group of three students steps forward as their three numbers are read (can get them to hold up number and H, T or O as their digit is read and to highlight the millions-thousands-ones). For example (9 students):



Again as for PV activities, construct following cards (can also have millions-thousands-ones under the H, T and O), get students to hold cards in order and other students to hold digit cards. Go both directions, teacher gives number \rightarrow students form this number, and teacher directs students to form a number \rightarrow other students read the number.

Millions	Millions	Millions	Thousands	Thousands	Thousands	Ones	Ones	Ones
н	Т	0	Н	Т	0	н	Т	0

Hand. Do the following as in PV activities (it is described for numbers up to 9 digits).

<u>Step 1</u>: Consider a nine-digit number such as 356 872 913, break digits into threes, i.e. 356 / 872 / 913, introduce and/or recognise that right-hand three digits are ones, middle three digits are thousands and left-hand three digits are millions. After looking at each part separately, combine the three parts and say the total number <u>pointing at group of three as you say the group</u> – three hundred and fifty-six millions, eight hundred and seventy-two thousands and nine hundred and thirteen. Repeat this for other numbers from 5 to 9 digits.

<u>Step 2</u>: Construct and place the following cards on the wall, get a student to place digits as follows in front of PV cards as below. Get the students to say the number shown, and then to write it on paper and/or enter it on a calculator. Allow students to experience place value in the following ways (<u>reversing</u>): (a) teacher places digits in front of PV cards and asks students to say and write/enter the number; (b) teacher says number and asks students to say PV and place the digits; and (c) teacher writes number and asks students to say number and place the digits.

Reality

Reflection

bstractio

Maths

Millions	Millions	Millions	Thousands	Thousands	Thousands	Ones	Ones	Ones
н	т	0	н	т	0	н	т	0
	3	4	5	0	7	3	7	2

<u>Step 3</u>: Provide a copy of the PVCs and digit cards below. Organise the students to place their PVCs side by side (ones on right, thousands in middle and millions on left). Use the digit cards to represent numbers from 4 to 9 digits. When the digits are placed, ensure the students do this: (a) say the number <u>moving hands to each PV</u> <u>position</u> as the digits are said, and <u>pointing to millions</u>, thousand and ones as they are said (or not said as the situation is for ones); and (b) write the number on paper and enter the number in their calculator.

It is important to place digits on PVCs and to use the kinaesthetic sense to embed an understanding of the patterns of three. It is also important to repeat this experience of PV in two ways (<u>reversing</u>): (a) teacher gives digits to be placed on the PVCs and asks students to read the number, write the numerals and enter the number on calculator; and (b) teacher says number or writes numerals and asks students to place digits on the chart and write/enter the number or say the number.

Pattern-of-threes PVCs

HUNDREDS	TENS	ONES				
MILLIONS						

HUNDREDS	TENS	ONES				
THOUSANDS						

HUNDREDS	TENS	ONES				
ONES						

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

<u>Step 4:</u> Reinforce reading and writing large whole numbers with number expanders. There are two special number expanders for large numbers – the millions and the mega number expander (for megabytes) – see below. These expanders have three digit positions and folds on ones, thousands and millions – showing the patterns of three. The number expander pleat folds at the coloured sections leaving 9 boxes for numbers. It then expands out to show the pattern of threes (e.g. 204 761 894 is 204 millions 761 thousands 894 ones).

Millions number expander



Mega number expander

		mega		kilo		ones
		mega		kilo		ones

Obviously, these number expanders are also useful for place value.

Mind. Shut eyes and imagine PVCs to help embed patterns from kinaesthetic experience to the mind. Assist this by, when placing material on PVC, always putting left hand on digits as they are said and moving left to right, saying the number as you go (e.g. three hundred and fifty-six millions, eight hundred and seventy-two thousands and nine hundred and thirteen ones as you move hand $L \rightarrow R$ across the digits and PV positions). Finally, write large numbers, look at them, break them into threes in the mind, assign macro-PV structure and say the number. Reversing, say the number and then write it down as groups of three numbers.

Give a real-world story: "Jim spent \$1 356 500 on his new mansion overlooking the ocean". Show the number on a PVC or millions number expander. Have students read the number. Say: Show me the ones part; show me the thousands part; show me the millions part. Say: How many are there in the ones section, how many are there in the thousands and millions section? <u>Swap roles</u> – the teacher says the number and the students show it on the PVC or number expander. Don't forget to ask the students to show the millions part of the number, the thousands part of the number and so on.

Mathematics

Practice. Use charts and worksheets with charts to practise representing, reading and writing numbers (chart \rightarrow language \rightarrow numerals **and** numerals \rightarrow language \rightarrow chart). Then use worksheets and calculators to practise reading and writing (numerals $\leftarrow \rightarrow$ language). For example, teacher calls digits \rightarrow students enter on calculator and say number, and teacher says number \rightarrow students enter on calculator and say digits pressed.

Spend time giving numbers for students to read and write. Use dice and cards to randomly select digits. Spend time giving varying numbers as digits for students to say as numbers; reverse and say number and students write as numerals or enter on calculator (e.g. the "Let your calculator do your talking" type activity). Give <u>special attention to teens and zeros</u>. Reinforce with the game "Target" – see next page.

Game: Target (reading and writing numbers, plus order and estimation)

Materials: Calculator, worksheet if necessary. Number of players: 2.

Directions: Give students a starting number and a target number, e.g. 37 and 9176. Enter $37 \times$ in calculator. Then press guess [-], guess [-], until get the target. (No pressing of "clear all".) Students take turns being the starting number provider. After 5 goes each, the winner is the student with the lowest number of guesses. Can be done with worksheet as below.

	Number	Target	Too high	Too low	Correct guess	Number of guesses
(a)						
(b)						

Connections. Connect to metrics (e.g. km, m and mm; kL, L and mL; tonne, kg and g, and so on) similarly to PV activities, but this time say the numbers when, for example, the ones are replaced with millimetre, the thousands with metres, and the millions with kilometres (e.g. three hundred and fifty-six million, eight hundred and seventy-two thousand and nine hundred and thirteen millimetres <u>is</u> three hundred and fifty-six kilometres, eight hundred and seventy-two metres and nine hundred and thirteen millimetres). More complex relationships exist for area and solid volume. These connections are an excellent way to build understanding of metric conversions.

Reflection

Applications. Use internet, papers and magazines to find examples of large numbers and get students to read and write these digits. Also set problems and investigations with large numbers that build understanding of whole numbers as quantity. For example:

"Spend \$100 000 000" – give catalogues of expensive items (such as cars, real estate, etc.) and give students a budget of up to \$100 million. May have to put restrictions (e.g. cannot buy two of the same thing).

Shop for data storage items that detail storage space in bytes (there are plenty of excellent online sites to visit). Have students write numbers onto expanders or PVC. Extend to ordering greatest to least amounts.

Look for examples of where big numbers would be used in the students' environment (e.g. bricks to build the school, leaves on a tree).

<u>Note</u>: Reverse these ideas – let the students make up their own prices for things and use to create their own catalogues.

Flexibility. Discuss and list different situations in which large numbers are used. For example, company profits, government budgets, Australia's deficit, expensive house prices, and so on.

Reversing. This has been done throughout the cycle but check at this point that students can go from PVs $\leftarrow \rightarrow$ number names, and number names $\leftarrow \rightarrow$ numerals.

Generalising. Look at large numbers and the pattern-of-threes structure that underlies them.

НТО	НТО	ΗΤΟ
millions	thousands	ones

Discuss how we get bigger numbers – elicit that the pattern of threes continues but with new names (e.g. billions, trillions, etc.). Discuss other names (e.g. kilo, mega, giga, tera, and so on). Look at the numbers in terms of indices – how could this be extended?

Changing parameters. What happens if the base is not 10? For example, days, hours, minutes and seconds – how do these relate to "place value"? For example, how many seconds in: 17 days, 19 hours, 35 minutes and 43 seconds?

A2 Counting, seriation and odometer for large numbers

A2.1 Counting

Big idea

Each large place-value position counts like the ones.

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Make single place flip cards as on right, ten cards 0 to 9 that can be flipped over, a larger card behind the ten cards with PV position written on the bottom part of the larger card (e.g. Ten Thousands). Five to nine students take a flip card each and set up in PV order $L \rightarrow R$. Teacher gives each position a starting digit. Students flip to this. Then a PV is chosen to count. The student in this position flips as students count. For forwards this means 5, 6, 7, 8, 9, 0 (with digit on LHS going up by 1), 1, 2, 3, 4, and so on. For backwards this means 2, 1, 0, 9 (with digit on LHS going back by 1)



Reality

Reflection

Abstractio

Math

Hand. Imitate the Body work above but with digit cards on a small PVC.

Hand out calculators, enter a large whole number (up to 8 digits), pick a PV (say ten thousands), add a 1 to that PV (i.e. +10 000), and keep entering =, =, =, and so on. Read the number at each equal or just the digit in the PV position. Repeat for counting backwards situations (i.e. -10 000). Keeping class together, clapping for = press, and together stating the number can be a good starting approach.

Write down each number as they press =. See examples of counting forward for ten-thousands position and counting backwards for hundred-thousands position:

376 581	4 276 401
386 581	4 176 401
396 581	4 076 401
406 581	3 976 401
416 581	3 876 401
426 581	3 776 401

Discuss that the counting in any PV position is like counting in the ones position. Show this by covering all of the number except the chosen PV position and the PV position on its left – this would make the LHS count forwards – example 37, 38, 39, 40, 41, 42 – same as counting by ones.

Mind. Shut eyes and imagine a number, pick a PV position, and count in that position forwards and backwards.
Mathematics

Practice. Repeatedly practise with and without calculator to count in any position forward and backward. Also use worksheets to continue patterns where a PV is counted forward and backward as below:

78 690 237, 78 790 237, 78 890 237, _____, ____, ____, ____,

Connections. Relate to counting in any situation, time, money, measures (km, m and mm), fractions, and so on.

Reflection

Applications. Look at where large numbers have counting in high PV positions – use internet and talk to people to find examples (e.g. population). Think of problems with counting in high PV positions.

Flexibility. Try to brainstorm all situations where counting in high PV positions is used – often in estimation situations. For example, estimating budgets and populations.

Reversing. Make sure you go from stating PV position \rightarrow counting forward and backward in that position, and counting forward and backward in a position \rightarrow identifying PV position.

Generalising. Continue work in Abstraction to ensure students see the counting generalisation for any place value.

Changing parameters. Extend the generalising of whole numbers above to counting in measures, fractions, and so on. Build a meaning for counting that applies to anything.

A2.2 Seriation process

Big idea

The ability to determine one more or less in each place value (e.g. 10 000 more than 53 000, 100 000 less than 409 000).

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Use the single place flip cards as in 2.1 but simply flip forward and back one flip to find the one PV forward and backward.

Hand. Use digit cards on a small PVC and calculators as in 2.1 but flip only once and press only one = so that you get one PV forward and backward. Try to get the students to think seriation for 10 000 PV for 345 672 is the same as seriation for the digit in the PV position. Here the digit is 4 – so 1 before and after for 4 is 3 and 5; this means 10 000 before and after for 345 672 is 335 672 and 355 672. It is also possible to develop "99 boards" for, say, 300 thousand to 399 thousand and to use the 99 board technique.

Spend extra time looking at one PV more and less for examples that are next to points where numbers change from 0 to 9 and vice versa. In other words, for example:

one 10 000 more and less

for 3 597 320, for 2 608 234, for 5 991 405, for 6 008 431, for 69 992 430 and for 70 002 765

Mind. Again shut eyes and imagine as in 2.1 but again only one PV number forward and backward.

Reality

Reflection

Abstraction

Maths

Mathematics

Practice. Give numbers and PV positions and ask for the one before and after in that PV position. Look especially at the 0, 00, 9 and 99 type numbers. Special activities to reinforce seriation are as follows (need to find PV to seriate first – in these examples it is 10 000).

• Give 3×3 squares with number in middle and one other number and ask for other numbers:

448 275	
458 275	

• Give two numbers outside the middle one and ask for other numbers:

	528 104	
537 104		

• Complete the "jigsaw" piece – the PV is given on top LHS. What would happen if we changed the PV to 100 000 or 1 000?

10 000 PV



Connections. Relate to one before and one after in many other situations (e.g. measures).

Reflection

Applications. Look for application of seriation for high PVs in large numbers. Often used in estimation.

Flexibility. Try to find a wide variety of situations where one more/less large PV are used (e.g. estimation examples).

Reversing. Go both ways – $PV \rightarrow$ one more/less, and one more/less $\rightarrow PV$.

Generalising. Develop a generalisation for seriation – has four parts:

- one PV more and digit in PV between 0 and 8 \rightarrow same number but PV position increased by one
- one PV more and digit in PV at 9 → same number but PV position reduced to 0 and PV position on left increased by one
- one PV less and digit in PV between 1 and 9 \rightarrow same number but PV position decreased by one
- one PV less and digit in PV at 0 → same number but PV position increased to 9 and PV position on left decreased by one

Changing parameters. As for 2.1, looking at measures, fractions, etc.

A2.3 Odometer principle

Big idea

The ability to determine one more or less in each place value (e.g. 10 000 more than 53 000, 100 000 less than 409 000).

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Use the same material as in 2.1, but as you state numbers give only the position digit and focus on the change at 9 (forward) and 0 (backward). For example: (a) counting forward by 10 000s: 384 761, 394 761, 404 761, 414 761, and so on, becomes 8, 9, 0, 1, and so on; and (b) counting backward by 100 000s: 4 281 034, 4 181 034, 4 081 034, 3 981 034, and so on, becomes 2, 1, 0, 9, and so on.

Hand. Repeat the activities from 2.1, but again focus on the change to the PV digits at 9 and 0. Add in an extra activity of making an odometer – take foam cups and put ten digits 0 to 9 and lines regularly around the top of each cup (so the cup is divided into tenths) – put each cup inside the other and then you can turn each cup like an odometer.

Mind. Shut eyes and imagine but focus on the counting passing 9 forward or 0 backward.

Mathematics

Practice. Follow ideas from 2.1 and 2.2 – but always get students to see that the digits increase 7, 8, 9 then drop to 0 while PV on left increases by one; and digits decrease 2, 1, 0 then increase to 9 while the PV on left decreases by one.

Connections. Once again show how this odometer idea works for fractions, measures, and so on, as well as whole numbers.

Reflection

Applications. Look at problems that apply to odometer situations in everyday life – for example, the odometer in a car.

Flexibility. Try to find as many odometer situations as you can.

Reversing. Try always to have lessons that go from: PV and number \rightarrow counting pattern; and counting pattern \rightarrow PV and starting number.

Generalising. Develop a generalisation for odometer – has two parts: (a) PV, starting number and counting forward \rightarrow 7, 8, 9, 0, 1, and so on, PV to left increasing by one; and (b) PV, starting number and counting backward \rightarrow 3, 2, 1, 0, 9, and so on, PV to left decreasing by one.

Changing parameters. Look at action of odometer in measures such as time (e.g. years, months, days, hours, minutes, seconds), fractions, and other situations.



A3 Multiplicativity and renaming for large numbers

A3.1 Multiplicative relationship concept

Big idea

The principle of multiplicativity applies no matter how large the number is.

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Give students cards and organise to stand in line. Get other students with number cards (one, two and more digits) to stand in front of place-value cards and to move left and right. Other students use their calculators to determine what multiplication or division (\times 10, \div 10, \times 100, \div 1000, \div 1000) is the same as one, two, and three places to left or right.



Hand. Students follow what happens at the front with their small PVCs and digit cards, and relate changes to multiplication and division on the calculator. Then, activity can just be with the small PVCs, digit cards and calculators. Then it can move to slide rule – see end of this section.

Mind. Encourage the students to find and write down patterns in movements and their relation to $\times 10$, $\div 10$, $\times 100$, $\div 100$, $\times 1000$, and $\div 1000$. Ask students for a pattern (i.e. move left one place is $\times 10$ and move right one place is $\div 10$). Spend time on the three PV position movements ($\times 1000$, and $\div 1000$) – that is, movements across the macrostructure (from ones to thousands to millions).

Mathematics

Practice. Worksheets – these should operate in both directions, giving the multiplication or division and asking for the movement, and giving the movement and asking for the \times and \div 10 or 100 or 1000.

Reflection

Validation. Students communicate their understandings of patterns discovered in Mathematics above.

Generalising. The understanding that moving left is $\times 10$ and moving right is $\div 10$ has to be generalised to where the structure of the number system is understood in terms of any adjacent place-value positions relating $\times 10$ and $\div 10$, with this relationship being continuous across all place values and bi-directional in application. There also has to be a similar generalisation across the macrostructure, that is, that one move left across macrostructure is $\times 1000$ and move right is $\div 1000$.



Changing parameters. Encourage students to see that other number ideas are also related in a similar way but possibly with different × and \div . For example weeks–days relate ×7 and \div 7, hours and minutes relate ×60 and \div 60. Metrics relate via 10 but, of course, the mm–m, the g–kg and mL–L relate ×1000 and \div 1000. The relationship between metric units is highly connected to movements across the macrostructure (and this is one of the reasons it is very important to teach this).

A3.2 Renaming process

Big ideas

Students must be comfortable working with the larger place values.

Abstraction



Use M-TH-O number expanders to show, e.g. that 273 365 478 is 273 M, 365 TH and 478 O but also 273 M and 365 478 O as well as 273 365 478 O – that is, renaming across the macrostructure. Number expanders are shown below.

Use the full number expander to relate changes from one PV position to the next, e.g. 3 Hun TH and 14 Ten TH is the same as 4 Hun TH and 4 Ten TH.

Use the H-T-O number expander for ones, thousands and millions to show renaming within macrostructure – make sure you go both ways, e.g. 273 thousand is 1 hundred thousand, 16 ten thousands and 13 one thousands; 2 hundred thousands, 25 ten thousands and 34 one thousands is 484 thousand.

Number expanders

Make large copies of the following number expanders.

Full number expander:



M-TH-O number expander:



H-T-O number expander for Thousands (similar ones can be made for Ones and Millions):



A4 Continuous–discrete, rank, ordering and rounding for large numbers

A4.1 Continuous–discrete

Big idea

The principle of continuous and discrete applies no matter how large the number is.

Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Look at large whole numbers with respect to discrete and continuous entities. Undertake some investigations involving the students and large numbers. For an example that can represent both, look at the investigation to find out if students were to hold hands across Australia, how many students would you need? In terms of discrete, the investigation will give a number of students. In terms of continuous, the investigation will provide a measure of distance across Australia in terms of student fathoms. Discuss the differences in both numbers (e.g. 0 means nothing in discrete but the start of the measuring in continuous; number can be naturally applied to students as objects because they are discrete but distance needs a unit to be repeated across the length to enable number to be applied).

Hand. Investigate large numbers in discrete situations (e.g. population, money, animals in the wild, TV set production, and so on). Investigate large numbers in continuous situations (e.g. length/distance, area, volume, mass, and so on). Discuss differences.

Mind. Shut eyes and imagine number being applied to discrete things (i.e. forming groups, and groups of groups, and so on) and to continuous things (i.e. using units and conversions between units).

Discuss what things can be counted and what cannot. Point out that, normally, items have to be discrete (individual and separated) to be counted. Point out that the world is full of discrete things (chairs, people, animals, days, grains of sand, etc.) but that some things have had to be "changed" to be countable. For example, the beach cannot be counted but grains of sand can; the length of a building cannot be counted but the number of bricks long can. Discuss length – it is not countable unless a unit of measure is used. Length is continuous (as is area, volume, mass, time, and so on) but units make it discrete. Ask why we want to turn the continuous into the discrete? (So we can apply number to it.)

Mathematics

Practice. Do activities where students look at situations (can be represented by pictures on a worksheet) and classify them as discrete or continuous applications of number.

Connections. Gather all discrete and continuous examples into two separate groups. Connect the various examples in each classification.

Reflection

Applications. Set problems with regard to number in discrete and continuous situations.



Flexibility. Try to brainstorm all the major discrete and continuous situations. In particular, look for number situations that can be both continuous and discrete depending on how they are perceived.

Reversing. Go both ways: starting with a situation \rightarrow determining if discrete or continuous; and starting with one of discrete or continuous \rightarrow constructing a situation.

Generalising. Try to identify characteristics of each of the classifications (discrete and continuous) that can be generalised across all examples of each classification. Get students to generalise that discrete can be counted but continuous needs a way (e.g. a unit of measure) to change it into discrete for a number to be used on it.

Changing parameters. Look for situations that are both discrete and continuous. Generalise the characteristics of such dual situations. Is it true that all discrete situations can be perceived as continuous and vice versa?

A4.2 Rank and number-line concept

Big idea

The concept that the further along the number line a number is, the larger it is. This RAMR cycle is about comprehending the size of these large numbers and working with them. We CANNOT use the rule that the more numerals in a number, the larger it is as this concept cannot be transferred to decimal numbers.



Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Put a start and end number on two students (say 0 on one student and 100 000 000 on the other) and put them at the front (0 on left looking towards the front). Get the two students to hold a rope between them (reasonably taut). Give other students numbers on paper between 0 and 100 000 000 and pegs and they have to peg on rope where they think this number would be. Get other students to help more accurately place the number. Discuss where numbers would be (e.g. in middle, near an end). Repeat for other starting and ending numbers. Go both ways (i.e. reverse) – give number \rightarrow students place; give position \rightarrow students guess number.

If students having difficulties, either (a) divide line into ten parts and stick paper at 10 000 000, 20 000 000, and so on, so students can see in which ten million interval to place number; or (b) round numbers to nearest thousand or million so students can use knowledge from hundreds-tens-ones (e.g. 54 678 203 is 55 million (rounding) and this is a little over halfway between 0 and 100 million, so 54 678 203 is a little over halfway between 0 and 100 million, so 54 678 203 is a little over halfway between 0 and 100 million.

Hand. Repeat the above for number lines with various ending numbers, e.g. 0 to 100 000. Give students numbers to place on the line. Point to a position on the line and get students to estimate the number.

If students are having difficulties determining points, divide the line into 10 sections (e.g. mark 10 000, 20 000, and so on) so students have references from which to place and estimate numbers. Teachers can even construct a ten-straw line with each straw representing 10 000 units (see below). Stick numbers on ends of straws as shown.



Mind. Have students imagine the line in their mind and then use the imagined line to place numbers.

Mathematics

Practice. Have students practise placing numbers on number lines and determining what number would be in a position on a number line using worksheets.

Connections. Connect number to measures, particularly of distance. Have students use mm rulers or tape measures to measure things. Discuss how to measure – align 0 with start, read answer off tape. Notice numbers are at the end of the spaces not in the middle of the spaces. Reverse the activity – ask students to find a measure on lines; ask students to measure an object.

Reflection

Applications. Set problems of numbers on number lines – use measurement situations.

Flexibility. Brainstorm all the number-line situations for large numbers – how about plans for a building in millimetres?

Reversing. Ensure you always go both ways: number \rightarrow position on line; and position on line \rightarrow number. Ensure that students construct number lines as well as interpret them.

Generalising. Relate placing large whole numbers on number lines to placing rounded numbers on number lines (see Body). Enable students to see, for example, that placing 27 608 112 on a 0 to 100 000 000 number line is the same as putting 28 on a 0 to 100 number line, and placing 456 789 on a 0 to 100 000 000 number line is the same as putting 5 on a 0 to 1 000 number line.

Changing parameters. If students are capable, expand number lines to include non-decimal measures – for example, nautical miles which are in whole numbers, minutes and tenths of minutes.

A4.3 Comparison and order process

Big idea

Numbers can be ordered by their distance along the number line. The concept that the further along the number line a number is, the larger it is. This RAMR cycle is about comprehending the size of these large numbers and working with them.



Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Repeat the body idea from 4.2 but, this time, use two numbers and state which is larger or three numbers and state order smallest to largest. If comparing/ordering numbers like 567 891 and 602 349, use a 0 to 1 000 000 rope. Place the two numbers; the one further from 0 is the larger (the other is the smaller). If necessary, stick 100 000, 200 000, 300 000, and so on, on the rope and then use these to place the two numbers. Spend time on numbers that cross macrostructures – for example 2 090 452 and 897 650. For these numbers use a 0 to 10 000 000 rope (divided into 1 million intervals if necessary). It can then be seen that 2 090 452 is between 2 and 3 million while 897 650 is between 0 and 1 million.

Hand. Repeat the hand ideas from 4.2 but, again, place two or more numbers on the number line and use the one further from 0 to find the larger of two numbers and the one furthest from 0 to find the largest of three

numbers. The straw construction is probably not necessary. Ensure that students do all of these: (a) place numbers and compare/order; (b) place one number and find a number that is larger/smaller; and (c) place two or more numbers and find the number that is largest, smallest or a number in between.

Try to understand what makes a number larger by recording sets of two numbers aligned underneath each other by PV, using placement on number line to find larger, ticking larger, then looking at examples to find a pattern. For example:

 478 302 ✓
 3 456 721
 45 706
 262 897 ✓
 34 502 761 ✓

 456 892
 4 034 561 ✓
 67 329 ✓
 95 092
 9 678 891

These examples give the pattern that the number with the biggest number in the highest PV position is the larger (taking into account that 95 092 is the same as 095 092 and therefore has a 0 in the 100 000 PV position). Also look at the 478 302 and 456 892 example to see that if the left digits are the same, you have to look at the second-left digits and so on.

Mind. Repeat the mind ideas from 4.2, but imagine two or more numbers and then compare or put in order.

Mathematics

Practice. Practise comparing/ordering numbers by using number lines. Discuss the pattern from Hand above, and use this to compare/order numbers. Do worksheets where students circle larger or smaller – do worksheets where students put numbers in order from largest to smallest and vice versa – do worksheets where students find numbers between, bigger or smaller than given numbers. Play large-number versions of games "Target" (see A1.3), and "Chance number" and "Chance order" (see below and next page).

Game: "Chance number" (comparison) – 6-digit version

Materials: Digit cards, 6-digit "Chance number" board (as on right), digit cards to fit into board, card deck (0–9 only).

Directions: Nine versions. (1) Teacher (or another student) deals six cards, students use numbers to make smaller/larger number with digit cards on game board (specify whether they have to try to make smallest or largest number for each game). Winner is student with smallest/largest number, as applicable. (2) Eight cards are dealt from which to choose six. Winner is same as (1). (3) Teacher (or another student) deals six cards one at a time, students use first number to place a digit card on board in PV of own choice, other numbers fill the other positions. Student who makes higher/lower number (as applicable) scores 1 point, 0 otherwise. Winner is largest score after five games. (4) Same as (3) but, at end, students can give up a number and take the value of a seventh dealt card. (5) Same as (3) but students can give up three numbers and another three cards dealt (one at a time) – can set rule that numbers cannot be risked from the same place value. (6) Same as (3) but score if closest to 500 000. (7) Same as (3) but score if student beats the teacher who is also playing. (8) Same as (3) but score only the LH digit on the board. (9) Any mix of the above.

Game: "Chance order" (comparison) – 6-digit version

Materials: Digit cards, 6-digit "Chance order" board – greater than or less than version (as on right), card deck (0–9 only), digit cards.



Directions: Six versions. (1) Teacher (or student) deals 12 cards, students use numbers to make the left-hand 6-digit number less than the right-hand number with digit cards on game board. Score 1 point if left-hand 6-digit number is correctly less than right-hand 6-digit number. Score 2 points if smaller 6-digit number is the largest possible (while still being less than the RH number). The winner has highest score after five games. (2) Teacher (or student) deals 12 cards **one at a time**, students use first number to place a digit card on any PV on LH or RH side of board, students continue making choices and placing digits on board before next card called. Score 1 if LH number less than RH number and 0 if not. The winner is highest score after five games. (3) Teacher (or another student) deals 12 cards **one at a time**, students use first number to place a digit card on board as in (2), students continue making choices and placing digits on board before next card called. Score 1 if LH number less than RH number and 0 if not. The winner is highest score after five games. (4) Same as (3), but score sum of two LH digits. (5) Same as (1) to (4) but students have to make the LH 6-digit number.

Connections. Reinforce the relation between order being furthest along a line with order being number with biggest digit in largest PV. Relate this to ordering in measures.

Reflection

Applications. Apply order to real-world situations where numbers have to be larger.

Flexibility. Try to brainstorm all situations where order of large numbers matters in the world.

Reversing. Go both ways: (a) numbers \rightarrow order; and (b) order (and some numbers) \rightarrow all numbers.

However, teachers should direct students to also spend time developing/constructing number lines for different types of numbers (e.g. 7-digit numbers); as well as determining/interpreting where numbers are on lines that have been given to them.

This is the **construction-interpretation** big idea and is very important. This is because construction leads to better interpretation and a lot of interesting learnings. One of these is to discuss how best to construct number lines. So when constructing number lines, discuss what works best (too many points and it's too crowded; not enough points and it is too hard to read). And do, for example, 70 000–90 000 lines as well as 0–100 000 lines.

Generalising. Spend time eliciting the pattern for order – three steps:

- align place values and add zeros if necessary so that there is a digit in each PV;
- look at left-most or highest PV, larger digit gives large number; and
- *if left-most digits the same, move to second left, third left, and so on, until you have a larger digit (and thus a larger number).*

Overall, build the idea that, for comparison and order, the larger PVs matter. Note that the games "Chance number" and "Chance order" reinforce this.

Changing parameters. Does the above generalisation still work when not in a base of ten (e.g. hours, minutes, seconds)?

A4.4 Rounding and estimation process

Big idea

Number lines enable values to be determined to which numbers under consideration are nearest. This RAMR cycle gives practice using the larger place values.



Reality

Where possible, find real-life contexts to embed the activities in; for example, using relevant objects or situations.

Abstraction

Body. Use the same rope materials as in earlier activities but with the rope divided into 10 parts with end points marked (as below for 5-digit numbers). Numbers are placed on the rope and then decisions made as to which ten thousand they are closest to. For example, 37 352 is closer to 40 000 than 30 000, so we say it is 40 000 rounded to the nearest ten thousand. Can also determine to which 5 000 it is closest – here, 37 352 is closer to 35 000 than to 40 000, so an estimate to nearest 5 000 is 35 000.

0	10 000	20 000	30 000	40 000	and so on	90 000	100 000

Hand. Prepare materials as we have earlier but with parts marked in initially. This way students can not only place the number but can also see to what other numbers the original number is closest. Try to discover a pattern from the place values for rounding and estimation. Take a number, say 368 921, compare it with PVs aligned to nearest 100 000s and 10 000s as below and star the correct rounding:

400 000 *	370 000 *
368 921	368 921
300 000	360 000

Two things have to be discovered by looking at a lot of examples:

- to find PV above and below, go to that position in the number and increase digit by one for above, and leave digit as is for below (e.g. for 6 540 872: 7 000 000 > 6 540 872 > 6 000 000); and
- to find closest rounded number with respect to given PV, look at the digit to the right of the PV digit and go to the lower digit if below 5 or go to the higher digit if 5 and above (e.g. for 6 540 872, the 540 872 means that 7 million is the nearest million, but the 40 872 means that 65 hundred thousand is the nearest hundred thousand).

Mind. Imagine these things in the mind – develop pictures of being nearer other numbers due to digits in the largest place values.

Mathematics

Practice. Practise rounding and estimating – first with number lines and then without, as the pattern from the Hand subsection above becomes more evident. Encourage students with difficulties to place things in <u>place-value alignment</u>, and to use seriation to obtain the above and below numbers for different PVs.

500	000	nearest 100 000	480	000	nearest 10 000
475	982		475	982	
400	000		470	000	

Connections. Connect this to measures and even to fractions.

Reflection

Applications. Apply these ideas to problems in everyday life.

Flexibility. Brainstorm all situations where we would want to round large numbers. Also look at when rounding is better than accuracy and how we would choose the level of accuracy required. This is part of the accuracy vs exactness big idea.

Reversing. <u>Don't forget this</u> – make sure teaching goes: number \rightarrow estimate **and** estimate \rightarrow number (e.g. have activities where students select numbers to which the estimate is applicable). For example:

- Round 347 229 to the nearest 10 000
- Which of the following numbers could be rounded to 560 000 if rounding to the nearest 10 000 547 821, 564 929, 59 876, 555 009, and so on.

Generalising. Ensure that the pattern from practice above is well known. It has two sections: (a) choosing PV above and below, and (b) deciding to which of these two the number is closer. It uses seriation with respect to the PV to find the PV above. It uses the PV one position to the right to determine whether the number goes down or up (i.e. 0, 1, 2, 3 and 4 is down, and 5, 6, 7, 8, and 9 is up).

Changing parameters. Again, does this work when not in a base 10 situation (e.g. nautical miles)?

A5 Equivalence for large numbers

Equivalence in large numbers is a fairly simple extension of equivalence in small numbers. As we discussed in section 3.5, equivalence of whole numbers depends on the purpose of the numbers. If the purpose is to provide a measure of value, then the normal rules for zeros follow (e.g. 043 = 43). However, if the purpose is to identify, then the normal rules do not apply (e.g. for the numerical part of a car's registration, 043 is very different to 43 - in fact, 43 would be seen as a partial plate). We will now briefly look at large whole numbers from the point of view of numbers as value and other purposes of number.

A5.1 Equivalence of whole numbers depends on point of view

- 1. Whole numbers as value. For large numbers, the rules for equivalence when looking at value are as follows:
 - zeros placed before the other numerals do not change value, for example: 3 245 678 = 003 245 678;
 - placing a zero elsewhere does change the value of the number, and this is true if more than one zero is involved, for example: 87 432 ≠ 807 432, 870 432, 874 032, 874 302 or 874 320; and
 - removing a zero does not change a number's value if the zero was before the other numerals, otherwise it changes the value, for example: 065 289 = 65 289 but 650 289 ≠ 65 289.
- 2. **Other purposes of number**. The first point about purposes for number other than value is that adding and removing zeros anywhere else than before the other numerals **does change** the number (the same rule as for value). Thus, we are only looking here at zeros **before** the other numerals.

What is being proposed is that there are some situations where zeros before the other numerals does change the number. These situations are as follows.

Identity numbers. The first situation is when numbers are used to identify things (and people). An
identity card with a number 000568 is often different to one with 568. In fact the difference is one of
belonging. The 000568 identity card probably has to have 6 digits (3 digits is not acceptable). So for
small numbers, we add the zeros so that the card is valid.

- *Phone numbers*. A second example is phone numbers. Phoning 0467 234 431 is very different to phoning 467 234 431.
- Order numbers. A third example is using numbers to order files in computer systems that use alphabetic systems to order. Since AB is before B then 10 is before 2. This necessitates ensuring that zeros are placed in front of numbers because ordering files labelled 4587 and 43311 gives a different result to ordering numbers 04587 and 43311.

Thus in the modern digital age, there are many situations where a number such as 045876 is not equivalent to 45876.

A5.2 RAMR lesson for equivalence with large whole numbers

Big idea

It is possible for a number to be the same and the numeral to be different. Equivalence of whole numbers depends on the point of view.



Reality

Prerequisites. Check that students understand equivalence for 2-and 3-digit numbers. Check that students have the understanding that different reasons for numbers can change whether numbers are equivalent.

Local knowledge. Try to find something in life where students are labelled with a number and how we can change those labels so they are different (e.g. student ID numbers).

Abstraction

Body. Set up mat as a PVC for less than 9 digits. Get students with bibs to stand in positions and read the numbers. Have a zero pushing in between numbers – read the numbers, are they different because of the zero? Where can the zero stand that does not make the number different?

Set up a 9-digit number with zeros using bibs. Ask students with zeros what the zero does to the number. Is it necessary to be there? What value does the number have without it?

Hand. Redo the steps in the RAMR lesson in section 3.5 using larger numbers – thousand and millions – use the computer to order 08 874, 09 874, 10 874 and so on. Use digit cards on a PVC to show how adding zeros within the numerals moves some of the numerals on the left of the extra zeros to higher place-value positions, changing the value of the numbers. Then show that this does not happen if the zero is on the left of all the numerals, that is, 007 891 is equivalent to 7 891.

Spend time looking for situations where a zero added to the left of the numerals does change things, such as phone numbers, identity numbers, computer ordering, and so on.

Discuss the role of zero in numbers. Look at where zeros change and zeros don't change in numbers. Look at examples like this: Circle YES or NO

83 654 \rightarrow 830 654 number changed in value – YES/NO?

056 218 \rightarrow 56 218 number changed in value – YES?NO

Then get students to construct these on PVC with digit cards: (a) give students a number to start with (say, 304) and ask them to do something to that number that changes it and something that does not change it; and (b) give students a number (say, 407) and ask them for a starting number and what was done to it that changes value and does not change value. Ask students for more than one answer.

Sort the following numbers into groups where the number is the same:

024 361 000 0 240 361 0 024 361 2 004 361 24 361 024 361

Mind. Imagine a number with no zeros – imagine putting in zeros – when/where does the zero make a difference? Imagine a number with zeros – start removing them, when does it make a difference?

Mathematics

Practice. Practise situations where zeros change and don't change numbers. Examples can be: (a) circle change – tick no change questions; (b)sorting numbers with some zeros into same-number groups; or (c) students use zeros to make change or no change.

Connections. Look where we have situations like this in mathematics where zeros are important; for example, length, mass, time, identification numbers, invoice numbers, and so on.

Reflection

Validation/Applications. Ask students to explore the world and find everywhere that numbers are expressed in different numerals to the normal 6, 11, 25, 327, etc. For example 24-hour time for plane timetables, and military operations when time and angles (bearings) are written in a specific pattern. Make up problems about these.

Flexibility. When numerals are different – why? What is the generalisation? Why is it done? [If numbers are to be ordered, and they go to a certain number of digits, then must fill all digits with zeros on left even though one is an early number; e.g. 00006732 is the 6732^{nd} person but needs the eight digits as this is what computer uses.]

Reversing. Reverse everything. If 164 has to be written as 000164, then what number is 002060?

Generalising. Generalise the ways numerals can be changed with a zero without changing the number (e.g. adding 0s at the start of a whole number) and changing the number (e.g. 0s in between digits or at end of digits of a whole number).

Changing parameters. What if we have decimal numbers – does this change the rules? How?



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